J. & R. 2.3)

$v(\vec{p}, y) = y p_1^\alpha p_2^\beta \quad , \alpha, \beta < 0.$

We have

$$u(\vec{x}) = \min_{\vec{p}} v(\vec{p}, 1) \text{ s.t. } \vec{p}\vec{x} = 1$$

$$\mathcal{L} = y p_1^\alpha p_2^\beta + \lambda(1 - p_1 x_1 - p_2 x_2)$$

$$\frac{\partial \mathcal{L}}{\partial p_1} = \frac{\alpha y p_1^\alpha p_2^\beta}{p_1} - \lambda x_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial p_2} = \frac{\beta y p_1^\alpha p_2^\beta}{p_2} - \lambda x_2 = 0$$

$$\Rightarrow \frac{\alpha}{\beta} \frac{p_2}{p_1} = \frac{x_1}{x_2} \Rightarrow p_2 = \frac{x_1}{x_2} p_1 \frac{\beta}{\alpha}$$

$$\Rightarrow 1 = p_1 x_1 + p_1 x_1 \beta/\alpha \Rightarrow 1 = p_1 x_1 (1 + \beta/\alpha)$$

$$\Rightarrow p_1^* = \frac{1}{x_1(1+\beta/\alpha)}$$

$$\Rightarrow p_2^* = \frac{x_1}{x_2} \cdot \frac{1}{x_1(1+\beta/\alpha)} \cdot \frac{\beta}{\alpha} = \frac{1}{x_2(1+\alpha/\beta)}$$

$$\Rightarrow u(\vec{x}) = \left(\frac{1}{x_1(1+\beta/\alpha)}\right)^\alpha \left(\frac{1}{x_2(1+\alpha/\beta)}\right)^\beta$$

$$= A x_1^{-\alpha} x_2^{-\beta} \text{ where } A = (1+\beta/\alpha)^{-\alpha} (1+\alpha/\beta)^{-\beta}$$

1

J. 2R. 2. 6)

$e(p_1, p_2, u) = \frac{u\, p_1 p_2}{p_1 + p_2}$

Since $e(p, v(p,y)) = y$, we have

$v(p,y) = \frac{y(p_1+p_2)}{p_1 p_2} \Rightarrow v(p,1) = \frac{p_1+p_2}{p_1 p_2}$

$\Rightarrow u(\vec{x}) = \min_{p_1, p_2} \frac{p_1+p_2}{p_1 p_2}$ s.t. $p_1 x_1 + p_2 x_2 = 1$

$\mathcal{L} = \frac{p_1+p_2}{p_1 p_2} + \lambda(1 - p_1 x_1 - p_2 x_2)$

$\frac{\partial \mathcal{L}}{\partial p_1} = \frac{p_1 p_2 - (p_1+p_2)p_2}{p_1^2 p_2^2} - \lambda x_1 = 0$

$\Rightarrow \frac{-1}{p_1^2} = \lambda x_1$

similarly: $\frac{-1}{p_2^2} = \lambda x_2$

$\Rightarrow \frac{p_2}{p_1} = \sqrt{\frac{x_1}{x_2}}$

$\Rightarrow p_2 = p_1 \sqrt{x_1/x_2}$

$\Rightarrow p_1 x_1 + (p_1 \sqrt{x_1/x_2})x_2 = 1$

$\Rightarrow p_1^* = \frac{1}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}, \quad p_2^* = \frac{1}{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}$

$u(\vec{x}) = \left[\frac{1}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})} + \frac{1}{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}\right] \cdot \sqrt{x_1 x_2}(\sqrt{x_1}+\sqrt{x_2})^2$

$\boxed{u(\vec{x}) = (\sqrt{x_1} + \sqrt{x_2})^2}$