Jehle R. 1.62)

ATT: Symmetry line added at bottom of page.

Hicks' 3rd Law:

$$\sum_{j=1}^{n} \frac{\partial x_i^h(\vec{p}, u)}{\partial p_j} p_j = 0$$

a) PROOF: Since $\vec{p}\vec{x}(\vec{p}, y) = y$, we have

$$\sum_{i=1}^{n}\left[p_i \frac{\partial x_i}{\partial p_j}\right] + x_j = 0$$

Using the Slutsky decomposition:

$$\sum_{i=1}^{n}\left[p_i \frac{\partial x_i^h}{\partial p_j} - x_j\, p_i \frac{\partial x_i}{\partial y}\right] + x_j = 0. \qquad [*]$$

Again, since $\vec{p}\vec{x}(\vec{p}, y) = y$, we have

$$\sum_i p_i \frac{\partial x_i}{\partial y} = 1.$$

Inserting into [*], we get:

$$\sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial p_j} = 0.$$

By symmetry of the substitution matrix:

$$\sum_{i=1}^{n} p_i \frac{\partial x_j^h}{\partial p_i} = 0 \qquad \underline{QED.}$$

b) Verify with
$$u(\vec{x}) = A\prod_{i=1}^{n} x_i^{\alpha_i}.$$

We had found that

$$x_j^h = \frac{u}{A}\left[\prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_j}{p_j}$$

$$\Rightarrow \frac{\partial x_j^h}{\partial p_k} = \frac{u}{A}\frac{\alpha_k}{p_k}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_j}{p_j}\ ,\ k \neq j$$

$$\frac{\partial x_j^h}{\partial p_j} = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\left[\left(\frac{\alpha_j}{p_j}\right)^2 - \frac{\alpha_j}{p_j^2}\right]$$

$$\Rightarrow \frac{\partial x_j^h}{\partial p_k}p_k = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_k\alpha_j}{p_j}$$

$$\frac{\partial x_j^h}{\partial p_j}p_j = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\left(\frac{\alpha_j^2 - \alpha_j}{p_j}\right)$$

$$\Rightarrow \sum_{j=1}^{n}\frac{\partial x_i^h}{\partial p_j}p_j = \frac{u}{A}\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\underbrace{\left\{\sum_{j\neq i}\left[\frac{\alpha_j\alpha_i}{p_i}\right] + \frac{\alpha_i^2 - \alpha_i}{p_i}\right\}}$$

$$\overset{*}{=} \frac{\alpha_i}{p_i}\sum_{j\neq i}\alpha_j + \frac{\alpha_i^2}{p_i} - \frac{\alpha_i}{p_i}$$

$$= \frac{\alpha_i}{p_i}\sum_{j=1}^{n}\alpha_j - \frac{\alpha_i}{p_i} = 0. \checkmark$$