Graduate studies in Affine Algebraic Geometry

I wrote the text below for students who are considering the possibility of doing an M.Sc. or a Ph.D. under my supervision, in order to give them some idea of what my research area is. The field in which I work is called *Affine Algebraic Geometry*.

You are probably aware of the fact that polynomial rings (in n variables, over a field) are objects of fundamental importance in both algebra and algebraic geometry. However, you might be surprised to read that polynomial rings are far from being understood. For instance, (i) the automorphisms of $\mathbb{C}[x_1, \dots, x_n]$ are not understood when $n > 2$ (the case $n = 2$ was solved in 1942); (ii) no good characterization of polynomial rings is known (given a ring R, it is not known how to determine whether R is isomorphic to a polynomial ring); (iii) let R be a ring which contains $\mathbb{C}$ and consider the polynomial ring $R[t]$ in one variable over R; if $R[t]$ is isomorphic to the polynomial ring $\mathbb{C}[x_1, \dots, x_{n+1}]$, does it follow that R is isomorphic to $\mathbb{C}[x_1, \dots, x_n]$? (this is called Zariski's Cancellation Problem, and has been open for the last 50 years or so). Here I only mention these three questions, but there is a much longer list of *fundamental open problems about polynomial rings.*

Affine Algebraic Geometry (AAG) studies commutative rings by using an array of techniques coming from algebra and algebraic geometry, and sometimes from other fields such as topology, differential geometry or the theory of singularities. Although AAG is ultimately interested in all commutative rings, it is especially attracted by those rings and those questions which have something to do with the fundamental problems about polynomial rings. One could say that AAG revolves around polynomial rings.

To speak about AAG, one can use the algebraic language (as I did up to this point) or the geometric language. For instance, studying the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ is equivalent to studying the algebraic variety $\mathbb{C}^n$, and the already mentioned Cancellation Problem can be stated as follows: if X is a complex algebraic variety satisfying $X \times \mathbb{C} \cong \mathbb{C}^{n+1}$, does it follow that $X \cong \mathbb{C}^n$? Also, instead of saying that we want to classify certain rings, we can speak of classifying certain affine algebraic varieties (which allows us to use geometric tools for doing the job). Students working with me will learn a good deal of commutative algebra and algebraic geometry; being able to go back and forth between the two viewpoints turns out to be very useful in this field.

One approach for studying an affine $\mathbb{C}$-variety X consists in describing the actions of the algebraic group $G_a = (\mathbb{C}, +)$ on X, or equivalently, describing the locally nilpotent derivations on the coordinate algebra of X. Over the last 15 years or so, this method developed rapidly and proved to be quite successful; it is now an active research area in AAG. A good portion of my work is devoted to developing and exploiting this theory.

When $n > 2$, it is a challenging open problem (closely related to the above-mentioned problem (i)) to describe the G_a-actions on $\mathbb{C}^n$, or equivalently, the locally nilpotent derivations of $\mathbb{C}[x_1, \dots, x_n]$. My graduate students made several contributions to this problem, and there are many more oportunities for future theses.

AAG tends to be particularly interested in algebraic varieties which have properties in common with the variety $\mathbb{C}^n$ (for instance the property of being rational, or factorial, or of having a very large group of automorphisms, etc). As $\mathbb{C}^n$ admits an abundance of G_a-actions, we are therefore interested in *affine algebraic varieties which admit many G_a-actions.* I participate in a large-scale effort (involving many people) whose aim is to classify those varieties, at least in low dimension. This is a relatively new field, with lots of possibilities for PhD projects.