J.&R. 3.23)

Superadditive fctns:

$$f(\vec{z}^1+\vec{z}^2) \geq f(\vec{z}^1) + f(\vec{z}^2).$$

i) Show that cost fctns are super-additive in $\vec{w}$, i.e.

$$c(\vec{w}^1+\vec{w}^2; y) \geq c(\vec{w}^1; y) + c(\vec{w}^2; y)$$

By homo. of deg. 1 in $\vec{w}$, we have

$$c(t\vec{w}) = t\, c(\vec{w}), \quad \forall t \geq 0.$$

By concavity in $\vec{w}$, we have

$$c(t\vec{w}^1 + (1-t)\vec{w}^2) \geq t\,c(\vec{w}^1) + (1-t)\,c(\vec{w}^2) \quad \forall t \in (0,1).$$

Combining both properties, we get

$$c(t\vec{w}^1 + (1-t)\vec{w}^2) \geq c(t\vec{w}^1) + c((1-t)\vec{w}^2) \quad \forall t \in (0,1).$$

The proof is complete by the fact that any vector $\vec{w}$ can be expressed as either $\vec{w} = t \cdot \frac{1}{t}\vec{w}$ or $\vec{w} = (1-t)\cdot\left(\frac{1}{1-t}\right)\vec{w}$, $\forall t \in (0,1)$.

QED

(ALTERNATIVE PROOF:)

3.23 Superadditive functions

Consider any input prices $\vec{w}_1$ and $\vec{w}_2$.

We have:

$$c(\vec{w}_1+\vec{w}_2, y) = (\vec{w}_1+\vec{w}_2)\vec{x}^{12} = \vec{w}_1\vec{x}^{12} + \vec{w}_2\vec{x}^{12}$$

$$c(\vec{w}_1, y) = \vec{w}_1\vec{x}^{1}$$

$$c(\vec{w}_2, y) = \vec{w}_2\vec{x}^{2}$$

where $\vec{x}^{12}$, $\vec{x}^{1}$ and $\vec{x}^{2}$ are the cost min. input bundles for $\vec{w}_1+\vec{w}_2$, $\vec{w}_1$ and $\vec{w}_2$ respectively, given y.

By the definition of cost minimization, we have

$$\vec{w}_1\vec{x}^{12} \geq \vec{w}_1\vec{x}^{1} \quad \text{and} \quad \vec{w}_2\vec{x}^{12} \geq \vec{w}_2\vec{x}^{2}.$$

$$\Rightarrow (\vec{w}_1+\vec{w}_2)\vec{x}^{12} \geq \vec{w}_1\vec{x}_1 + \vec{w}_2\vec{x}_2$$

$$\Rightarrow c(\vec{w}_1+\vec{w}_2, y) \geq c(\vec{w}_1, y) + c(\vec{w}_2, y).$$

$\forall\, \vec{w}_1, \vec{w}_2 \geq 0.$ <u>QED</u>

ii) Show that $c(\vec{w}, y)$ is non-decreasing in $\vec{w}$, i.e.

$$\vec{w}^0 \geq \vec{w}^1 \Rightarrow c(\vec{w}^0) \geq c(\vec{w}^1).$$

PROOF:

$$\vec{w}^0 \geq \vec{w}^1 \Rightarrow \vec{w}^0 = \vec{w}^1 + \Delta\vec{w}^1$$

where $\Delta\vec{w}^1 \geq 0$.

By super-additivity, we have

$$c(\vec{w}^0) = c(\vec{w}^1 + \Delta\vec{w}^1) \geq c(\vec{w}^1) + c(\Delta\vec{w}^1)$$

$$\geq c(\vec{w}^1)$$

QED

3.28 (1)

J.&R. 3.28) $x_i(w_1, w_2, w_3; y)$

a) $\frac{\partial x_2}{\partial w_1} > 0$ and $\frac{\partial x_3}{\partial w_1} > 0$.

This is consistent.

Let $c(\vec{w}, y) = A w_1^{\alpha} w_2^{\beta} w_3^{1-\alpha-\beta} y$.

$\Rightarrow x_2(\vec{w}, y) = \beta A w_1^{\alpha} w_2^{\beta-1} w_3^{1-\alpha-\beta} y$.

$\Rightarrow \frac{\partial x_2}{\partial w_1} = \alpha\beta A w_1^{\alpha-1} w_2^{\beta-1} w_3^{1-\alpha-\beta} y > 0$ ✓

And similarly for x_3.

b) $\frac{\partial x_2}{\partial w_1} > 0$ and $\frac{\partial x_3}{\partial w_1} < 0$.

This is consistent. Example?

c) $\frac{\partial x_i}{\partial y} < 0$, $i = 1, 2, 3$.

Inconsistent with the fact that the cost fctn is strictly increasing in y.

d) $\frac{\partial x_i}{\partial y} = 0$. This is consistent with perfect substitutes:

$f(\vec{x}) = ax_1 + bx_2 + cx_3$.

$a, b, c > 0$.

e) $\frac{\partial(x_1/x_2)}{\partial x_3} = 0.$

Consistent with Leontief technology:

$f(x) = \min\{x_1, x_2, x_3\}$

J.&R. 3.32) For fixed w, we have

$$AC(y) \equiv \frac{c(y)}{y}$$

$$\Rightarrow \frac{\partial AC}{\partial y} = \frac{c'(y)y - c(y)}{y^2} = \frac{1}{y}\left(c'(y) - \frac{c(y)}{y}\right)$$

$$= \frac{1}{y}\left(MC(y) - AC(y)\right)$$

$$\Rightarrow \frac{\partial AC}{\partial y} \gtreqless 0 \quad \text{iff} \quad MC(y) \gtreqless AC(y) \text{ . QED.}$$

J&R 3.44)

$$y = x_1^{\alpha} x_2^{\beta}$$

$$\max_{x_1, x_2} \pi = p x_1^{\alpha} x_2^{\beta} - w_1 x_1 - w_2 x_2$$

$$\frac{\partial \pi}{\partial x_1} = \frac{\alpha p x_1^{\alpha} x_2^{\beta}}{x_1} - w_1 = 0$$

$$\frac{\partial \pi}{\partial x_2} = \frac{\beta p x_1^{\alpha} x_2^{\beta}}{x_2} - w_2 = 0.$$

$$\Rightarrow \frac{\alpha}{\beta} \frac{x_2}{x_1} = \frac{w_1}{w_2} \Rightarrow x_1 = \frac{\alpha}{\beta} x_2 \frac{w_2}{w_1}$$

$$\Rightarrow \beta p \left(\frac{\alpha}{\beta}\right)^{\alpha} x_2^{\alpha} \left(\frac{w_2}{w_1}\right)^{\alpha} x_2^{\beta-1} = w_2$$

$$\Rightarrow x_2^* = \left[\beta^{1-\alpha} p \alpha^{\alpha} \frac{w_2^{\alpha-1}}{w_1^{\alpha}} \right]^{\frac{1}{1-\beta-\alpha}}$$

$$x_1^* = \left[\alpha^{1-\beta} p \beta^{\beta} \frac{w_1^{\beta-1}}{w_2^{\beta}} \right]^{\frac{1}{1-\beta-\alpha}}$$

$$\Rightarrow \pi(p, w_1, w_2) = \ldots$$

J.&R. 3.50)

Let $y^0 = y(p^0, \vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$ and $\vec{x}^0 = \vec{x}(p^0, \vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$ be the SR supply and input demands at $(p^0, \vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$.

$\Rightarrow \pi(p^0, \vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}}) = p^0 y^0 - \vec{w}^0 \vec{x}^0 - \vec{\bar{w}} \vec{\bar{x}}$.

Let $y^1 = y(tp^0, t\vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$ and $\vec{x}^1 = \vec{x}(tp^0, t\vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$.

$\Rightarrow \pi(tp^0, t\vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}}) = tp^0 y^1 - t\vec{w}^0 \vec{x}^1 - \vec{\bar{w}} \vec{\bar{x}}$.

If y^0 and $\vec{x}^0$ does not maximize profit at tp^0 and $t\vec{w}^0$, we have:

$tp^0 y^1 - t\vec{w}^0 \vec{x}^1 - \vec{\bar{w}} \vec{\bar{x}} > tp^0 y^0 - t\vec{w}^0 \vec{x}^0 - \vec{\bar{w}} \vec{\bar{x}}$

$\Rightarrow \quad p^0 y^1 - \vec{w}^0 \vec{x}^1 > p^0 y^0 - \vec{w}^0 \vec{x}^0$.

This implies that y^0 and $\vec{x}^0$ do not max. profits at $(p^0, \vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$.

A contradiction.

Hence, $\vec{y}^0 \vec{x}^0$ must maximize profits at $(tp^0, t\vec{w}^0, \vec{\bar{w}}, \vec{\bar{x}})$.

QED.