J&R #2.16)

G1: Completeness axiom.
G2: Transitivity axiom.

$A = \{a_1, \ldots, a_n\}$: set of (certain) outcomes

① Show that there exists $a_j \in A$ such that $a_j \succsim a_i$, $\forall a_i \in A$.

i) Let's begin with $n=2$: $A = \{a_1, a_2\}$.

By completeness, we have either $a_1 \succsim a_2$ or $a_2 \succsim a_1$. Hence, there is a best outcome.

ii) Take now $n=3$: $A = \{a_1, a_2, a_3\}$.

By completeness, we have

$$a_1 \succsim a_2 \text{ or } a_2 \succsim a_1$$
$$a_2 \succsim a_3 \text{ or } a_3 \succsim a_2$$
$$a_1 \succsim a_3 \text{ or } a_3 \succsim a_1$$

Completeness then implies any of the following possibilities:

$$a_1 \succsim a_2 \text{ with} \rightarrow \quad a_2 \succsim a_3 \text{ and } a_1 \succsim a_3$$
$$a_3 \succsim a_2 \text{ and } \checkmark$$
$$\checkmark \text{ and } a_3 \succsim a_1$$
$$a_2 \succsim a_1 \text{ with} \rightarrow \quad a_2 \succsim a_3 \text{ and } \checkmark$$

By transitivity, we must rule out all cases where $a_i \succsim a_j$, $a_j \succsim a_k$ and $a_k \succsim a_i$. We are thus left with

$$a_1 \succsim a_2,\ a_2 \succsim a_3 \text{ and } a_1 \succsim a_3$$
$$a_1 \succsim a_2,\ a_3 \succsim a_2 \text{ and } a_1 \succsim a_3$$
$$a_1 \succsim a_2,\ a_3 \succsim a_2 \text{ and } a_3 \succsim a_1$$

$a_2 \succsim a_1$, $a_2 \succsim a_3$ and $a_1 \succsim a_3$
$a_2 \succsim a_1$, $a_3 \succsim a_2$ and $a_3 \succsim a_1$
$a_2 \succsim a_1$, $a_2 \succsim a_3$ and $a_3 \succsim a_1$

All six cases imply an ordering such that
$a_i \succsim a_j \succsim a_k$.

iii) Take any $n$: $A = \{a_1, a_2, \ldots, a_n\}$

Assume not. Then $\forall a_j$, $\exists a_i$ s.t.
$a_i \succsim a_j$ and $\exists a_k$ s.t. $a_k \precsim a_j$.

Take any $a_i$, $a_j$, $a_k$ s.t. $a_i \succsim a_j$
$a_j \succsim a_k$

J&R #2.19)

Axiom G3: Continuity

$\forall g \in G,\ \exists \alpha \in [0,1]$ s.t. $g \sim (\alpha \circ a_1, (1-\alpha)\circ a_n)$.

Show that $\alpha$ is unique if the following axiom holds:

G4: Monotonicity

$\forall \alpha, \beta \in [0,1]$, $(\alpha \circ a_1, (1-\alpha)\circ a_n) \succsim (\beta \circ a_1, (1-\beta)\circ a_n)$ iff $\alpha \geq \beta$.

DEMO: Let $g \sim (\alpha \circ a_1, (1-\alpha)\circ a_n)$.

Assume that $\alpha$ is not unique. Then there exists $\beta \in [0,1]$, $\beta \neq \alpha$, such that

$$g \sim (\beta \circ a_1, (1-\beta)\circ a_n).$$

By transitivity, we have:

$$(\alpha \circ a_1, (1-\alpha)\circ a_n) \sim (\beta \circ a_1, (1-\beta)\circ a_n). \quad [*]$$

Without loss of generality, take $\beta > \alpha$. According to the monotonicity axiom, we should have

$$(\alpha \circ a_1, (1-\alpha)\circ a_n) \not\succsim (\beta \circ a_1, (1-\beta)\circ a_n),$$

which contradicts [*]. Hence, $\alpha$ must be unique.

J&R 2.22) Show that



G2: Transitivity.
G3: Continuity.
G4: Monotonicity.
$\Rightarrow$ G1: Completeness

PROOF: Take any two gambles $g, g' \in \mathcal{G}$.

By the continuity axiom, we know that there exist $\alpha, \beta \in [0,1]$ such that

$$g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$$
$$g' \sim (\beta \circ a_1, (1-\beta) \circ a_n)$$

By monotonicity, we have that

$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ iff $\alpha \geq \beta$

and ( — " — ) $\precsim$ ( — " — ) iff $\beta \geq \alpha$.

By transitivity, this implies that

$$g \succsim g' \text{ iff } \alpha \geq \beta$$
$$g' \succsim g \text{ iff } \beta \geq \alpha$$

Since $\alpha$ and $\beta$ both exist, then either $\alpha \geq \beta$ or $\beta \geq \alpha$ so that either $g \succsim g'$ or $g' \succsim g$, which is the completeness axiom.

QED

J&R 2.25)

VNM utility fctn: $U(m) = a + bm + cm^2$.

a) The agent is risk averse iff $U(m)$ is strictly concave. (p. 106)

$\Rightarrow U''(m) < 0$.

$\Rightarrow 2c < 0 \Rightarrow \boxed{c < 0}$

We also want utility to increase with wealth, i.e. $U'(m) > 0$.

$U'(m) = b + 2cm > 0$

$\Rightarrow \boxed{b > 0}$ since $c < 0$.

As for parameter a, its value is irrelevant since VNM utility fctns are unique up to positive affine transformations.

$\Rightarrow \boxed{a \gtrless 0}$

b) As long as $U'(m) > 0$, i.e.

$U'(m) = b + 2cm > 0 \Rightarrow \boxed{m < -\frac{b}{2c}}$

c) i) Certainty Equivalent (CE) is defined as the certain wealth level such that the person is indifferent with the gamble; i.e.

$U(g) = U(CE)$.

where $g = \left(\frac{1}{2} \circ (m+h), \frac{1}{2} \circ (m-h)\right)$

We have

$$U(CE) = a + bCE + c(CE)^2$$

$$U(g) = \tfrac{1}{2}U(m+h) + \tfrac{1}{2}U(m-h)$$

$$= \tfrac{1}{2}\left(a + b(m+h) + c(m+h)^2\right)$$

$$+ \tfrac{1}{2}\left(a + b(m-h) + c(m-h)^2\right)$$

$$= a + bm + \tfrac{1}{2}c\,\underbrace{\left((m+h)^2 + (m-h)^2\right)}_{\substack{= m^2+2mh+h^2+m^2-2mh+h^2 \\ = 2(m^2+h^2)}}$$

$$\Rightarrow U(g) = a + bm + c(m^2+h^2) = U(CE)$$

$$E(g) = \tfrac{1}{2}(m+h) + \tfrac{1}{2}(m-h) = m$$

$$\Rightarrow U(E(g)) = U(m) = a + bm + cm^2$$

Hence,

$$U(CE) = a + bm + c(m^2+h^2) < a + bm + cm^2 = U(E(g))$$

since $c < 0$.

$\Rightarrow CE < E(g)$ since $U'(m) \geq 0$. QED

ii) Risk premium ($P$) is defined as an amount of wealth that one is willing to forgo in order to avoid risk and receive the expected gain from a gamble; i.e.

$$U(E(g) - P) = U(g).$$

$$\Rightarrow U(E(g) - P) = U(CE)$$

Since $E(g) > CE$ and $U'(m) > 0$, it must be the case that $P > 0$. <u>QED</u>

<u>NB</u> One can also verify that

$$a + b(m-P) + c(m-P)^2 = a + bm + c(m^2 + h^2) \Rightarrow P > 0.$$

$$\Rightarrow -bP + \cancel{cm^2} - 2cmP + cP^2 = \cancel{cm^2} + ch^2$$

$$\Rightarrow \underbrace{\left[-(b+2cm) + cP\right]}_{<0} P = \underbrace{ch^2}_{<0} \qquad [**]$$

$\Rightarrow P > 0.$ <u>QED</u>

d) Decreasing absolute risk aversion (DARA):

$$\Rightarrow R_a'(m) = \frac{\partial}{\partial m}\left[\frac{-U''(m)}{U'(m)}\right] < 0$$

$\Rightarrow$ As an individual becomes richer, she becomes <u>more willing</u> to accept a given gamble. $\Rightarrow P'(m) < 0.$

We have:

$$R_a(m) = \frac{-2c}{b+2cm}$$

$$\Rightarrow R_a'(m) = \frac{(2c)^2}{(b+2cm)^2} > 0 \Rightarrow \text{ARA increases with wealth}$$

<u>NB</u> Verify by showing that the risk premium increases with wealth. By implicit differentiation, we have, from [**]:

$$\frac{dP}{dm} = -\frac{-2cP}{-(b+2cm)+2cP} > 0 \quad \text{QED}$$

J&R 2.27) $u(w) = \alpha + \beta \ln(w)$

DARA $\Rightarrow R_a'(w) < 0$.

We have $u'(w) = \dfrac{\beta}{w}$

$$u''(w) = \frac{-\beta}{w^2}$$

$$\Rightarrow R_a(w) = -\frac{u''(w)}{u'(w)} = -\frac{-\cancel{\beta}/w^2}{\cancel{\beta}/w} = \frac{1}{w}$$

$$\Rightarrow R_a'(w) = \frac{-1}{w^2} < 0. \quad \underline{\text{QED}}$$

J. & R. 2.29)

We have: $u(E(g)) > u(g) \Rightarrow$ risk averse
$u(E(g)) = u(g) \Rightarrow$ risk neutral
$u(E(g)) < u(g) \Rightarrow$ risk loving

We must show that

$$CE < E(g) \Leftrightarrow u(E(g)) > u(g).$$

where CE is defined as $u(g) = u(CE)$.

i) Show that
$$CE < E(g) \Rightarrow u(E(g)) > u(g):$$
Let $CE < E(g)$. Then $u(CE) < u(E(g))$ since $u'(w) > 0$. But $u(CE) = u(g)$. Hence, $u(g) < u(E(g))$. QED

ii) Show that
$$u(E(g)) > u(g) \Rightarrow CE < E(g):$$
$u(E(g)) > u(g) \Rightarrow u(E(g)) > u(CE)$ by def. of CE.

Since $u'(w) > 0$, we have $E(g) > CE$.

QED.

J. & R. 2.38)

$$u(x_t) = -\frac{1}{2}(2-x_t)^2 \Rightarrow u'(x_t) = 2-x_t,\ u''(x_t) = -1$$

$\beta \equiv$ discount factor

$x_0 + \beta x_1 = y_0 + \beta y_1$ : intertemporal budget constraint

a) $y_0 = y_1 = 1$ : Certain income levels.

Problem of consumer:

$$\max_{\{x_0, x_1\}} V_0 = u(x_0) + \beta u(x_1)$$

$$\text{s.t.} \quad x_0 + \beta x_1 = 1 + \beta$$

Substituting the budget constraint, we have:

$$\max_{x_0} V_0 = u(x_0) + \beta u\left(\frac{1+\beta-x_0}{\beta}\right)$$

$$\text{FOC}: \frac{\partial V_0}{\partial x_0} = u'(x_0^*) - \beta u'(x_1^*) \cdot \frac{1}{\beta} = 0$$

$$\Rightarrow u'(x_0^*) = u'(x_1^*)$$

$$\Rightarrow \boxed{x_0^* = x_1^* = 1}$$ Perfect consumption smoothing.

$$\Rightarrow V_0^* = -\frac{1}{2} + \beta\left(-\frac{1}{2}\right) = -\frac{(1+\beta)}{2}$$ : Lifetime level of utility

b) $y_0 = 1$

$$y_1 = \begin{cases} y_1^H & \text{with probab. } \frac{1}{2} \\ y_1^L & \text{" " } \frac{1}{2} \end{cases}$$

$y_1^H = \frac{3}{2}$, $y_1^L = \frac{1}{2}$.

The intertemporal budget constraint:

$$x_0 + \beta x_1 = y_0 + \beta y_1$$

$$\Rightarrow x_1 = (y_0 + \beta y_1 - x_0)\frac{1}{\beta}$$

$$\Rightarrow x_1 = \begin{cases} \frac{1}{\beta}(y_0 - x_0 + \beta y_1^H) \equiv x_1^H \text{ with prob. } 1/2 \\ \frac{1}{\beta}(y_0 - x_0 + \beta y_1^L) \equiv x_1^L \quad '' \quad '' \quad 1/2 \end{cases}$$

The problem of the expected-utility maximizer is thus:

$$\max_{x_0} E(V_0) = u(x_0) + \beta\left\{\frac{1}{2} u\left(\frac{y_0 - x_0 + \beta y_1^H}{\beta}\right) + \frac{1}{2} u\left(\frac{y_0 - x_0 + \beta y_1^L}{\beta}\right)\right\}$$

$$\frac{\partial}{\partial x_0} E(V_0) = u'(x_0) - \beta\left\{\frac{1}{2} u'(x_1^H)\frac{1}{\beta} + \frac{1}{2} u'(x_1^L)\frac{1}{\beta}\right\} = 0$$

$$\Rightarrow u'(x_0) = \frac{1}{2}\left\{u'(x_1^H) + u'(x_1^L)\right\}$$

We have $u'(x_t) = 2 - x_t$.

$$\Rightarrow 2 - x_0 = \frac{1}{2}\left\{2 - x_1^H + 2 - x_1^L\right\}$$

$$\Rightarrow 2x_0 = x_1^H + x_1^L = $$
$$= \frac{2}{\beta}(y_0 - x_0) + y_1^H + y_1^L$$

$$\Rightarrow 2x_0 = \frac{2}{\beta}(1 - x_0) + \frac{3}{2} + \frac{1}{2} = \frac{2}{\beta}(1 - x_0) + 2$$

$$\Rightarrow \beta(x_0 - 1) = 1 - x_0 \Rightarrow \boxed{\begin{aligned} & x_0^* = 1 \\ \Rightarrow\ & x_1^H = y_1^H = 3/2 \\ & x_1^L = y_1^L = 1/2 \end{aligned}}$$

$$\frac{9}{8} + \frac{1}{8} = \frac{10}{8} = \frac{5}{4}$$

$$\Rightarrow E(V_0^0) = -\frac{1}{2} + \beta\left(\frac{-1}{2}\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} + \frac{-1}{2}\left(\frac{3}{2}\right)^2 \cdot \frac{1}{2}\right)$$

$$= -\frac{1}{2}\left(1 + \frac{5}{4}\beta\right)$$

$\Rightarrow E(U_0'') < -\frac{1}{2}(1+\beta) = U_0^*$

Even though the expected consumption levels are the same in a) and b), the expected utility level is lower in b) because the consumer is risk-averse ($u''(x_t) < 0$).