J & R
1.38) The expenditure fctn obtained from a CES utility fctn is

$$e(\vec{p}, u) = u(p_1^r + p_2^r)^{1/r}$$

where $r = \rho/(\rho-1)$, $0 \neq \rho < 1$.

Theorem 1.7 says that $e(\vec{p}, u)$ must have the following properties:

① $e(\vec{p}, u) = 0$ when $u(\vec{x})$ is lowest?

Since $u(\vec{x}) = (x_1^\rho + x_2^\rho)^{1/\rho}$, the lowest value is when $\vec{x} = \vec{0}$, which yields $u = 0$. Substituting into $e(\vec{p}, u)$ above, we do get $e(\vec{p}, 0) = 0$. ✓

② Continuous in $\mathbb{R}^n_{++} \times U$?

$u(p_1^r + p_2^r)^{1/r}$ is continuous in u, p_1 and p_2. ✓

③ The function is increasing and unbounded above in u, $\forall \vec{p} \gg 0$?

$$\frac{\partial e}{\partial u} = (p_1^r + p_2^r)^{1/r} > 0 \Rightarrow \text{increasing} \checkmark$$

$$\lim_{u \to \infty} e = +\infty. \Rightarrow \text{unbounded above} \checkmark$$

④ $e(\vec{p}, u)$ is increasing in $\vec{p}$?

Is $u(p_1^{0r} + p_2^{0r})^{1/r} \geq u(p_1^{1r} + p_2^{1r})^{1/r}$, $\forall \vec{p}^0 \geq \vec{p}^1$?

Whether $r>0$ or $r<0$, we have the following:

$$\left(p_1^{0r}+p_2^{0r}\right)^{1/r} \geq \left(p_1^{1r}+p_2^{1r}\right)^{1/r} \quad \forall\ \vec{p}^{\,0} \geq \vec{p}^{\,1}. \checkmark$$

⑤ Homogeneous of deg. 1 in $\vec{p}$?

$$e(t\vec{p},u) = u\left((tp_1)^r+(tp_2)^r\right)^{1/r} = u\left[t^r\left(p_1^r+p_2^r\right)\right]^{1/r}$$

$$= t\,u\left(p_1^r+p_2^r\right)^{1/r} = te(\vec{p},u). \checkmark$$

⑥ Concave in p?

Is the Hessian matrix of $e(\vec{p})$ negative semi-definite?

We have:

$$\frac{\partial e}{\partial p_1} = u\cdot\frac{1}{r}\left(p_1^r+p_2^r\right)^{\frac{1}{r}-1}\cdot r p_1^{r-1} = \frac{e(\vec{p},u)}{p_1^{1-r}\left(p_1^r+p_2^r\right)} > 0$$

$$\frac{\partial^2 e}{\partial p_1^2} = \frac{1}{p_1^{1-r}\left(p_1^r+p_2^r\right)}\frac{\partial e}{\partial p_1} - \frac{e(\vec{p},u)}{\left(p_1^{1-r}\left(p_1^r+p_2^r\right)\right)^2}\cdot\left[(1-r)p_1^{-r}\left(p_1^r+p_2^r\right)+r\right]$$

$$= \frac{e(\vec{p},u)}{\left(p_1^{1-r}\left(p_1^r+p_2^r\right)\right)^2}(1-r)\left(1-\frac{p_1^r+p_2^r}{p_1^r}\right) < 0 \quad \checkmark$$

It remains to be verified that

$$\begin{vmatrix} \frac{\partial^2 e}{\partial p_1^2} & \frac{\partial^2 e}{\partial p_1 \partial p_2} \\ \frac{\partial^2 e}{\partial p_2 \partial p_1} & \frac{\partial^2 e}{\partial p_2^2} \end{vmatrix} > 0$$

...