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$$u(\vec{x}) = A\prod_{i=1}^{n} x_i^{\alpha_i}, \quad A>0, \quad \sum_{i=1}^{n}\alpha_i = 1.$$

$$\Rightarrow u(\vec{x}) = A x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

a) Marshallian demand:

$$\max_{\{x_i\}} u(\vec{x}) \quad \text{s.t.} \quad \vec{p}\,\vec{x} = y.$$

$$\mathcal{L} = u(\vec{x}) + \lambda[y - \vec{p}\,\vec{x}]$$

$$\mathcal{L}_j = \frac{\partial u}{\partial x_j} - \lambda p_j = 0, \quad \forall j = 1, \ldots, n.$$

where $\dfrac{\partial u}{\partial x_j} = \dfrac{\alpha_j A\prod_{i=1}^{n} x_i^{\alpha_i}}{x_j}$

Note that $\lim_{x_j \to 0} \dfrac{\partial u}{\partial x_j} = +\infty$. Hence, $x_j^* > 0, \forall j$.

$$\Rightarrow \frac{\alpha_j u(\vec{x}^*)}{x_j^*} = \lambda p_j, \quad \forall j.$$

$$\Rightarrow \frac{\dfrac{\alpha_j u(\vec{x}^*)}{x_j^*}}{\dfrac{\alpha_k u(\vec{x}^*)}{x_k^*}} = \frac{p_j}{p_k}, \quad \forall j, k.$$

$$\Rightarrow \frac{\alpha_j x_k^*}{\alpha_k x_j^*} = \frac{p_j}{p_k} \Rightarrow x_j^* = \left(\frac{\alpha_j}{\alpha_k}\right)\left(\frac{p_k}{p_j}\right) x_k^*.$$

Inserting in the budget constraint:

$$p_1 x_1^* + \frac{\alpha_2}{\alpha_1} \cdot p_1 x_1^* + \frac{\alpha_3}{\alpha_1} p_1 x_1^* + \dots + \frac{\alpha_n}{\alpha_1} p_1 x_1^* = y$$

$$\Rightarrow x_1^* = \frac{y}{\frac{p_1}{\alpha_1} \sum_{i=1}^{n} \alpha_i} = \frac{\alpha_1 y}{p_1}$$

And similarly for all x_j^*:

$$x_j(\vec{p}, y) = \alpha_j \frac{y}{p_j}$$: This is the marshallian demand.

b) Indirect utility fctn:

$$v(\vec{p}, y) = u(\vec{x}(\vec{p}, y))$$

$$= A\left(\frac{\alpha_1 y}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2 y}{p_2}\right)^{\alpha_2} \cdots \left(\frac{\alpha_n y}{p_n}\right)^{\alpha_n}$$

$$= A\, y \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$$: This is the indirect utility fctn.

c) The expenditure fctn:

$$v(\vec{p}, e(\vec{p}, u)) = u$$

$$\Rightarrow \quad A\, e(\vec{p}, u) \prod_i \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} = u$$

$$\Rightarrow e(\vec{p}, u) = \frac{u}{A} \prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$$

d) The Hicksian demand:

By Shephard's Lemma, we have

$$x_j^h(\vec{p}, u) = \frac{\partial e}{\partial p_j}$$

$$= \frac{u}{A}\left[\prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right] \frac{\alpha_j}{p_j}$$