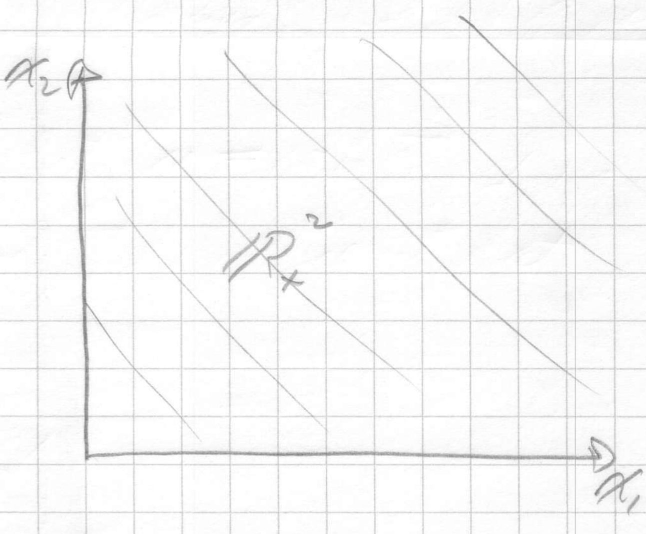
J. & R.

#1.1) $X = \mathbb{R}^2_+$



Assumptions 1.1:

① $\emptyset \neq X$: This is true since $\mathbb{R}^2_+ \neq \emptyset$; that is, $\mathbb{R}^2_+$ is not empty by definition.

✓② X is closed: We must verify that $\mathbb{R}^2_+$ is closed. The definition of a closed set is (see p. 422):

"A set S is closed in $\mathbb{R}^n$ if its complement, S^c, is an open set."

So we must define the complement X^c of $\mathbb{R}^2_+$ in $\mathbb{R}^2$. X^c includes all three quadrants to the South and West of $\mathbb{R}^2_+$ exclusive of the points on the $(0, x_2)$ and $(x_1, 0)$ axes.

Thus defined, is X^c open? An open set is defined as (see p. 420):

"$S \subset \mathbb{R}^n$ is an open set in $\mathbb{R}^n$ if $\forall \vec{x} \in S$, $\exists$ some $\varepsilon > 0$ s.t. $B_\varepsilon(\vec{x}) \subset S$."

Since X^c does not include any point on its boundaries, it respects the definition of an open set. Hence, $\mathbb{R}^2_+$ is closed. QED

1.1) cont'd

3 X is convex: (Def. p. 412)

"$S \subset \mathbb{R}^m$ is a convex set if $\forall \vec{x}^1 \in S$ and $\vec{x}^2 \in S$, we have $t\vec{x}^1 + (1-t)\vec{x}^2 \in S$, $\forall t \in [0,1]$."

Take any two elements $\vec{x}^1 \in \mathbb{R}^2_+$ and $\vec{x}^2 \in \mathbb{R}^2_+$. Then

$$t\vec{x}^1 + (1-t)\vec{x}^2 = \vec{x}^t = (tx_1^1 + (1-t)x_1^2,\ tx_2^1 + (1-t)x_2^2)$$
$$= (x_1^t, x_2^t)$$

Since $x_1^1, x_1^2, x_2^1, x_2^2$ are all ≥ 0, it must be the case that x_1^t and $x_2^t \geq 0$. Hence $\vec{x}^t \in \mathbb{R}^2_+$. QED

4 $\vec{0} \in X$. This is true since $(0,0) \in \mathbb{R}^2_+$ by definition.

J&R. #1.3:

✓a) i) Show that $\succ$ is not complete.

Def. of $\succ$: "The binary relation $\succ$ on X is defined as: $\vec{x}^1 \succ \vec{x}^2$ iff $\vec{x}^1 \succsim \vec{x}^2$ and $\vec{x}^2 \not\succsim \vec{x}^1$.

Def. of completeness for $\succ$: $\forall\, \vec{x}^1$ and $\vec{x}^2 \in X$ either $\vec{x}^1 \succ \vec{x}^2$ or $\vec{x}^2 \succ \vec{x}^1$.

By the definition of $\succ$, if $\vec{x}^1 \succsim \vec{x}^2$ <u>and</u> $\vec{x}^2 \succsim \vec{x}^1$ then neither $\vec{x}^1 \succ \vec{x}^2$ nor $\vec{x}^2 \succ \vec{x}^1$. So $\succ$ is not complete. <u>QED</u>

In plain words, the definition of $\succ$ does not rule out the fact that there are some indifferent bundles in X. Since these bundles <u>cannot</u> be compared using the definition of $\succ$ we conclude that $\succ$ is incomplete in X.

ii) Show that $\sim$ is not complete.

Def. of $\sim$: "The binary relation $\sim$ on X is defined as $\vec{x}^1 \sim \vec{x}^2$ iff $\vec{x}^1 \succsim \vec{x}^2$ and $\vec{x}^2 \succsim \vec{x}^1$."

Completeness of $\sim$: $\forall\, \vec{x}^1, \vec{x}^2 \in X$, either $\vec{x}^1 \sim \vec{x}^2$ or $\vec{x}^2 \sim \vec{x}^1$.

Take $\vec{x}^1$ and $\vec{x}^2 \in X$ s.t. $\vec{x}^1 \succsim \vec{x}^2$ and $\vec{x}^2 \not\succsim \vec{x}^1$. Then neither $\vec{x}^1 \sim \vec{x}^2$ nor $\vec{x}^2 \sim \vec{x}^1$. So the relation $\sim$ is not complete. <u>QED</u>

1.3) b) Show that for any $\vec{x}^1, \vec{x}^2 \in X$, only one of the following holds: $\vec{x}^1 \succ \vec{x}^2$ or $\vec{x}^2 \succ \vec{x}^1$ or $\vec{x}^1 \sim \vec{x}^2$.

Stratgy of proof: Show that any pair of relation cannot hold simultaneously.

① Assume that $\vec{x}^1 \succ \vec{x}^2$ and $\vec{x}^2 \succ \vec{x}^1$ holds.
Then, by the definition of $\succ$, we have $\vec{x}^1 \succsim \vec{x}^2$ and $\vec{x}^1 \not\succsim \vec{x}^2$. A contradiction.
$\Rightarrow \vec{x}^1 \succ \vec{x}^2$ and $\vec{x}^2 \succ \vec{x}^1$ cannot hold simultaneously.

② Assume that $\vec{x}^i \succ \vec{x}^j$ and $\vec{x}^i \sim \vec{x}^j$ holds for $i \neq j$.
Then, we have, by the def. of $\succ$ and $\sim$, $\vec{x}^j \not\succsim \vec{x}^i$ and $\vec{x}^j \succsim \vec{x}^i$. A contradiction.
$\Rightarrow \vec{x}^i \succ \vec{x}^j$ and $\vec{x}^i \sim \vec{x}^j$ cannot hold simultaneously.

We have shown that all three pairs of relation cannot hold.

QED

J. & R. #1.4)

a) Show that $\succ$ is transitive; that is,

$$x^1 \succ x^2 \succ x^3 \Rightarrow x^1 \succ x^3.$$

We have:

$x^1 \succ x^2 \Rightarrow x^1 \succsim x^2$ and $x^2 \not\succsim x^1$.

$x^2 \succ x^3 \Rightarrow x^2 \succsim x^3$ and $x^3 \not\succsim x^2$.

By transitivity of $\succsim$, we thus have $x^1 \succsim x^3$.

We still need to show that $x^3 \not\succsim x^1$. Assume not. Then $x^3 \succsim x^1$. Since $x^1 \succsim x^2$, we have, by transitivity, $x^3 \succsim x^2$. This contradicts $x^2 \succ x^3$. Hence, $x^3 \not\succsim x^1$. QED

b) Show that $\sim$ is transitive; that is,

$$x^1 \sim x^2 \sim x^3 \Rightarrow x^1 \sim x^3.$$

We have:

$$x^1 \sim x^2 \Rightarrow x^1 \succsim x^2 \text{ and } x^2 \succsim x^1$$

$$x^2 \sim x^3 \Rightarrow \underbrace{x^2 \succsim x^3}_{\Rightarrow x^1 \succsim x^3 \text{ (by transitivity of } \succsim)} \text{ and } \underbrace{x^3 \succsim x^2}_{\Rightarrow x^3 \succsim x^1 \text{ (idem)}}$$

$\Rightarrow x^1 \sim x^3$ by definition of $\sim$. QED

#1.4) cont'd

c) Show that $x^1 \sim x^2 \succsim x^3 \Rightarrow x^1 \succsim x^3$.

We have:

$x^1 \sim x^2 \Rightarrow x^1 \succsim x^2$ and $x^2 \succsim x^1$

Hence, $x^1 \succsim x^2 \succsim x^3$.

Thus, $x^1 \succsim x^3$ by transitivity of $\succsim$.

✓ J. & R. #1.7:

$\succsim(\vec{x}^0) \equiv \{\vec{x} \mid \vec{x} \in X, \vec{x} \succsim x^0\}$

This set is convex if $\forall \vec{x}^1, \vec{x}^2 \in \succsim(\vec{x}^0)$

then $t\vec{x}^1 + (1-t)\vec{x}^2 \in \succsim(\vec{x}^0)$ $\forall t \in [0,1]$.

(See definition of convex set, p. 412.)

Take any $\vec{x}^1, \vec{x}^2 \in \succsim(\vec{x}^0)$. We have

$\vec{x}^1 \succsim \vec{x}^0$ and $\vec{x}^2 \succsim \vec{x}^0$.

By <u>completeness</u>, we have either

$\vec{x}^1 \succsim \vec{x}^2$ or $\vec{x}^2 \succsim \vec{x}^1$ or both.

Without loss of generality, suppose $\vec{x}^1 \succsim \vec{x}^2$.

Then, by <u>axiom 5'</u>, we have

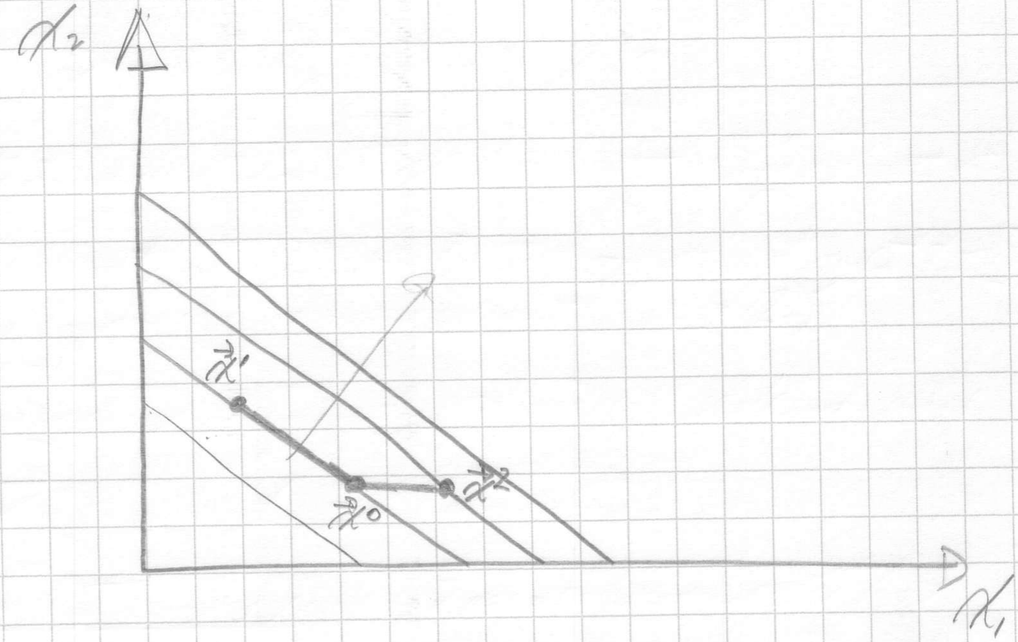
$t\vec{x}^1 + (1-t)\vec{x}^2 \succsim \vec{x}^2$, $\forall t \in [0,1]$.

Since $\vec{x}^2 \succsim \vec{x}^0$, <u>transitivity</u> gives

$t\vec{x}^1 + (1-t)\vec{x}^2 \succsim \vec{x}^0$.

Therefore, $t\vec{x}^1 + (1-t)\vec{x}^2 \in \succsim(\vec{x}^0)$. <u>QED</u>

J. & R. #1.8)



a) Axiom 5': (Convexity) If $\vec{x}_1 \succsim \vec{x}_0$, then $t\vec{x}^1 + (1-t)\vec{x}^0 \succsim \vec{x}^0$, $\forall t \in [0,1]$.

proof that convexity holds:

If $\vec{x}^1 \succsim \vec{x}^0$, then either $\vec{x}^1 \succ \vec{x}^0$ or $\vec{x}_1 \sim \vec{x}^0$.

① Take $\vec{x}^1 \sim \vec{x}^0$, then both bundles are on the same indifference straight line and any convex combination of the two must also lie on that same line; that is, $t\vec{x}^1 + (1-t)\vec{x}^0 \sim x_1 \sim x_0$.

② Take $\vec{x}^2 \succ \vec{x}^0$, then $\vec{x}^1$ is to the north-east of the indifference straight line that runs thru $\vec{x}^0$. By graphical inspection, that along any point on the line connecting $\vec{x}^0$ with $\vec{x}^2$, we must have a (weakly) prefered bundle.

#1.8) cont'd

b) Axiom 5: (Strict convexity)

If $\vec{x}^1 \neq \vec{x}^0$ and $\vec{x}^1 \succsim \vec{x}^0$, then

$t\vec{x}^1 + (1-t)\vec{x}^0 \succ \vec{x}^0 \ \forall t \in (0,1)$.

This axiom does not hold.

PROOF: Take $\vec{x}^1$ and $\vec{x}^0$ on the same indifference straight line.

Then $\vec{x}^1 \sim \vec{x}^0$ and any convex combination also lies on the same indifference line. Hence,

$t\vec{x}^1 + (1-t)\vec{x}^0 \sim \vec{x}^0 \ \forall t \in (0,1)$.

Axiom 5 is thus violated. QED