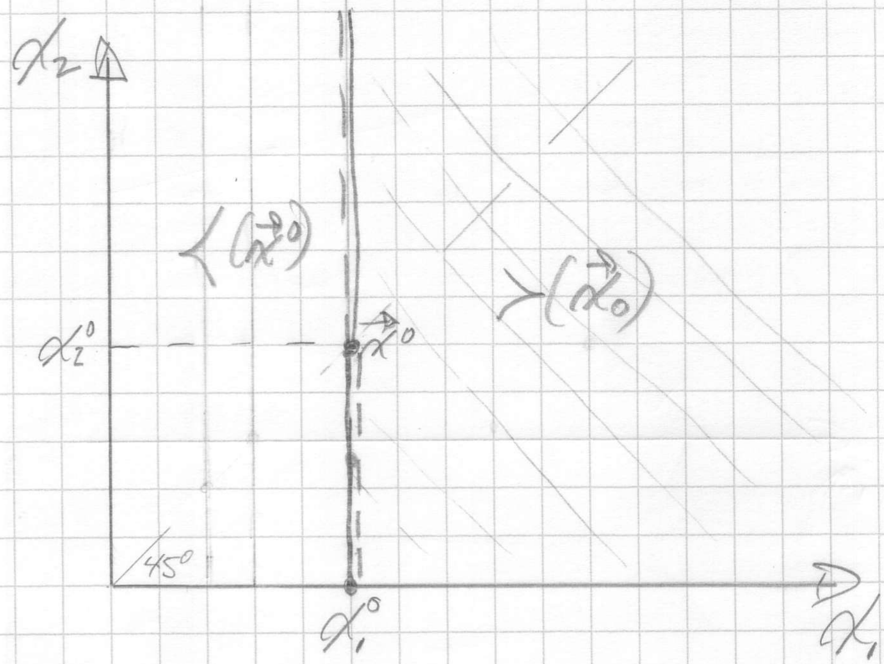
J&R. #1.13

Lexicographic preferences:

$\vec{x}^1 \succsim \vec{x}^2$ iff $x_1^1 > x_1^2$

or $x_1^1 = x_1^2$ and $x_2^1 \geq x_2^2$

a)



① All points east of x_1^0 are preferred to $\vec{x}^0$.

② Bundles with $x = x_1^0$ for which $x_2 > x_2^0$ are also preferred to $\vec{x}^0$.

③ Bundles with $x_1 = x_1^0$ for which $x_2 < x_2^0$ are worse than $\vec{x}_0$.

④ All points west of x_1^0 are worse than $\vec{x}^0$.

⇒ No indifference curve can be drawn.

b) (Work in progress.)

J&R 1.18) $x_1^* > 0$ and $x_2^* = 0$.

The consumer's problem is:

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq y.$$
$$x_1 \geq 0, \; x_2 \geq 0.$$

$$\Rightarrow \mathcal{L} = u(x_1, x_2) + \lambda(y - p_1 x_1 - p_2 x_2) + \mu_1 x_1 + \mu_2 x_2$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u}{\partial x_1} - \lambda^* p_1 + \mu_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda^* p_2 + \mu_2 = 0.$$

$$y - p_1 x_1^* - p_2 x_2^* = 0.$$ (assuming strict monotonicity)

$$\mu_1 x_1^* \geq 0, \; x_1^* \geq 0, \; \mu_2 x_2^* \geq 0, \; x_2^* \geq 0.$$

If $x_2^* = 0$ and $x_1^* > 0$, then $\mu_1 = 0$ and $\mu_2 \geq 0$

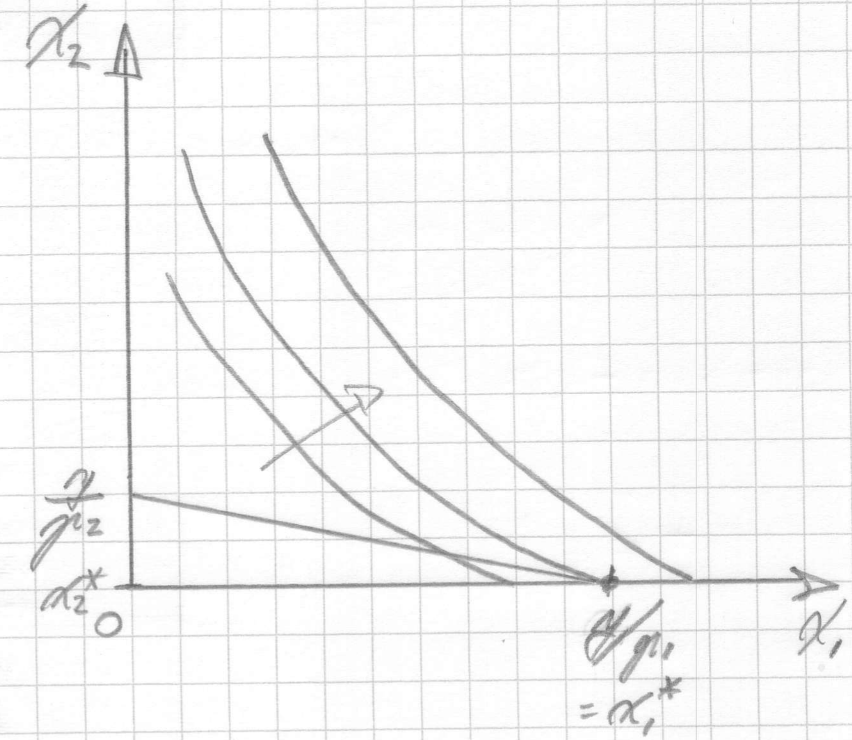
$$\Rightarrow \frac{\partial u(x_1^*, 0)}{\partial x_1} - \lambda^* p_1 = 0$$

$$\frac{\partial u(x_1^*, 0)}{\partial x_2} - \lambda^* p_2 = -\mu_2 \leq 0.$$

$$\Rightarrow \frac{\frac{\partial u(x_1^*, 0)}{\partial x_1}}{\frac{\partial u(x_1^*, 0)}{\partial x_2}} = \frac{p_1}{p_2 - \frac{\mu_2}{\lambda^*}} \geq \frac{p_1}{p_2}$$

Interpretation: At $x_1 = x_1^*$ and $x_2 = 0$, if I forgo one unit of good 1 to buy p_1/p_2 units of good 2, my utility does not increase since $\frac{\partial u}{\partial x_2} \cdot \frac{p_1}{p_2} \leq \frac{\partial u}{\partial x_1}$.

1.18) cont'd.



J&R #1.19)

Prove that the utility fctn is invariant to positive monotonic transforms.

We have to show that for any function $f(u)$ which is strictly increasing, then, for any bundle $\vec{x}^1, \vec{x}^2 \in \mathbb{R}^m_+$, if $u(\vec{x}^1) \geq u(\vec{x}^2)$ then

$f(u(\vec{x}^1)) \geq f(u(\vec{x}^2))$.

Assume not. Then we can have

$u(\vec{x}^1) \geq u(\vec{x}^2)$

and $f(u(\vec{x}^1)) < f(u(\vec{x}^2))$.

This violates the fact that $f(u)$ is strictly increasing.

QED

J. & R. 1.23)

① Show that

$u(\vec{x})$ strictly increasing $\Leftrightarrow$ $\succsim$ strictly monotonic.

DEFINITIONS:

Strict monotonicity of $\succsim$: If $\vec{x}^0 \geqq \vec{x}^1$ then $\vec{x}^0 \succsim \vec{x}^1$ and if $\vec{x}^0 \gg \vec{x}^1$ then $\vec{x}^0 \succ \vec{x}^1$.

Strictly increasing $u(\vec{x})$ (p. 437):

$u(\vec{x}^0) \geqq u(\vec{x}^1)$ if $\vec{x}^0 \geqq \vec{x}^1$.

$u(\vec{x}^0) > u(\vec{x}^1)$ if $\vec{x}^0 \gg \vec{x}^1$.

PROOF:

1st part: ($\Leftarrow$) Strict monotonicity of $\succsim$ $\Rightarrow$ $u(\vec{x})$ strictly increasing.

We must show that

a) $\vec{x}^0 \geqq \vec{x}^1 \Rightarrow u(\vec{x}^0) \geqq u(\vec{x}^1)$ since $\vec{x}^0 \succsim \vec{x}^1$.

and b) $\vec{x}^0 \gg \vec{x}^1 \Rightarrow u(\vec{x}^0) > u(\vec{x}^1)$ since $\vec{x}^0 \succ x^1$.

This is true by the very definition of a strictly increasing function $u(\vec{x})$.

QED for $\Leftarrow$.

1.23) cont'd.

2^{nd} part; ($\Rightarrow$) $u(\vec{x})$ strictly increasing

$\Rightarrow \succsim$ are strictly monotonic.

If $u(\vec{x})$ is strictly increasing, then it tells us that

a) $u(\vec{x}^0) \geq u(\vec{x}^1) \;\forall\; \vec{x}^0 \geq \vec{x}^1$. Hence it implies $\vec{x}^0 \succsim \vec{x}^1 \;\forall\; \vec{x}^0 \geq \vec{x}^1$.

b) $u(\vec{x}^0) > u(\vec{x}^1) \;\forall\; \vec{x}^0 \gg \vec{x}^1$. Hence, it implies $\vec{x}^0 \succ \vec{x}^1 \;\forall\; \vec{x}^0 \gg \vec{x}^1$.

The above two implications correspond to the definition of $\succsim$ being strictly monotonic.

QED for $\Rightarrow$.

1.23) cont'd

② Show that

$u(\vec{x})$ quasi-concave $\Longleftrightarrow$ $\succsim$ convex.

Definition: Let $\vec{x}^t = t\vec{x}' + (1-t)\vec{x}^0$.

Convexity of preferences $\succsim$:

$$\vec{x}' \succsim \vec{x}^0 \Rightarrow \vec{x}^t \succsim \vec{x}^0, \ \forall t \in [0,1]$$

Quasi-concavity of $u(\vec{x})$:

$$u(\vec{x}^t) \geq \min[u(\vec{x}'), u(\vec{x}^0)], \ \forall t \in [0,1].$$

PROOF:

Part 1: ($\Longleftarrow$) Convex preferences $\Rightarrow u(\vec{x})$ quasi-concave.

Assume not. Then, for some $t \in [0,1]$, we have $\vec{x}^t \succsim \vec{x}^0$ while $u(\vec{x}^t) < u(\vec{x}^0)$. This contradicts the fact that $u(\vec{x})$ represents preferences. QED for $\Leftarrow$.

Part 2: ($\Longrightarrow$) $u(\vec{x})$ quasi-concave $\Rightarrow$ convex preferences.

Assume not. Then, we have $u(\vec{x}^t) \geq u(\vec{x}^0) \ \forall t \in [0,1]$ while $\vec{x}^t \prec \vec{x}^0$ for some $t \in [0,1]$. This contradicts the fact that $u(\vec{x})$ represents preferences. QED for $\Rightarrow$.

1.23) cont'd

③ $u(\vec{x})$ strictly quasi-concave

$\iff \succsim$ strictly convex.

Definitions: Let $x^t = t\vec{x}^1 + (1-t)\vec{x}^0$.

$u(\vec{x})$ strictly quasi-concave:

$\forall \vec{x}^1 \neq \vec{x}^0$, $u(x^t) > \min[u(\vec{x}^1), u(\vec{x}^0)]$

$\forall t \in (0,1)$.

Strictly convex preferences:

$\forall \vec{x}^1 \neq \vec{x}^0$ and $\vec{x}^1 \succsim \vec{x}^0 \Rightarrow \vec{x}^t \succ x^0$, $\forall t \in (0,1)$.

PROOF:

Part 1: ($\Leftarrow$) Strict convexity of $\succsim$

$\Rightarrow u(\vec{x})$ strictly quasi-concave.

Assume not. Then we have $\vec{x}^t \succ \vec{x}^0$ $\forall t \in (0,1)$ while, for some $t \in (0,1)$, $u(x^t) \leq u(x^0)$. This contradicts the fact that $u(x)$ represents preferences.

QED for $\Leftarrow$.

Part 2: ($\Rightarrow$) $u(\vec{x})$ strictly quasi-concave

$\Rightarrow$ strict convexity of $\succsim$.

Assume not. Then we have

$u(x^t) > u(\vec{x}^0)$ $\forall t \in (0,1)$ while

for some $t \in (0,1)$, $\vec{x}^t \precsim \vec{x}^0$: a contradiction.

QED for $\Rightarrow$.

J. & R. 1.34)

Show that

$e(t\vec{p}, u) = t\, e(\vec{p}, u)$.

PROOF:

We have,

$e(\vec{p}, u) = \min_{\vec{x}} \vec{p}\vec{x}$ s.t. $u(\vec{x}) = u$.

$e(t\vec{p}, u) = \min_{\vec{x}} t\vec{p}\vec{x}$ s.t. $u(\vec{x}) = u$.

Let $e(\vec{p}, u) = \vec{p}\vec{x}^*$

and $e(t\vec{p}, u) = t\,\vec{p}\,\vec{x}'$.

It suffices to show that $t\vec{p}\vec{x}' = t\vec{p}\vec{x}^*$.

Assume not. Then, since $\vec{x}'$ minimizes expenditure at $t\vec{p}$ and u, we must have

$$t\,\vec{p}\,\vec{x}' < t\,\vec{p}\,\vec{x}^*.$$

$$\Rightarrow \quad \vec{p}\,\vec{x}' < \vec{p}\,\vec{x}^*.$$

This contradicts the fact that $\vec{x}^*$ minimizes expenditures at $\vec{p}$ and u.

Hence, $t\,\vec{p}\,\vec{x}' = t\,\vec{p}\,\vec{x}^*$.

QED.