ON WEIGHTED PROJECTIVE PLANES AND THEIR AFFINE RULINGS

DANIEL DAIGLE AND PETER RUSSELL

Following the terminology of [2], we say that an algebraic surface X satisfies (†) if:

(†) X is a complete normal rational surface, X is affine ruled and rank(Pic X_s) $= 1$,

where X_s denotes the smooth locus of X; we say that X satisfies (‡) if:

(‡) X satisfies (†) and every singular point of X is a cyclic quotient singularity.

(Here, and throughout this paper, all algebraic varieties are over an algebraically closed field $\mathbf{k}$ of characteristic zero.) As we will see in Section 1, the weighted projective planes $\mathbb{P}(a,b,c)$ satisfy (‡).

Paper [2] investigates the problem of finding all affine rulings of a given surface X satisfying (†). In particular, it shows that if X satisfies (‡) then the problem reduces to that of describing a certain set $\mathbb{T}_0(X)$ of triples (m, T_1, T_2), where m is a positive integer and each T_i is a $2 \times h_i$ matrix with entries in $\mathbb{N}$ $(0 \le h_i \le 2)$. The aim of the present paper is to give an explicit description of the set $\mathbb{T}_0(X)$ in the case where X is a weighted projective plane; this is achieved by Corollary 7.1 and Propositions 7.3, 7.4 and 7.7. Thus [2] and this paper solve the above mentioned problem for weighted projective planes.

Let us also point out the following characterization of weighted projective planes, which we prove in the form of Corollary 6.12, below (see 1.19 for the notion of *resolution graph* of a normal surface):

Theorem. *Let X be a complete normal rational surface which is affine ruled and satisfies* rank(Pic X_s) $= 1$*. If X has the same resolution graph as the weighted projective plane $\mathbb{P}(a,b,c)$, then X is isomorphic to $\mathbb{P}(a,b,c)$.*

Although this paper relies heavily on the results and concepts developed in [2], it is almost completely self-contained, thanks to Section 2, which is essentially an outline of those parts of [2] which are directly needed here. However, it may be necessary to consult [2] in order to fully understand how to recover the affine rulings from the description of $\mathbb{T}_0(X)$ given in this paper. (First, starting from $\mathbb{T}_0(X)$, one uses 5.17 and 5.39 of [2] to construct the larger set $\mathbb{T}(X)$; then, as explained in 5.3 of [2], one has a "recipe" for constructing all affine rulings of X.)

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We also refer to the introduction of [2] for a discussion of related problems and applications. For instance, the results of this paper enable one to describe all curves C on $\mathbb{P} = \mathbb{P}(a, b, c)$ satisfying $\bar{\kappa}(\mathbb{P} \setminus C) = -\infty$, and all locally nilpotent derivations of $\mathbf{k}[X, Y, Z]$ which are homogeneous with respect to weights a, b, c for X, Y, Z.

1. Preliminaries on weighted projective planes

Let a_0, a_1, a_2 be positive integers and consider the weighted projective plane

$$\mathbb{P} = \mathbb{P}(a_0, a_1, a_2) = \operatorname{Proj} A,$$

where $A = \mathbf{k}[X_0, X_1, X_2]$ is graded by assigning weight a_i to X_i. Note that $\mathbb{P}$ is a complete normal rational surface and that $\mathbb{P}(a_0, a_1, a_2) \cong \mathbb{P}(\dot{a}_0, \dot{a}_1, \dot{a}_2)$, where $\dot{a}_i = a_i/d$, $d = \gcd(a_0, a_1, a_2)$. Moreover, if we assume that a_0, a_1, a_2 are relatively prime then:

1.1 ([3], 1.3.1)**.** For distinct $i, j, k \in \{0, 1, 2\}$, let $\alpha_i = \gcd(a_j, a_k)$ and $a_i' = \frac{a_i}{\alpha_j \alpha_k}$. Then a_0', a_1', a_2' are pairwise relatively prime and $\mathbb{P}(a_0, a_1, a_2) \cong \mathbb{P}(a_0', a_1', a_2')$.

Since our results depend only on the isomorphism type of $\mathbb{P}$, and not on a specific projective structure, we will assume throughout:

1.2. a_0, a_1, a_2 are pairwise relatively prime.

1.3. By a *coordinate system* of $\mathbb{P}$, we mean an ordered triple (f_0, f_1, f_2) of homogeneous elements of A satisfying $A = \mathbf{k}[f_0, f_1, f_2]$. (Then $(a_0, a_1, a_2) = (\deg f_{\tau 0}, \deg f_{\tau 1}, \deg f_{\tau 2})$ for some permutation τ of $0, 1, 2$, and $X_i \mapsto f_{\tau i}$ gives an automorphism of A as a graded $\mathbf{k}$-algebra.)

If $F \in A$ is homogeneous, let $V(F) \subset \mathbb{P}$ denote the zero locus of F.

1.4. Given a coordinate system (X_0, X_1, X_2) of $\mathbb{P}$, let $R_i = V(X_i) \subset \mathbb{P}$ (an irreducible rational curve) and let $q_i \in \mathbb{P}$ be the point $R_j \cap R_k$ (where $\{i, j, k\} = \{0, 1, 2\}$).

Lemma 1.5. *Given a coordinate system (X_0, X_1, X_2) of $\mathbb{P}$, the rational maps $\varphi_i : \mathbb{P} \to \mathbb{P}^1$ $(i = 0, 1, 2)$ defined by*

$$\varphi_0 = \frac{X_1^{a_2}}{X_2^{a_1}}, \quad \varphi_1 = \frac{X_2^{a_0}}{X_0^{a_2}}, \quad \varphi_2 = \frac{X_0^{a_1}}{X_1^{a_0}}$$

induce three affine rulings of $\mathbb{P}$.

Proof. Note that q_0 is the only fundamental point of φ_0 in $\mathbb{P}$. The general fibre of φ_0 is $C = V(\alpha X_1^{a_2} - \beta X_2^{a_1})$, $\alpha, \beta \in \mathbf{k}^*$, which is irreducible since $\gcd(a_1, a_2) = 1$. Since $C \setminus \{q_0\} \cong \mathbb{A}^1$, φ_0 induces an affine ruling of $\mathbb{P}$. □

Definition 1.6. The three affine rulings of 1.5 are said to be *standard with respect to* (X_0, X_1, X_2). An affine ruling of $\mathbb{P}$ is *standard* if it is standard with respect to some coordinate system.

Lemma 1.7. *Let $\mathbb{P}_s$ be the smooth locus of $\mathbb{P}$. Then* $\operatorname{Pic} \mathbb{P}_s = \mathbb{Z}$.

Proof. We have $\mathrm{Pic}(\mathbb{P}_s) = \mathrm{Cl}(\mathbb{P}_s) = \mathrm{Cl}(\mathbb{P})$, where "Cl" denotes divisor class group. Using the fact that A is an $\mathbb{N}$-graded U.F.D., one obtains a degree function $\mathrm{Cl}(\mathbb{P}) \to \mathbb{Z}$ which is in fact an isomorphism. □

By the above results, $\mathbb{P}$ satisfies (†); we will show in 1.20 that $\mathbb{P}$ satisfies a condition stronger than (‡). Also recall:

1.8 ([2], 1.16)**.** *A surface satisfying* (‡) *cannot have more than* 3 *singular points.*

Linear chains

1.9. We use the standard definitions for blowing-up, contraction and equivalence of weighted graphs (but note that, in weighted graphs, we do not allow multiple edges between a given pair of vertices). A *linear chain* is a weighted tree without branch points; an *admissible chain* is a linear chain in which every weight is strictly less than -1. The empty graph is regarded as an admissible chain.

1.10. Let $\mathcal{G}$ be a weighted graph, $v_1, \dots, v_n$ its vertices and ω_i the weight of v_i. Recall that the *determinant* of $\mathcal{G}$ is defined by $\det(\mathcal{G}) = \det(-A)$, where A denotes the "intersection matrix" of $\mathcal{G}$, i.e., the $n \times n$ matrix with entries $A_{ii} = \omega_i$ and, if $i \neq j$, $A_{ij} = 1$ (resp. 0) if v_i, v_j are neighbors (resp. are not neighbors).

1.11. Let $\mathcal{G}$ be a weighted tree, v a vertex of weight $\Omega(v)$ in $\mathcal{G}$, $\mathcal{G}_1, \dots, \mathcal{G}_n$ the branches of $\mathcal{G}$ at v and v_i the vertex of $\mathcal{G}_i$ which is a neighbor of v in $\mathcal{G}$. If $d_i = \det \mathcal{G}_i$ and $d_i' = \det(\mathcal{G}_i - \{v_i\})$, then

$$\det \mathcal{G} = -\Omega(v)\, d_1 \cdots d_n - \sum_{i=1}^{n} d_1 \cdots d_{i-1} d_i'\, d_{i+1} \cdots d_n\,.$$

Definition 1.12. Let $\mathcal{A}$ be the linear chain $\overset{w_1}{\underset{v_1}{\bullet}}\!\!-\!\!\overset{w_2}{\underset{v_2}{\bullet}}\!\!-\cdots-\!\!\overset{w_{n-1}}{\underset{v_{n-1}}{\bullet}}\!\!-\!\!\overset{w_n}{\underset{v_n}{\bullet}}$ $(w_i \in \mathbb{Z},\ n \geq 0)$. We say that $\mathcal{A}$ has *discriminant* δ and *subdiscriminants* δ^* and δ_* to indicate that $\det(\mathcal{A}) = \delta$ and that $\{\det(\mathcal{A} \setminus \{v_1\}), \det(\mathcal{A} \setminus \{v_n\})\} = \{\delta^*, \delta_*\}$ (equality of sets). If $\mathcal{A}$ is empty, it has discriminant 1 and subdiscriminants 0 and 0; if $\mathcal{A}$ consists of a single vertex, its subdiscriminants are 1 and 1.

1.13. If $\mathcal{A}$ is a linear chain with discriminant δ and subdiscriminants δ^* and δ_*, then $\delta^*\delta_* \equiv 1 \pmod{\delta}$.

1.14. Let $\mathcal{A}$ be an *admissible* chain with discriminant δ and let s be one of the subdiscriminants of $\mathcal{A}$. Then $0 \leq s < \delta$; also, $\mathcal{A}$ is empty $\iff \delta = 1 \iff s = 0$. Moreover, $\mathcal{A}$ is completely determined by the outer Euclidian algorithm on (δ, s): write $r_0 = \delta$, $r_1 = s$, $r_{i-1} = q_i r_i - r_{i+1}$ $(0 \leq r_{i+1} < r_i,\ i = 1, \dots, n)$ and $r_{n+1} = 0$; then $\mathcal{A}$ is

$$\overset{-q_1}{\bullet}\!\!-\!\!\overset{-q_2}{\bullet}\!\!-\cdots-\!\!\overset{-q_{n-1}}{\bullet}\!\!-\!\!\overset{-q_n}{\bullet}\,.$$

1.15. Let $\mathcal{A}$ and $\mathcal{A}'$ be two linear chains.

(1) If $\mathcal{A}$ and $\mathcal{A}'$ are equivalent as weighted graphs, then they have the same discriminant δ and, modulo δ, the same subdiscriminants.

(2) Assume that $\mathcal{A}$ and $\mathcal{A}'$ are equivalent to admissible chains. If $\mathcal{A}$ and $\mathcal{A}'$ have the same discriminant δ and if some subdiscriminants s of $\mathcal{A}$ and s' of $\mathcal{A}'$ satisfy $s \equiv s' \pmod{\delta}$, then $\mathcal{A}$ and $\mathcal{A}'$ are equivalent weighted graphs.

1.16. Let a, b, c be pairwise relatively prime positive integers.

(1) There is a unique integer $c' = c'(a, b)$ with $0 \leq c' < c$ and $b \equiv ac' \bmod c$. (Note that $c' = 0$ if and only if $c = 1$.)

(2) Define the integer $c'' = c''(a, b)$ by $c'(a, b)c'(b, a) = 1 + c''c$. (Note that $c = 1 \Rightarrow c'' = -1$ and $c \neq 1 \Rightarrow 0 \leq c'' < c' < c$.)

One also defines integers $a'(b, c)$, $a'(c, b)$, $a''(b, c)$, $b'(a, c)$, etc. Note that each one of these is a function of the *three* variables a, b, c.

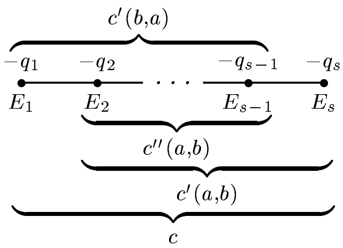
Definition 1.17. Consider an unordered triple $[\delta_0, \delta_1, \delta_2]$, where $\delta_0, \delta_1, \delta_2$ are pairwise relatively prime positive integers. We define the weighted graph $\mathcal{G}_{[\delta_0,\delta_1,\delta_2]}$ to be the disjoint union $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_i$ is the unique admissible chain with discriminant δ_i and subdiscriminants $\delta_i'(\delta_{i+1}, \delta_{i+2})$ and $\delta_i'(\delta_{i+2}, \delta_{i+1})$ (with indices computed modulo 3). Note that each $\mathcal{A}_i$ is allowed to be empty.

Cyclic quotient singularities

Let $\omega_c \subset \mathbf{k}^*$ be the group of c-th roots of unity.

Lemma 1.18 ([4]). *Let a, b, c be pairwise relatively prime positive integers. Let ω_c act on $\mathbf{k}[[\xi, \eta]]$ with weights $a, b \bmod c$ for ξ, η and let $\mathcal{X} = \operatorname{Spec} \mathbf{k}[[\xi, \eta]]^{\omega_c}$.*

(1) *The exceptional locus of the minimal resolution of the singularity of $\mathcal{X}$ is an admissible chain $E = E_1 + \cdots + E_s$ of rational curves with dual graph:*



where the braces give the determinants of the indicated subtrees.

(2) *The proper transform of the image of $V(\eta)$ (resp. $V(\xi)$) meets E normally in E_1 (resp. E_s).*

1.19. The *resolution graph* of a normal surface X is the dual graph of E in $\hat{X}$, where E is the exceptional locus of the minimal resolution of singularities $\pi : \hat{X} \to X$ of X. Let x be a cyclic quotient singularity of X and recall that the resolution locus $\pi^{-1}(x)$ of x is an admissible chain $\mathcal{A}$. We define the *discriminant* and *subdiscriminants* of the singularity x to be those of $\mathcal{A}$. A smooth point is regarded as a cyclic quotient singularity of discriminant 1. If the singularity x is determined by ω_c acting with weights a and b (where a, b, c are pairwise relatively prime) then Lemma 1.18 says that x has discriminant c and subdiscriminants $c'(a, b)$ and $c'(b, a)$.

Singularities of $\mathbb{P}$

Choose a coordinate system (X_0, X_1, X_2) of $\mathbb{P}$ and consider the open neighbourhood $D_+(X_2)$ of q_2 in $\mathbb{P}$. As noted in the proof of 1.3.3 of [3], $D_+(X_2)$ is isomorphic to the quotient $\mathbb{A}^2/\omega_{a_2}$, where the action is given by $t(u_0, u_1) = (t^{a_0}u_0, t^{a_1}u_1)$ (with $t \in \omega_{a_2}$, $(u_0, u_1) \in \mathbb{A}^2$). So q_2 is a cyclic quotient singularity of $\mathbb{P}$ and, by 1.19, q_2 has discriminant a_2 and subdiscriminants $a_2'(a_0, a_1)$ and $a_2'(a_1, a_0)$; note that the image in $\mathbb{P}$ of the line "$u_i = 0$" is part of R_i ($i = 0, 1$). Similar remarks hold for q_0 and q_1, so we obtain:

1.20. (1) For each $i = 0, 1, 2$, $\mathbb{P}$ has a cyclic quotient singularity at q_i, of discriminant a_i and subdiscriminants $a_i'(a_{i+1}, a_{i+2})$ and $a_i'(a_{i+2}, a_{i+1})$.
(2) $\operatorname{Sing}\mathbb{P} \subseteq \{q_0, q_1, q_2\}$.
(3) q_i is a smooth point if and only if $a_i = 1$.

It follows that $\mathbb{P}$ is a surface of type $[a_0, a_1, a_2]$, according to:

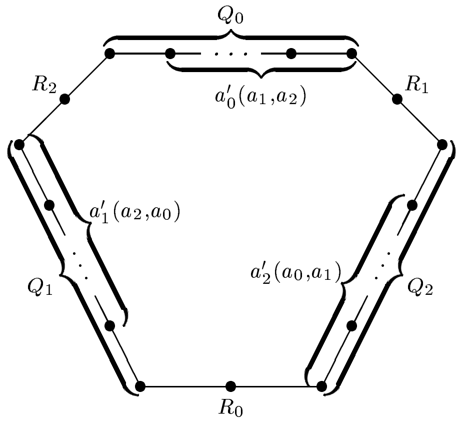
Definition 1.21. Let $[\delta_0, \delta_1, \delta_2]$ be an unordered triple of pairwise relatively prime positive integers. By a *surface of type* $[\delta_0, \delta_1, \delta_2]$, we mean a surface satisfying (‡) and whose resolution graph is $\mathcal{G}_{[\delta_0,\delta_1,\delta_2]}$.

Remark. A surface X satisfying (‡) may or may not have a type as defined in 1.21. If X has a type, we sometimes say that it has *tuned singularities.*

Remark. We will show in 6.12 that every surface of type $[\delta_0, \delta_1, \delta_2]$ is isomorphic to $\mathbb{P}(\delta_0, \delta_1, \delta_2)$.

Let $\hat{\mathbb{P}} \to \mathbb{P}$ be the minimal resolution of singularities and Q_i the exceptional locus above q_i. By the above, R_0 and R_1 meet the chain Q_2 normally at opposite ends. (With some abuse of notation, we use the same letter to denote R_i and its proper transform in $\hat{\mathbb{P}}$.) More precisely, we have the first part of the following lemma. The second part will be proved in section 3 (but will not be needed).

Lemma 1.22. (1) *$R = R_0 + Q_1 + R_2 + Q_0 + R_1 + Q_2$ is a "ring" of rational curves with dual graph*



(2) *$-R$ is a canonical divisor of $\hat{\mathbb{P}}$.*

2. Graphs, tableaux and rulings

This section gathers some of the definitions and results of [2] and (we hope) organizes them in a coherent way. It also includes a few items which are not found in [2].

Graphs and tableaux

2.1. Given weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, the symbol $\mathcal{G} \leftarrow \mathcal{G}'$ indicates that $\mathcal{G}'$ is obtained from $\mathcal{G}$ by blowing-up once. In that case, if V (resp. V') denotes the set of vertices of $\mathcal{G}$ (resp. $\mathcal{G}'$) then V can be viewed as a subset of V' and $V' \setminus V$ contains a single vertex, say e. We call e the vertex *created* by $\mathcal{G} \leftarrow \mathcal{G}'$. This e has weight -1 and has at most two neighbors in $\mathcal{G}'$; if it has one neighbor v_1 (resp. two neighbors v_1, v_2) then, regarding v_1 (resp. v_1, v_2) as a vertex of $\mathcal{G}$, we say that $\mathcal{G} \leftarrow \mathcal{G}'$ is the blowing-up of $\mathcal{G}$ *at the vertex* v_1 (resp. *at the edge* $\{v_1, v_2\}$). A blowing-up at a vertex (resp. at an edge) is also called a *sprouting* (resp. *subdivisional*) blowing-up. In reverse, we say that $\mathcal{G}$ is obtained by contracting (or blowing-down) $\mathcal{G}'$ at e. Given a sequence $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ of blowings-up, we may also speak of the contraction "$\mathcal{G}_n \geq \mathcal{G}_0$" of weighted graphs.

2.2. Let $n \geq 1$. By a *weighted n-tuple*, we mean an ordered n-tuple $S = (\mathcal{G}, v_1, \dots, v_{n-1})$ where $\mathcal{G}$ is a weighted graph and $v_1, \dots, v_{n-1}$ are distinct vertices of $\mathcal{G}$.

When $n = 1$, S is simply a weighted graph; when $n = 2$, it is called a *weighted pair*. The following is the only weighted n-tuple with $n > 2$ that we will need:

Notation 2.3. Given $x \in \mathbb{Z}$, let $\mathcal{G}_{(x)}$ denote the weighted triple $(\mathcal{G}, v_1, v_2)$, where $\mathcal{G}$ is the weighted graph $\overset{0}{\underset{v_1}{\bullet}} \!\!-\!\!-\!\!-\!\! \overset{x}{\bullet} \!\!-\!\!-\!\!-\!\! \overset{0}{\underset{v_2}{\bullet}}$.

2.4. If $(\mathcal{G}, v)$ is a weighted pair, we call v its *distinguished vertex*. By a *linear weighted pair*, we mean a weighted pair $(\mathcal{G}, v)$ satisfying: (i) $\mathcal{G}$ is a linear chain; and (ii) v has at most one neighbor in $\mathcal{G}$.

2.5. Let $(\mathcal{G}, v)$ be a weighted pair and $\mathcal{G} \geq \mathcal{G}'$ a contraction of weighted graphs such that v is not contracted (i.e., v is still a vertex of $\mathcal{G}'$). Then we write $(\mathcal{G}, v) \geq (\mathcal{G}', v)$ and call this a *contraction of weighted pairs*. The equivalence relation (on the set of weighted pairs) generated by $\geq$ is denoted "$\approx$", and is called "equivalence of weighted pairs".

2.6. Let $(\mathcal{G}, v)$ and $(\mathcal{G}', v')$ be weighted pairs. Suppose that $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ (i.e., $\mathcal{G} \leftarrow \mathcal{G}'$) and that the following hold: (i) The blowing-up $\mathcal{G} \leftarrow \mathcal{G}'$ is either at v or at an edge incident to v; and (ii) v' is the vertex of $\mathcal{G}'$ which is created by the blowing-up $\mathcal{G} \leftarrow \mathcal{G}'$. Then we say that $(\mathcal{G}', v')$ is a *blowing-up* of $(\mathcal{G}, v)$ and write $(\mathcal{G}, v) \leftarrow (\mathcal{G}', v')$.

Remark. A blowing-up of weighted pairs $(\mathcal{G}, v) \leftarrow (\mathcal{G}', v')$ cannot be undone by contracting $(\mathcal{G}', v')$ as in 2.5.

2.7. A *tableau* is a matrix $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)$ whose entries are integers satisfying $c_i \geq p_i \geq 1$ and $\gcd(p_i, c_i) = 1$ for all $i = 1, \dots, k$. We allow $k = 0$, in which case we say that T is the *empty tableau* and write $T = \mathbf{1}$. The set of all tableaux is denoted $\mathcal{T}$. We define a binary operation on the set $\mathcal{T}$ by:

$$\left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)\left(\begin{smallmatrix} p_{k+1} & \cdots & p_\ell \\ c_{k+1} & \cdots & c_\ell \end{smallmatrix}\right) = \left(\begin{smallmatrix} p_1 & \cdots & p_k & p_{k+1} & \cdots & p_\ell \\ c_1 & \cdots & c_k & c_{k+1} & \cdots & c_\ell \end{smallmatrix}\right).$$

Thus $\mathcal{T}$ is the free monoid on the set of columns $\binom{p}{c}$ where $p \le c$ are relatively prime positive integers.

2.8. Let $(\mathcal{G}_0, e_0)$ be a weighted pair and $\binom{p}{c} \in \mathcal{T}$. By *blowing-up* $(\mathcal{G}_0, e_0)$ *according to* $\binom{p}{c}$, we mean producing the sequence $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ defined as follows.

(1) Let $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ be the blowing-up at e_0 and let e_1 be the vertex of $\mathcal{G}_1$ so created. Define $\left(\begin{smallmatrix} u_1 & x_1 \\ v_1 & y_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e_1 & p \\ e_0 & c-p \end{smallmatrix}\right)$.

(2) If $i \ge 1$ is such that $(\mathcal{G}_i, e_i)$ and $\left(\begin{smallmatrix} u_i & x_i \\ v_i & y_i \end{smallmatrix}\right)$ have been defined, then:
   (a) If $y_i = 0$ then we set $n = i$ and stop.
   (b) If $y_i \neq 0$ then let $\mathcal{G}_{i+1}$ be the blowing-up of $\mathcal{G}_i$ at the edge $\{u_i, v_i\}$, let e_{i+1} be the vertex of $\mathcal{G}_{i+1}$ so created and define

$$\left(\begin{smallmatrix} u_{i+1} & x_{i+1} \\ v_{i+1} & y_{i+1} \end{smallmatrix}\right) = \begin{cases} \left(\begin{smallmatrix} e_{i+1} & x_i \\ v_i & y_i - x_i \end{smallmatrix}\right) & \text{if } x_i \le y_i, \\ \left(\begin{smallmatrix} u_i & x_i - y_i \\ e_{i+1} & y_i \end{smallmatrix}\right) & \text{if } x_i > y_i. \end{cases}$$

2.9. Let $(\mathcal{G}_0, e_0)$ be a weighted pair and $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right) \in \mathcal{T}$ a tableau.

(1) We define *the sequence* $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ *obtained by blowing-up* $(\mathcal{G}_0, e_0)$ *according to* T by induction on k:
   - If $k = 0$ (i.e., T is the empty tableau), then $n = 0$ (no blowing-up is performed).
   - If $k = 1$, then $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ is defined in 2.8.
   - If $k > 1$, then $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ is

   $$(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_m, e_m) \leftarrow (\mathcal{G}_{m+1}, e_{m+1}) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n),$$

   where $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_m, e_m)$ is the sequence obtained by blowing-up $(\mathcal{G}_0, e_0)$ according to $\binom{p_1}{c_1}$ and $(\mathcal{G}_m, e_m) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ is obtained by blowing-up $(\mathcal{G}_m, e_m)$ according to $\left(\begin{smallmatrix} p_2 & \cdots & p_k \\ c_2 & \cdots & c_k \end{smallmatrix}\right)$.

(2) Consider the sequence $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ obtained by blowing-up $(\mathcal{G}_0, e_0)$ according to T, as defined in part (1). Then we write $(\mathcal{G}_0, e_0)T = (\mathcal{G}_n, e_n)$. Hence, blowing-up is a right action of the monoid $\mathcal{T}$ on the set of weighted pairs.

2.10. Let S be a weighted n-tuple, with $n \ge 2$, and let $T \in \mathcal{T}$ be a tableau.

Write $S = (\mathcal{G}, v_1, \dots, v_{n-1})$ and let $(\mathcal{G}', e)$ denote the weighted pair $(\mathcal{G}, v_1)T$, as defined in part (2) of 2.9. Note that $v_2, \dots, v_{n-1}$ can be regarded as vertices of $\mathcal{G}' \setminus \{e\}$.

(1) Define $ST = (\mathcal{G}', e, v_2, \dots, v_{n-1})$, a weighted n-tuple.
(2) Define $S \ominus T = (\mathcal{G}' \setminus \{e\}, v_2, \dots, v_{n-1})$, a weighted $(n-1)$-tuple.
(3) Let $S \oplus T$ denote the unique connected component of $S \ominus T$ which contains no vertex of $\mathcal{G}$. We regard $S \oplus T$ as a weighted graph; actually, $S \oplus T$ is a (possibly empty) admissible chain. Note that $S \oplus T$ is empty when T is the empty tableau.

(4) Let $S \textcircled{\scriptsize$\uparrow$} T$ be the complement of $S \textcircled{\scriptsize$\downarrow$} T$ in $S \ominus T$. We regard $S \textcircled{\scriptsize$\uparrow$} T$ as a weighted $(n-1)$-tuple.

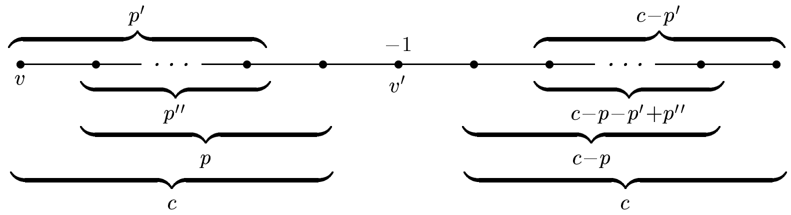
Note that $S \ominus T$ is the disjoint union of $S \textcircled{\scriptsize$\uparrow$} T$ and $S \textcircled{\scriptsize$\downarrow$} T$.

2.11. Given relatively prime positive integers a and b, define $\binom{a}{b}^* = \binom{x}{y}$, where x and y are the unique nonnegative integers which satisfy

$$\left|\begin{smallmatrix} x & a \\ y & b \end{smallmatrix}\right| = -1 \qquad \text{and} \qquad x < a \ \text{ or } \ y < b.$$

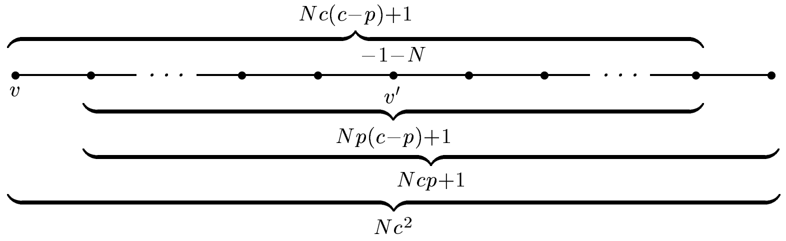
2.12 ([2], 3.23)**.** *Let $c > p > 0$ be relatively prime integers, let $\mathcal{G}$ be the weighted graph which consists of a single vertex v of weight zero, and let $(\mathcal{G}', v') = (\mathcal{G}, v)\binom{p}{c}$. Then $\mathcal{G}'$ has two branches at v', with determinants of subtrees as follows:*

$(\mathcal{G}', v')$:



where we define $\binom{p''}{p'} = \binom{p}{c}^*$. Note that these two branches are $(\mathcal{G}, v) \textcircled{\scriptsize$\uparrow$} \binom{p}{c}$ (left part of the picture) and $(\mathcal{G}, v) \textcircled{\scriptsize$\downarrow$} \binom{p}{c}$ (right).

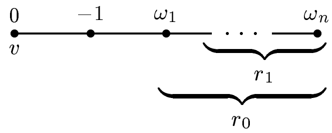
Moreover, if we let $(\mathcal{G}'', v'') = (\mathcal{G}', v')\binom{1}{N}$ (with $N \geq 1$) then the connected component of $\mathcal{G}'' \setminus \{v''\}$ containing v and v' is as follows:



(This connected component is the same thing as $(\mathcal{G}', v') \textcircled{\scriptsize$\uparrow$} \binom{1}{N} = (\mathcal{G}, v) \textcircled{\scriptsize$\uparrow$} \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$.)

2.13. Consider a weighted pair

$\mathcal{L}$:



where v is the distinguished vertex, $n \geq 0$, $\omega_i \leq -2$, and where r_0 and r_1 denote the determinants of the indicated subtrees (if $n = 1$ then $r_1 = 1$; if $n = 0$ then $r_0 = 1$ and $r_1 = 0$). Then $\mathcal{L}$ determines the 2×2 matrix $M(\mathcal{L}) = \left(\begin{smallmatrix} x & r_0 - r_1 \\ y & r_0 \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} r_0 - r_1 \\ r_0 \end{smallmatrix}\right)^*$. For each $\nu \geq 0$, let $\binom{p_\nu}{c_\nu} = M(\mathcal{L}) \cdot \binom{1}{\nu}$ (matrix product). Then define a subset $\mathcal{T}(\mathcal{L})$ of

$\mathcal{T}$ by:

$$\mathcal{T}(\mathcal{L}) = \left\{ \begin{pmatrix} p_\nu \\ c_\nu \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\nu} \;\middle|\; \nu \geq 0 \ \ (\text{resp. } \nu > 0) \right\}$$

if $\omega_i < -2$ for some i (resp. $\omega_i = -2$ for all i), and where $\binom{p_\nu}{c_\nu}\binom{1}{1}^\nu$ is a product in the monoid $\mathcal{T}$. We also define $\mathcal{T}_k(\mathcal{L}) = \left\{ T \in \mathcal{T} \mid T\binom{1}{1}^k \in \mathcal{T}(\mathcal{L}) \right\}$ for each $k \in \mathbb{N}$.

2.14. Given $\mathcal{L}$ as in 2.13, define $\mathcal{L}^t$: $\overset{0}{\bullet}\!\!-\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!-\!\!\overset{\omega_n}{\bullet}\ \cdots\ \overset{\omega_1}{\bullet}$. Also define $\mathcal{L}^{t^0} = \mathcal{L}$ and, for each $s > 0$, $\mathcal{L}^{t^s} = (\mathcal{L}^{t^{s-1}})^t$. By 3.24 of [2], $M(\mathcal{L}^t) = M(\mathcal{L})^t$.

2.15 ([2], 3.32)**.** *Given $\mathcal{L}$ as in 2.13 and $\binom{p}{c} \in \mathcal{T}$ such that $\binom{p}{c} \neq \binom{1}{1}$,*

$$\mathcal{L}\begin{pmatrix} p \\ c \end{pmatrix} \text{ contracts to a linear weighted pair } \iff \begin{pmatrix} p \\ c \end{pmatrix} \in \mathcal{T}_k(\mathcal{L}) \text{ for some } k \in \mathbb{N}.$$

Moreover, if $\binom{p}{c} \in \mathcal{T}_k(\mathcal{L})$ then $\mathcal{L}\binom{p}{c}\binom{1}{1}^k \approx \mathcal{L}^t$.

2.16. We will sometimes refer to the following conditions on a tableau $T \in \mathcal{T}$:

(1) $T = \mathbf{1}$ (the empty tableau);
(2) $T = \binom{p}{c}$, where $\binom{p}{c} \neq \binom{1}{1}$;
(3) $T = \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$, where $\binom{p}{c} \neq \binom{1}{1}$ and $N \geq 1$.

Given $T \in \mathcal{T}$ satisfying one of the above conditions (1–3), define $\check{T} \in \mathcal{T}$ by:[1]

$$\check{T} = \begin{cases} \mathbf{1}, & \text{if } T \text{ satisfies 2.16.1;} \\ \binom{p'}{c}, & \text{if } T \text{ satisfies 2.16.2, where } p' \text{ is given by } \binom{p''}{p'} = \binom{p}{c}^* \text{ (see 2.11);} \\ \left(\begin{smallmatrix} c-p & 1 \\ c & N \end{smallmatrix}\right), & \text{if } T \text{ satisfies 2.16.3.} \end{cases}$$

Note that if T satisfies condition 2.16.i (where $i \in \{1,2,3\}$) then so does $\check{T}$. If s is a positive integer, write $T^{(\vee s)} = \left(T^{(\vee(s-1))}\right)^{\vee}$, where $T^{(\vee 0)} = T$. Note that $T^{(\vee 2)} = T$.

Let $\mathbb{Z}^+$ denote the set of positive integers.

2.17. Let $\mathbb{T}(\ddagger)$ be the set of triples $(m, T_1, T_2) \in \mathbb{Z}^+ \times \mathcal{T} \times \mathcal{T}$ such that (i) T_1 satisfies one of the conditions (1–3) of 2.16; (ii) $T_2 \notin \binom{1}{1}\mathcal{T}$ (i.e., if T_2 is nonempty then its first column is not $\binom{1}{1}$); and (iii) each connected component of the weighted graph $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ shrinks to an admissible chain.

2.18. Define an order relation $>$ on the set $\mathbb{T}(\ddagger)$ by declaring that $(n, T_1, T_2) > (m, T_1', T_2')$ if $n = 1$ and the following holds (let $\mathcal{L} = \mathcal{G}_{(-1)} \mathbin{\bigcirc\!\!\!\!\uparrow} T_1'$):

> There exist an integer $s \geq 1$ and tableaux $X_1, \dots, X_s$ such that $T_1 = (T_1')^{(\vee s)}$, $T_2 = X_s \cdots X_1 T_2'$ and $X_i \in \mathcal{T}_{k_i}(\mathcal{L}^{t^i})$, where $k_1 = m - 1$ and $k_i = 0$ for all $i > 1$.

2.19. *Consider $\tau = (1, T_1, T_2) \in \mathbb{T}(\ddagger)$ and let $\mathcal{L}' = \mathcal{G}_{(-1)} \mathbin{\bigcirc\!\!\!\!\uparrow} T_1$. Then the following are equivalent:*

---

[1] In the second part of the definition of $\check{T}$, we could also define p' by $0 < p' < c$ and $pp' \equiv 1 \pmod{c}$.

(1) *τ is non-minimal in $\mathbb{T}(\ddagger)$;*
(2) *T_2 is nonempty and its first column belongs to $\mathcal{T}_k(\mathcal{L}')$ for some $k \in \mathbb{N}$.*

2.20. Given $(n, T_1, T_2), (m, T_1', T_2') \in \mathbb{T}(\ddagger)$, write $(n, T_1, T_2) \equiv (m, T_1', T_2')$ to indicate that $(\mathcal{G}_{(-n)} \ominus T_1)T_2 \approx (\mathcal{G}_{(-m)} \ominus T_1')T_2'$ (equivalence of weighted pairs). Note that "$\equiv$" is an equivalence relation on the set $\mathbb{T}(\ddagger)$. We have

$$\tau > \tau' \implies \tau \equiv \tau' \qquad (\text{all } \tau, \tau' \in \mathbb{T}(\ddagger))$$

by 5.18 of [2], but "$\equiv$" is *not* the equivalence relation generated by "$>$".

Affine rulings

2.21. Let X be a complete normal rational surface. By an *affine ruling* of X we mean a one-dimensional linear system Λ on X (without fixed components) which arises[2] from a morphism $p : U \to \Gamma$ where Γ is a curve, U is a nonempty open subset of X isomorphic to $\Gamma \times \mathbb{A}^1$ and p is the projection $\Gamma \times \mathbb{A}^1 \to \Gamma$.

2.22. Let Λ be an affine ruling of a surface X satisfying ($\ddagger$). By "resolving" (X, Λ), we mean constructing a pair $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$ as follows ([2], 1.5):

(1) Minimally resolve the singularities of X (write $\hat{X} \to X$). Let $\hat{\Lambda}$ be the strict transform of Λ on $\hat{X}$.
(2) Minimally resolve the base point of $\hat{\Lambda}$ (write $\tilde{X} \to \hat{X}$). Let $\tilde{\Lambda}$ be the strict transform of $\hat{\Lambda}$ on $\tilde{X}$.

Let $\rho : \tilde{X} \to X$ be the composition $\tilde{X} \to \hat{X} \to X$. The center of ρ is $\operatorname{Sing} X \cup \operatorname{Bs} \Lambda$ and $\rho^{-1}(\operatorname{Sing} X \cup \operatorname{Bs} \Lambda)$ is the support of a divisor D of $\tilde{X}$ with strong normal crossings.

2.22.1 ([2], 1.14)**.** We say that Λ is *basic* if each connected component of D is a linear chain.

Then Theorem 2.1 of [2] implies (in particular):

2.22.2. *Every surface satisfying ($\ddagger$) admits a basic affine ruling.*

Clearly, $\tilde{\Lambda}$ is base-point-free and its general member is $\mathbb{P}^1$, i.e., $\tilde{\Lambda}$ is a "$\mathbb{P}^1$-ruling" of $\tilde{X}$. Using that X satisfies ($\ddagger$), one shows ([2], 1.8 and 1.15):

(1) *Exactly one irreducible component H of D is a section of $\tilde{\Lambda}$.*
(2) *Each reducible $G \in \tilde{\Lambda}$ has exactly one (-1)-component C_G. Moreover, $H \cdot C_G = 0$ and $D = H + \sum_i (G_i^\# - C_{G_i})$, where the G_i are the reducible members of $\tilde{\Lambda}$ and where $G_i^\#$ is the reduced effective divisor of $\tilde{X}$ with same support as G_i.*
(3) *$\tilde{\Lambda}$ has at most two reducible members.*

Define $m > 0$ by $H^2 = -m$ and consider the Nagata ruled surface $\mathbb{F}_m$; let Λ_m be the standard ruling of $\mathbb{F}_m$ and Σ_m the negative section of Λ_m. Then well-known properties of $\mathbb{P}^1$-rulings imply:

[2] Note that Γ must be an open subset of $\mathbb{P}^1$, so p extends to a rational map $p' : X \to \mathbb{P}^1$ and p' determines a linear system Λ on X without fixed components.

(4) *By shrinking each G_i to a 0-curve, we get $\pi : \tilde{X} \to \mathbb{F}_m$, where the exceptional locus of π is disjoint from H, $\pi(H) = \Sigma_m$ and $\pi(G_i) \in \Lambda_m$.*

It follows from (2) that each member of Λ is irreducible (but not necessarely reduced). Via the isomorphism $\tilde{X} \setminus \operatorname{supp}(D) \cong X \setminus (\operatorname{Sing} X \cup \operatorname{Bs} \Lambda)$, each $F \in \Lambda$ determines an $\tilde{F} \in \tilde{\Lambda}$; moreover, $F \mapsto \tilde{F}$ is a bijection $\Lambda \to \tilde{\Lambda}$.

2.22.3 ([2], 2.4 and 2.5)**.** Define a *nonempty* subset Λ_* of Λ by declaring that it contains all $F \in \Lambda$ satisfying: (i) At most one element of $\Lambda \setminus \{F\}$ is not reduced; and (ii) all branching components of D are components of $\tilde{F}$.

Note that if $P_i \in \mathbb{F}_m$ is a point of the center of π then $\pi^{-1}(P_i)$ contains exactly one (-1)-curve (namely, C_{G_i}). Because of this property, π can be described by using a pair of Hamburger-Noether tableaux (one for each point of the center), say HN_1 and HN_2. Let T_i be the tableau obtained from HN_i by deleting the third row and dividing each column by its gcd ($T_i = \overline{\mathrm{HN}_i} \in \mathcal{T}$, see 3.6 of [2]). The triple (m, T_1, T_2) is then a partial description of π.

2.23 ([2], 5.1 and 5.2)**.** Given a triple (X, Λ, F), where X is a surface satisfying (‡), Λ is an affine ruling of X and F is an element of Λ_*, let us now define an element τ of $\mathbb{T}(\ddagger)$, called the *discrete part* of (X, Λ, F) (notation: $\operatorname{disc}(X, \Lambda, F) = \tau$). Consider the triple (m, T_1, T_2) constructed at the end of 2.22, but make sure[3] that the P_i's and G_i's have been labeled in such a way that the bijection $\Lambda \to \tilde{\Lambda}$ sends F to G_2. Then we define $\operatorname{disc}(X, \Lambda, F) = (m, T_1, T_2)$. It satisfies:

$$(m, T_1, T_2) \in \mathbb{T}(\ddagger) \text{ and } \big(\mathcal{G}_{(-m)} \ominus T_1\big) \ominus T_2 \text{ is the dual graph of } D \text{ in } \tilde{X},$$

so $\big(\mathcal{G}_{(-m)} \ominus T_1\big) \ominus T_2$ shrinks to the resolution graph of X (D and $\tilde{X}$ are as in 2.22 and $\mathbb{T}(\ddagger)$ was defined in 2.17).

2.24 ([2], 5.25)**.** Two triples as in 2.23 are *equivalent*, $(X, \Lambda, F) \sim (X', \Lambda', F')$, when there exists an isomorphism $X \to X'$ which transforms Λ into Λ' and F into F'. If this is the case then (X, Λ, F) and (X', Λ', F') have the same discrete part; so we may speak of the discrete part of the equivalence class $[X, \Lambda, F]$ of (X, Λ, F), and we have a set map

$$\operatorname{disc} : \mathbb{S}(\ddagger) \to \mathbb{T}(\ddagger) \qquad [X, \Lambda, F] \mapsto \text{discrete part of } [X, \Lambda, F]$$

where $\mathbb{S}(\ddagger)$ is the set of equivalence classes $[X, \Lambda, F]$. This map is in fact surjective and restricts to a bijection

$$\operatorname{disc} : \mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$$

where

$$\mathbb{T}_0(\ddagger) = \{\, (m, T_1, T_2) \in \mathbb{T}(\ddagger) \mid T_2 \text{ satisfies one of conditions (1–3) of 2.16} \}$$
$$\mathbb{S}_0(\ddagger) = \{\, [X, \Lambda, F] \in \mathbb{S}(\ddagger) \mid \Lambda \text{ is basic} \} = \operatorname{disc}^{-1}(\mathbb{T}_0(\ddagger)).$$

[3]This can always be arranged; it may involve *choosing* some of the P_i's and G_i's when $\tilde{\Lambda}$ has less than two reducible members. See [2] for details.

2.25. Given X satisfying (‡), define subsets $\mathbb{T}_0(X) \subset \mathbb{T}(X)$ of $\mathbb{T}(‡)$ by:

$$\begin{aligned}\mathbb{T}(X) &= \{\operatorname{disc}(X,\Lambda,F) \mid \Lambda \text{ is an affine ruling of } X \text{ and } F \in \Lambda_*\},\\ \mathbb{T}_0(X) &= \mathbb{T}(X) \cap \mathbb{T}_0(‡)\\ &= \{\operatorname{disc}(X,\Lambda,F) \mid \Lambda \text{ is a basic affine ruling of } X \text{ and } F \in \Lambda_*\}.\end{aligned}$$

Then 5.13 of [2] implies: *For any $\tau, \tau' \in \mathbb{T}(‡)$ satisfying $\tau \equiv \tau'$, we have*

$$\tau \in \mathbb{T}(X) \iff \tau' \in \mathbb{T}(X). \tag{1}$$

Moreover, if $\tau = \operatorname{disc}(X,\Lambda,F)$ then there exists an affine ruling Λ' of X and an element F' of Λ'_* such that $\operatorname{supp}(F) = \operatorname{supp}(F')$ *and* $\tau' = \operatorname{disc}(X,\Lambda',F')$. Note that these facts still hold if we replace the assumption $\tau \equiv \tau'$ by $\tau > \tau'$ (see 2.20). We also point out that 5.17 of [2] implies:

$$\textit{Given } \tau \in \mathbb{T}(X) \setminus \mathbb{T}_0(X), \textit{ there exists } \tau' \in \mathbb{T}_0(X) \textit{ such that } \tau > \tau'. \tag{2}$$

2.26. Noting that each element (m, T_1, T_2) of $\mathbb{T}_0(‡)$ satisfies exactly one of:

I: Each of T_1, T_2 has at most one column;
II.1: T_1 has at most one column but T_2 has two;
II.2: T_1 has two columns but T_2 has at most one;
III: each of T_1, T_2 has two columns,

we give the following two definitions:

(1) Given $\mathcal{P} \in \{\mathrm{I}, \mathrm{II.1}, \mathrm{II.2}, \mathrm{III}\}$[4] and pairwise relatively prime positive integers a_0, a_1, a_2, let $\mathbb{T}_{\mathcal{P}}(a_0,a_1,a_2)$ be the set

$$\{(m,T_1,T_2) \in \mathbb{T}_0(‡) \mid (m,T_1,T_2) \text{ satisfies } \mathcal{P} \text{ and } G_i \sim \mathcal{A}_i \text{ for } i = 0,1,2\},$$

where "$\sim$" is equivalence of weighted graphs, $G_0 = (\mathcal{G}_{(-m)} \oplus T_1) \oplus T_2$, $G_1 = \mathcal{G}_{(-m)} \oplus T_1$, $G_2 = (\mathcal{G}_{(-m)} \oplus T_1) \oplus T_2$ and where $\mathcal{A}_i$ is the unique admissible chain with discriminant a_i and subdiscriminants $a_i'(a_{i+1}, a_{i+2})$ and $a_i'(a_{i+2}, a_{i+1})$ (with indices computed modulo 3). Note that G_0, G_1 and G_2 are the connected components of $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$, with the understanding that G_1 and G_2 are allowed to be empty (G_0 is never empty).

(2) Let Λ be a basic affine ruling of a surface X satisfying (‡). Then it is easy to see that

$$\{\operatorname{disc}(X,\Lambda,F) \mid F \in \Lambda_*\} = \{(m,T_1,T_2),(m,T_2,T_1)\} \subset \mathbb{T}_0(X)$$

for some tableaux T_1 and T_2 and some $m \in \mathbb{Z}^+$. We say that Λ is a basic affine ruling *of type I* (resp. *II, III*) if, for $F \in \Lambda_*$, the discrete part (m, T_1, T_2) of (X, Λ, F) satisfies the above condition I (resp. II.1 or II.2, III).

[4] We mean that $\mathcal{P}$ is one of the four symbols I, II.1, II.2, III.

2.27. *Let X be a surface of type $[a,b,c]$, where a,b,c are pairwise relatively prime positive integers (see* 1.21*). If Λ is a basic affine ruling of X and $F \in \Lambda_*$ then $\tau = \operatorname{disc}(X,\Lambda,F)$ belongs to $\mathbb{T}_{\mathcal{P}}(a_0,a_1,a_2)$ for some $\mathcal{P} \in \{\mathrm{I}, \mathrm{II.1}, \mathrm{II.2}, \mathrm{III}\}$ and some permutation a_0,a_1,a_2 of a,b,c.* (Indeed, if we write $\tau = (m,T_1,T_2)$ then $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ is equivalent to the resolution graph of X, which is $\mathcal{G}_{[a,b,c]}$.)

2.28. *Let $\mathcal{P} \in \{\mathrm{I}, \mathrm{II.1}, \mathrm{II.2}, \mathrm{III}\}$ and let a_0,a_1,a_2 be pairwise relatively prime positive integers. If $(m,T_1,T_2) \in \mathbb{T}_{\mathcal{P}}(a_0,a_1,a_2)$ then the entry in the lower right corner of T_i is a_i.* (For $i \in \{1,2\}$ we may write $G_i = Z \oplus T_i = Z \oplus \binom{p_i}{c_i}$, where G_i is as in 2.26, Z is the weighted pair consisting of a single vertex of weight zero and $\binom{p_i}{c_i}$ is the rightmost column of T_i; then 2.12 gives $\det(G_i) = c_i$, so $c_i = a_i$.)

Note that 2.28 holds even when T_i is empty, in which case we use the following convention:

2.29. When a tableau T has at most one column, we sometimes abuse notation and write $T = \binom{p}{c}$ in all cases, with $p=0$ and $c=1$ when T is empty.

2.30. Suppose that $\tau = (m,T_1,T_2), \tau' = (m',T_1',T_2') \in \mathbb{T}(\ddagger)$ satisfy $\tau \equiv \tau'$ and consider G_0,G_1,G_2 determined by τ as in 2.26 and G_0',G_1',G_2' determined by τ' in a similar way. Then it is immediate that $G_1 \sim G_1'$ and that, for some permutation i,j of $0,2$, $G_0 \sim G_i'$ and $G_2 \sim G_j'$. In the special case where $\tau > \tau'$, we have:

If T_2' is nonempty (resp. empty), then $G_i \sim G_i'$ (resp. $G_i \sim G_{2-i}'$) for all $i = 0,1,2$.

If τ is a non-minimal element of $\mathbb{T}(\ddagger)$ then ([2], 5.21) there exists a unique $\tau^- \in \mathbb{T}(\ddagger)$ satisfying: (i) $\tau > \tau^-$ and (ii) no $\tau' \in \mathbb{T}(\ddagger)$ is such that $\tau > \tau' > \tau^-$. We call τ^- the *immediate predecessor* of τ.

Lemma 2.31. *Let τ be a nonminimal element of $\mathbb{T}(\ddagger)$, let τ^- be its immediate predecessor and suppose that $\tau \in \mathbb{T}_{\mathcal{P}}(a_0,a_1,a_2)$ for some $\mathcal{P} \in \{\mathrm{I}, \mathrm{II.1}, \mathrm{II.2}, \mathrm{III}\}$ and some pairwise relatively prime positive integers a_0,a_1,a_2. Then*

$$\tau^- \in \begin{cases} \mathbb{T}_{\mathrm{II.2}}(a_0,a_1,a_2), & \text{if } \mathcal{P} = \mathrm{III}, \\ \mathbb{T}_{\mathrm{I}}(a_0,a_1,a_2), & \text{if } \mathcal{P} = \mathrm{II.1}, \\ \mathbb{T}_{\mathcal{P}}(a_2,a_1,a_0), & \text{if } \mathcal{P} \in \{\mathrm{I}, \mathrm{II.2}\}. \end{cases}$$

Proof. Write $\tau = (1,T_1,T_2)$ and $\tau^- = (m,T_1',T_2')$ and recall that T_1 and T_1' have the same number of columns, and the number of columns of T_2' is strictly less than that of T_2. If $\mathcal{P}$ is I or II.2 then T_2' must be the empty tableau, so the assertion follows from 2.30.

Suppose that $\mathcal{P} = \mathrm{III}$ (resp. $\mathcal{P} = \mathrm{II.1}$). If T_2' is not empty then, again, the assertion follows from 2.30. Assume that T_2' is empty and note that $\tau^- \in \mathbb{T}_{\mathrm{II.2}}(a_2,a_1,a_0)$ (resp. $\tau^- \in \mathbb{T}_{\mathrm{I}}(a_2,a_1,a_0)$) by 2.30. Since $\tau > \tau^-$ and $T_2' = \mathbf{1}$, we have $T_2 \in \mathcal{T}_{m-1}(\mathcal{L}^t)$ by definition of "$>$" (where $\mathcal{L} = G_{(-1)} \oplus T_1'$); since T_2 has two columns, its rightmost column is therefore $\binom{1}{1}$ and we get $a_2 = 1$ by 2.28. Applying 2.28 to τ^- gives

$a_0 = 1$, so $(a_2, a_1, a_0) = (a_0, a_1, a_2)$ and consequently $\tau^- \in \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2)$ (resp. $\tau^- \in \mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2)$). $\square$

3. Basic affine rulings of type I

The following uses the convention of 2.29:

Lemma 3.1. *Let $\Lambda_0, \Lambda_1, \Lambda_2$ be the standard affine rulings of $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2)$ with respect to a coordinate system (X_0, X_1, X_2) (where Λ_i corresponds to the φ_i of 1.5). Let i, j, k be a permutation of $0, 1, 2$.*

(1) *For some $F \in (\Lambda_i)_*$, $\operatorname{supp} F = R_j$.*
(2) *The discrete part of $(\mathbb{P}, \Lambda_i, F)$ is $(z, \binom{x}{a_j}, \binom{y}{a_k})$, where (x, y, z) is the unique integral solution of $a_i = a_j a_k z - a_j y - a_k x$ with $0 \leq x < a_j$ and $0 \leq y < a_k$.*

Proof. It's enough to prove the case $(i, j, k) = (0, 1, 2)$. Consider $\Lambda = \Lambda_0$. Clearly, there exist $F_1, F_2 \in \Lambda_*$ such that $\operatorname{supp} F_i = R_i$. Consider $(\tilde{\mathbb{P}}, \tilde{\Lambda}) = (\mathbb{P}, \Lambda)^\sim$ and the morphisms $\tilde{\mathbb{P}} \to \hat{\mathbb{P}} \to \mathbb{P}$. Consider the divisor R of $\hat{\mathbb{P}}$ as in Lemma 1.22.

Since $\mathrm{Bs}(\Lambda) = \{q_0\}$, and since the strict transforms of R_1, R_2 on $\tilde{\mathbb{P}}$ belong to distinct members of $\tilde{\Lambda}$, we have:

(i) If $\tilde{\mathbb{P}} \to \hat{\mathbb{P}}$ is the identity map, then some component of Q_0 is a section of $\hat{\Lambda}$;
(ii) if $\tilde{\mathbb{P}} \to \hat{\mathbb{P}}$ is not the identity map, then it is centered at a point of Q_0 and is subdivisional for $R - R_0$.

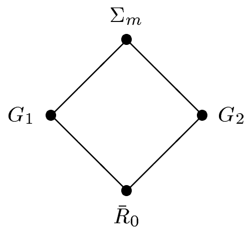
Hence, the divisor $H + \sum_i G_i^\#$ of $\tilde{\mathbb{P}}$ (notation as in 2.22) is a linear chain; it follows that Λ is basic of type I and that the discrete part of $(\mathbb{P}, \Lambda, F_1)$ has the form $(z, \binom{x}{c_1}, \binom{y}{c_2})$, with 2.29 in effect. Moreover, the connected components of the weighted graph

$$\left(\mathcal{G}_{(-z)} \ominus \binom{x}{c_1}\right) \ominus \binom{y}{c_2}$$

have determinants c_1, c_2 and $zc_1c_2 - c_1y - c_2x$, and are respectively equal to Q_1, Q_2 and to a chain which contracts to Q_0. So $c_1 = a_1$, $c_2 = a_2$ and $a_0 = za_1a_2 - a_1y - a_2x$. $\square$

Proof of Lemma 1.22. Let the notation be as in the above proof; we show that $-R$ is a canonical divisor of $\hat{\mathbb{P}}$.

Consider the inverse image $\tilde{R}$ of R in $\tilde{\mathbb{P}}$; let $\pi : \tilde{\mathbb{P}} \to \mathbb{F}_m$ be the contraction of the reducible members of $\tilde{\Lambda}$ to 0-curves and let $\bar{R}$ be the image of $\tilde{R}$ under π (we regard $\tilde{R}$ and $\bar{R}$ as reduced effective divisors—note that they have strong normal crossings). Since R has the shape of a ring, so does $\tilde{R}$ by (i) and (ii); thus $\bar{R}$ has the shape of a ring as well, and its dual graph is:



where G_1, G_2 are distinct members of the standard ruling Λ_m of $\mathbb{F}_m$ and Σ_m is the negative section of Λ_m. Since $\bar{R}_0 \cdot G_1 = 1$, $\bar{R}_0$ is a section of Λ_m, disjoint from Σ_m. It follows that $-\bar{R}$ is a canonical divisor of $\mathbb{F}_m$. Since $\tilde{R}$ is obtained from R (resp. $\bar{R}$) by subdivisional blowing-up, the assertion follows. $\square$

Proposition 3.2. (1) *The basic affine rulings of type I of $\mathbb{P}$ are precisely the standard affine rulings.*

(2) *Suppose that X satisfies (‡) and that the discriminants a_0, a_1, a_2 of its singularities are pairwise relatively prime. If X admits a basic affine ruling of type I, then $X \cong \mathbb{P}(a_0, a_1, a_2)$.*

Proof. Let Λ be a basic affine ruling of type I (of X), let $G \in \Lambda_*$ and let $\tau = (z, \binom{x}{c_1}, \binom{y}{c_2})$ be the discrete part of (X, Λ, G). The connected components of the weighted graph

$$\left(\mathcal{G}_{(-z)} \ominus \binom{x}{c_1}\right) \ominus \binom{y}{c_2}$$

have determinants c_1, c_2 and $zc_1c_2 - c_1y - c_2x$; so these must be equal to a_j, a_k and a_i respectively, for some permutation i, j, k of $0, 1, 2$. Then $\tau = (z, \binom{x}{a_j}, \binom{y}{a_k})$, where (x, y, z) is the unique integral solution of $a_i = a_j a_k z - a_j y - a_k x$ with $0 \le x < a_j$ and $0 \le y < a_k$. By Lemma 3.1, τ is also the discrete part of $(\mathbb{P}(a_0, a_1, a_2), \Lambda_i, F)$, where Λ_i is one of the standard rulings of $\mathbb{P}(a_0, a_1, a_2)$ and F is some element of $(\Lambda_i)_*$. Thus $[X, \Lambda, G]$ and $[\mathbb{P}(a_0, a_1, a_2), \Lambda_i, F]$ have the same image τ under the bijection $\mathbb{S}_0(‡) \to \mathbb{T}_0(‡)$ of 2.24. This proves both assertions of the proposition. $\square$

Remark. Let Λ be an affine ruling of $\mathbb{P}$. Then the morphisms $\mathbb{F}_m \leftarrow \tilde{\mathbb{P}} \to \hat{\mathbb{P}} \to \mathbb{P}$ defined in 2.22 induce a rational map $\mathbb{P} \to \mathbb{F}_m$. Let us make this rational map explicit in the case where $\Lambda = \Lambda_0$ (notation as in 3.1). Recall that the discrete part of $(\mathbb{P}, \Lambda_0, F)$ is $(x_0, \binom{x_1}{a_1}, \binom{x_2}{a_2})$ where (x_0, x_1, x_2) is the unique integral solution of $a_0 = a_1a_2x_0 - a_2x_1 - a_1x_2$ with $0 \le x_1 < a_1$ and $0 \le x_2 < a_2$ (in particular $m = x_0$). Let the notations Σ_m, G_1, G_2, $\bar{R}_0$ have the same meaning as before in this section. The divisors $mG_1 + \Sigma_m$, $mG_2 + \Sigma_m$ and $\bar{R}_0$ are members of the linear system $|mF + \Sigma_m|$ on $\mathbb{F}_m$. It is not difficult to see that the transform of $|mF + \Sigma_m|$ on $\mathbb{P}$ is the linear system $\mathcal{O}(ma_1a_2)$ of curves of degree ma_1a_2. Now $U = X_0X_1^{x_2}X_2^{x_1}$, $V_1 = X_1^{ma_2}$ and $V_2 = X_2^{ma_1}$ define curves in $\mathcal{O}(ma_1a_2)$. Also, $u_1 = U/V_1$ and $v = X_2^{a_1}/X_1^{a_2}$ are rational functions on $\mathbb{P}$ that give equations respectively for $\bar{R}_0$ and G_1 (at their intersection point) in $\mathbb{F}_m \setminus (\Sigma_m \cup G_2) \cong \mathbb{A}^2$.

4. Some results on weighted pairs

Lemma 4.1. *Consider a linear weighted pair $\mathcal{L} = (0, -1, \omega_1, \dots, \omega_n)$, where $n \geq 1$ and $\omega_j \leq -2$ for all j and where the distinguished vertex is the one of weight 0. Let $i \in \{1, \dots, n\}$ and let $x, y \in \mathbb{Z}$ be such that $x + y = \omega_i$ and $x \leq -2$. Then there exists a unique column $\binom{p}{c} \in \mathcal{T}$ such that the weighted pair $\mathcal{L}\binom{p}{c}$ contracts to:*

$$(\omega_1, \dots, \omega_{i-1}, x, 0, y, \omega_{i+1} \dots, \omega_n), \tag{3}$$

where the distinguished vertex is the one of weight 0, $\mathcal{L} \oplus \binom{p}{c} = (\omega_1, \dots, \omega_{i-1}, x)$ and $\mathcal{L} \oplus \binom{p}{c}$ contracts to $(y, \omega_{i+1}, \dots, \omega_n)$.

Remark. We will refer to $\binom{p}{c}$ as "the column determined by $\mathcal{L}$, i, x and y, as in 4.1".

Notation 4.2. The following conventions are used in the proof of Lemma 4.1.

(1) Write $\mathcal{C} = (c_1, \dots, c_m)$ to indicate that $\mathcal{C}$ is the linear chain $\overset{c_1}{\bullet}\!\!-\!\!-\cdots-\!\!-\!\!\overset{c_m}{\bullet}$ ($c_i \in \mathbb{Z}$). To indicate that we have a string of n consecutive -2, say $c_{i+1} = \dots = c_{i+n} = -2$, we may write $\mathcal{C} = (c_1, \dots, c_i, [n], c_{i+n+1}, \dots)$. Note that each admissible chain has a unique representation of the form $([n_0], z_1, [n_1], \dots, z_h, [n_h])$, with $h \geq 0$, $n_i \geq 0$ and $z_i \leq -3$.

(2) Consider a blowing-up $\mathcal{C} \leftarrow \mathcal{C}'$ of weighted pairs, where the underlying weighted graph of $\mathcal{C}$ is $(c_1, \dots, c_m)$. The notation $\mathcal{C} = (c_1, \dots, c_{i-1}, c_i^*, c_{i+1}, \dots, c_m)$, where $*$ is one of the three symbols ℓ, r, s, means:
   (a) The distinguished vertex of $\mathcal{C}$ is the one of weight c_i.
   (b) If $* = \ell$ (resp. $* = r$, $* = s$) then $\mathcal{C}$ is blown-up at the edge $\overset{c_{i-1}}{\bullet}\!\!-\!\!-\!\!\overset{c_i}{\bullet}$ (resp. the edge $\overset{c_i}{\bullet}\!\!-\!\!-\!\!\overset{c_{i+1}}{\bullet}$, the vertex $\overset{c_i}{\bullet}$). Note that ℓ, r and s remind us of "left", "right" and "sprouting" respectively.

   (When $*$ is not one of ℓ, r, s, but is really just "$*$", we mean only (a).)

(3) Suppose that we blow-up a weighted pair $\mathcal{G}_0$ according to some tableau, thus producing a sequence $\mathcal{G}_0 \leftarrow \dots \leftarrow \mathcal{G}_N$ of blowings-up. Suppose that for some $k < N$ the graph

$$\mathcal{G}_k = (\dots, c_{i-1}, c_i^*, c_{i+1}, \dots) \quad (* \in \{\ell, r, s\})$$

   has a weight $c_j = -1$ (where $j \neq i$), and let $\overline{\mathcal{G}}_k$ be the contraction of $\mathcal{G}_k$ at the vertex of weight c_j. If one of the following holds:
   (a) $|j - i| > 1$;
   (b) $j = i + 1$ and $* \neq r$;
   (c) $j = i - 1$ and $* \neq \ell$,

   we say that the contraction $\mathcal{G}_k \geq \overline{\mathcal{G}}_k$ is "allowed". In that case, the blowings-up $\mathcal{G}_k \leftarrow \dots \leftarrow \mathcal{G}_N$ can be performed on $\overline{\mathcal{G}}_k$, giving $\overline{\mathcal{G}}_k \leftarrow \dots \leftarrow \overline{\mathcal{G}}_N$, and we have a contraction of weighted pairs $\mathcal{G}_N \geq \overline{\mathcal{G}}_N$.

Proof of Lemma 4.1. We use the conventions of 4.2. If $\binom{p}{c}$ exists then

$$\mathcal{L}\binom{p}{c} = (\omega_1, \dots, \omega_{i-1}, x, -1^*, \dots),$$

so 2.12 implies that $\det(\omega_1,\dots,\omega_{i-1},x) = c$ and $\det(\omega_1,\dots,\omega_{i-1}) = c - p'$, where $p' \in \{1,\dots,c-1\}$ is the inverse of p modulo c. Thus $\binom{p}{c}$ is unique, if it exists.

To show that $\binom{p}{c}$ exists, it suffices to construct a sequence $\mathcal{L}_0 \leftarrow \cdots \leftarrow \mathcal{L}_k$ of blowings-up of weighted pairs satisfying (where e_j is the distinguished vertex of $\mathcal{L}_j$):

(i) $\mathcal{L}_0 = \mathcal{L}$;

(ii) $\mathcal{L}_0 \leftarrow \mathcal{L}_1$ is the blowing-up at e_0 and, for each $j > 0$, $\mathcal{L}_j \leftarrow \mathcal{L}_{j+1}$ is a blowing-up at an edge incident to e_j;

(iii) $\mathcal{L}_k$ contracts to (3) in such a way that the following holds: if $\mathcal{A}$ and $\mathcal{B}$ are the branches of $\mathcal{L}_k$ at e_k, where $\mathcal{B}$ contains the vertices of $\mathcal{L}$, then $\mathcal{A} = (\omega_1,\dots,\omega_{i-1},x)$ and $\mathcal{B}$ contracts to $(y,\omega_{i+1},\dots,\omega_n)$.

Consider the natural number $N = N(\mathcal{L},i) = |\{ j < i \mid \omega_j \le -3 \}|$. If $N = 0$ then

$$\mathcal{L} = (0^s, -1, [m], \omega_i, \dots, \omega_n), \tag{4}$$

where $m = i-1 \ge 0$, and this contracts to $((m+1)^s, \omega_i + 1, \dots, \omega_n)$. Performing a blow-up of type "s" followed by $m+1$ blows-up of type "r" gives:

$$([m], -2, -1^\ell, -1, \omega_i + 1, \dots, \omega_n).$$

This contracts to $([m], -2, 0^\ell, \omega_i + 2, \dots, \omega_n)$, which is the desired tree (3) if $x = -2$. If $x < -2$ then performing $-2-x > 0$ blows-up of type "ℓ" gives:

$$([m], x, -1^*, [-3-x], -1, \omega_i + 2, \omega_{i+1}, \dots, \omega_n),$$

which contracts to

$$([m], x, 0^*, \omega_i - x, \omega_{i+1}, \dots, \omega_n).$$

This proves the case $N = 0$.

If $N > 0$ then we may write $\mathcal{L} = (0^s, -1, \omega_1, \dots, \omega_j, [m], \omega_i, \dots, \omega_n)$, where $\omega_j \le -3$ and $m = i-j-1 \ge 0$. Since $N(\mathcal{L},j) = N-1$, there exists (by induction, with $y = -1$) a column $\binom{p_1}{c_1}$ such that $\mathcal{L}\binom{p_1}{c_1}$ contracts to

$$(\omega_1, \dots, \omega_{j-1}, \omega_j + 1, 0^\ell, -1, [m], \omega_i, \dots, \omega_n). \tag{4'}$$

Note how (4') is similar to (4) and let us apply the above argument to (4'). We may contract (4') to

$$(\omega_1, \dots, \omega_{j-1}, \omega_j + 1, (m+1)^\ell, \omega_i + 1, \dots, \omega_n)$$

and perform a blow-up of type "ℓ" followed by $m+1$ blows-up of type "r":

$$(\omega_1, \dots, \omega_j, [m], -2, -1^\ell, -1, \omega_i + 1, \dots, \omega_n).$$

This contracts to $(\omega_1, \dots, \omega_j, [m], -2, 0^\ell, \omega_i + 2, \dots, \omega_n)$, which is the desired tree if $x = -2$. If $x < -2$, perform $-2-x > 0$ blows-up of type "ℓ". □

Definition 4.3. Consider a linear weighted pair $\mathcal{L} = (0, -1, \omega_1, \dots, \omega_n)$, where $n \ge 0$, $\omega_j \le -2$ for all j and where the distinguished vertex is the one of weight 0. We define tableaux $\operatorname{cont}(\mathcal{L}, \nu; x, y) \in \mathcal{T}$ for certain values of $\nu, x, y \in \mathbb{Z}$. The first case is:

$$\operatorname{cont}(\mathcal{L}, 0; x, -1) = \mathbf{1} \text{ (the empty tableau) for all } x \in \mathbb{Z}.$$

Write $\{i \mid 1 \le i \le n \text{ and } \omega_i \le -3\} = \{i_1, \dots, i_h\}$ $(1 \le i_1 < \cdots < i_h \le n)$. Given $(\nu; x, y) \in \mathbb{Z}^3$ satisfying

$$1 \le \nu \le h, \quad x \le -2, \quad y \le -1 \quad \text{and} \quad x + y = \omega_{i_\nu}, \tag{5}$$

let $\binom{p}{c}$ be the unique column determined by $\mathcal{L}$, $i = i_\nu$, x and y as in Lemma 4.1. Then define

$$\operatorname{cont}(\mathcal{L}, \nu; x, y) = \begin{pmatrix} p \\ c \end{pmatrix}.$$

We also define a subset $\operatorname{Cont}(\mathcal{L})$ of $\mathcal{T}$ by

$$\operatorname{Cont}(\mathcal{L}) = \{\mathbf{1}\} \cup \{\operatorname{cont}(\mathcal{L}, \nu; x, y) \mid (\nu; x, y) \text{ satisfies } (5)\}$$

and a map $\operatorname{Cont}(\mathcal{L}) \to \operatorname{Cont}(\mathcal{L}^t)$ $(C \mapsto \tilde{C})$ by:

$$\tilde{C} = \begin{cases} \operatorname{cont}(\mathcal{L}^t, h - \nu; x', -1) \text{ (for suitable } x'), & \text{if } C = \operatorname{cont}(\mathcal{L}, \nu; x, -1); \\ \operatorname{cont}(\mathcal{L}^t, h - \nu + 1; y, x), & \text{if } C = \operatorname{cont}(\mathcal{L}, \nu; x, y) \text{ and } y \le -2. \end{cases}$$

This makes sense because, given $\mathcal{L}$ and $C \in \operatorname{Cont}(\mathcal{L}) \setminus \{\mathbf{1}\}$, there is a unique triple $(\nu; x, y)$ satisfying $\operatorname{cont}(\mathcal{L}, \nu; x, y) = C$. Note that $\mathbf{1} \mapsto \operatorname{cont}(\mathcal{L}^t, h; x', -1)$ (for suitable x') and $\operatorname{cont}(\mathcal{L}, h; x, -1) \mapsto \mathbf{1}$.

We call $\tilde{C}$ the *$\mathcal{L}$-dual of C*. It is easily verified that $\operatorname{Cont}(\mathcal{L}) \to \operatorname{Cont}(\mathcal{L}^t)$ is bijective and that its inverse is $C \mapsto \mathcal{L}^t$-dual of C.

Lemma 4.4. *Consider a linear weighted pair $\mathcal{L} = (0, -m, \omega_1, \dots, \omega_n)$, where $m \in \mathbb{Z}$, $n \ge 0$, $\omega_j \le -2$ and where the distinguished vertex is the one of weight 0. Let $T = \left(\begin{smallmatrix} p & \cdots \\ c & \cdots \end{smallmatrix}\right)$ be a tableau with at least two columns and such that $\binom{p}{c} \neq \binom{1}{1}$. Suppose that the weighted graph $\Gamma = \mathcal{L} \oplus T$ contracts to an admissible chain $\mathcal{A}$ satisfying $|\mathcal{A}| \le |\mathcal{L}|$. Then $m = 1$ and one of the following holds:*

1. *$\binom{p}{c} \in \mathcal{T}_k(\mathcal{L})$, for some $k > 0$;*
2. *$\binom{p}{c} \in \operatorname{Cont}(\mathcal{L})$.*

Proof. Consider the sequence of blowing-ups of linear chains

$$\mathcal{L} = \mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \cdots \leftarrow \mathcal{G}_N = \mathcal{L}\begin{pmatrix} p \\ c \end{pmatrix}$$

produced by blowing-up $\mathcal{L}$ according to $\binom{p}{c}$. Note that $|\Gamma| \ge |\mathcal{G}_N| > |\mathcal{L}| \ge |\mathcal{A}|$, so Γ contains a vertex of weight -1. Since $\binom{p}{c} \neq \binom{1}{1}$, this implies that $m = 1$. Using the conventions of 4.2, we may write

$$\mathcal{L} = \mathcal{G}_0 = (0^s, -1, [n_0], z_1, [n_1], \dots, z_h, [n_h]), \tag{6}$$

where $h \ge 0$, $z_j \le -3$ and $n_j \ge 0$; also, it is allowed (4.2.3) to contract (6) to:

$$((n_0 + 1)^s, z_1 + 1, [n_1], \dots). \tag{7}$$

Since Γ contracts to an admissible chain, the number n_0+1 must be decreased by blowing-up until it becomes negative, i.e., the next n_0+2 trees must be:

$$(-1^r, n_0, z_1+1, [n_1], \dots), \dots, ([n_0], -2, -1^*, -1, z_1+1, [n_1], \dots), \tag{8}$$

where $* \in \{\ell, r, s\}$. Since the last chain in (8) contains $|\mathcal{L}|+1$ vertices, the condition $|\mathcal{A}| \le |\mathcal{L}|$ implies that $* \neq r$. Then we may contract the last chain to

$$([n_0], -2, 0^*, z_1+2, [n_1], \dots) \quad (\text{with } * \in \{\ell, s\}). \tag{9}$$

In the special case where $h=0$, the chains (7) and (9) are simply $((n_0+1)^s)$ and $([n_0], -2, 0^*)$ (with $* \in \{\ell, s\}$) respectively, and the latter implies that $\mathcal{L}\binom{p}{c}$ shrinks to $([n_0], x, 0^*)$ for some $x \le -2$; let $k = -1-x > 0$ then $\mathcal{L}\binom{p}{c}\binom{1}{1}^k \approx \mathcal{L}^t$, so condition (1) holds. So we may assume that $h > 0$. Then, by (9), there exists $j \le N$ such that $\mathcal{G}_j$ contracts to a chain of the form:

$$([n_0], \dots, [n_{i-1}], x, 0^*, y, [n_i], \dots, [n_h]) \quad (\text{with } * \in \{\ell, s\}) \tag{10}$$

where $1 \le i \le h$, $x \le -2$ and $x+y=z_i$. Let us assume that j is maximal with respect to this property. Note that $y<0$, because $* \neq r$ and Γ shrinks to an admissible chain. It suffices to prove:

Claim. $j = N$ or $\mathcal{L}\binom{p}{c} \ge ([n_0], z_1, [n_1], \dots, z_h, [n_h], x, 0^*)$ for some $x \le -2$.

Indeed, if $j=N$ then $\binom{p}{c} = \operatorname{cont}(\mathcal{L}, i; x, y)$ and if $\mathcal{L}\binom{p}{c} \ge ([n_0], z_1, [n_1], \dots, z_h, [n_h], x, 0^*)$ then $\mathcal{L}\binom{p}{c}\binom{1}{1}^k \approx \mathcal{L}^t$, with $k=-1-x>0$.

To prove the claim, we may assume that $j<N$; then $* \neq s$ in (10), so $* = \ell$ and the tree which immediately follows (10) is:

$$([n_0], \dots, [n_{i-1}], x-1, -1^\star, -1, y, [n_i], \dots, [n_h]) \quad (\text{with } \star \in \{\ell, r, s\}). \tag{11}$$

Note that $\star = r$, otherwise it would be allowed to shrink (11) to

$$([n_0], \dots, [n_{i-1}], x-1, 0^\star, y+1, [n_i], \dots, [n_h]) \quad (\text{with } \star \in \{\ell, s\}),$$

contradicting the assumption that j is maximal. Since the chain (11) contains $|\mathcal{L}|+1$ vertices and $\star = r$, it follows that $y=-1$ (and $x = z_i+1$), so (10) is:

$$([n_0], \dots, [n_{i-1}], z_i+1, 0^\ell, -1, [n_i], \dots, [n_h]). \tag{12}$$

Note the similarity between (12) and (6); the above argument applied to (12) shows that one of the following two conditions holds:

(i) $i=h$, in which case (12) contracts to $([n_1], \dots, [n_{h-1}], z_h+1, (n_h+1)^\ell)$; as in the case $h=0$, this implies that condition (1) is satisfied and we are done in this case.

(ii) Some $\mathcal{G}_{j'}$ (with $j'>j$) contracts to

$$([n_0], \dots, [n_i], x', 0^*, y', [n_{i+1}], \dots, [n_h]) \quad (\text{with } * \in \{\ell, s\})$$

where $i+1 \le h$, $x' \le -2$ and $x'+y' = z_{i+1}$. By maximality of j, this is impossible.

This proves the claim and hence the lemma. □

5. Basic affine rulings of type II

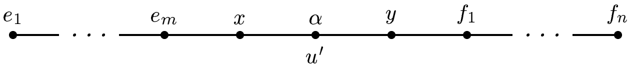
Proposition 5.1. *Let a_0, a_1, a_2 be pairwise relatively prime positive integers and let $\tau \in \mathbb{T}_{\text{II.1}}(a_0, a_1, a_2)$. Then τ is not minimal in $\mathbb{T}(\ddagger)$ and its immediate predecessor belongs to $\mathbb{T}_{\text{I}}(a_0, a_1, a_2)$.*

We have to establish two lemmas before proving this, but let us first give:

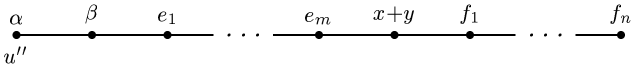
Corollary 5.2. *Let X be a surface of type $[a, b, c]$, where a, b, c are pairwise relatively prime positive integers. Then every basic affine ruling of type II of X reduces to one of type I. In particular, if X admits a basic affine ruling of type II then $X \cong \mathbb{P}(a, b, c)$.*

Proof. The last assertion follows from Proposition 3.2. Let Λ be a basic affine ruling of type II of X. Then, for some $F \in \Lambda_*$, $\tau = \text{disc}(X, \Lambda, F)$ belongs to $\mathbb{T}_{\text{II.1}}(a_0, a_1, a_2)$ for some permutation a_0, a_1, a_2 of a, b, c (see 2.27). By Proposition 5.1, there exists $\tau' \in \mathbb{T}_{\text{I}}(a_0, a_1, a_2)$ such that $\tau > \tau'$. We have $\tau' \in \mathbb{T}(X)$ by 2.25, so there exists an affine ruling Λ' of X and an element F' of Λ'_* such that $\tau' = \text{disc}(X, \Lambda', F')$; note that Λ' is of type I. (In the language of [2], Λ' is obtained from Λ by "reduction".) □

Lemma 5.3. *Let*

$\mathcal{A}'$:


and

$\mathcal{A}''$:


be two linear chains, where $m, n \geq 0$. If $\mathcal{A}'$ and $\mathcal{A}''$ are equivalent to the same admissible chain $\mathcal{A}$, then one of the linear chains $X = (e_1, \dots, e_m, x)$ and $Y = (y, f_1, \dots, f_n)$ shrinks to the empty graph.

Proof. If some e_i or f_i is -1, then we may blow-down $\mathcal{A}'$ and $\mathcal{A}''$ at the corresponding vertex; this produces linear chains $\bar{\mathcal{A}}'$ and $\bar{\mathcal{A}}''$ which still satisfy the hypothesis of the lemma (with possibly different values of m, n, x, y, β) and where the new X and Y are obtained from the old ones by blowing-down. We may therefore assume that $e_i < -1$ and $f_i < -1$ for all i. Since $\mathcal{A}'$ contracts to an admissible chain, $x, y < 0$ and consequently $x + y \leq -2$; since $\mathcal{A}''$ contracts to an admissible chain, α and β are negative and at most one of them is -1. Thus at most one weight in $\mathcal{A}''$ is -1.

Given a linear chain $\mathcal{C}$, let $w(\mathcal{C})$ denote the sum of the weights in $\mathcal{C}$. Note that if we blow-down $\mathcal{C}$ at a vertex v of weight -1 then $w(\mathcal{C})$ increases by $n(v) + 1$ where $n(v) \in \{0, 1, 2\}$ is the number of neighbors of v in $\mathcal{C}$.

Since $\mathcal{A}'$ and $\mathcal{A}''$ have the same number of vertices and contract to the same chain $\mathcal{A}$, there exist two sequences of linear chains:

$$\mathcal{S}' : \ \mathcal{A}' = \mathcal{A}'_0, \dots, \mathcal{A}'_s = \mathcal{A} \quad \text{and} \quad \mathcal{S}'' : \ \mathcal{A}'' = \mathcal{A}''_0, \dots, \mathcal{A}''_s = \mathcal{A}$$

(of the same length s) where each $\mathcal{A}'_i$ (resp. $\mathcal{A}''_i$) is obtained from $\mathcal{A}'_{i-1}$ (resp. $\mathcal{A}''_{i-1}$) by blowing-down one vertex v'_{i-1} (resp. v''_{i-1}). Note that $\mathcal{S}''$ is unique and $\{n(v''_{i-1})\}_{i=1}^{s}$ is nonincreasing; also, we may choose $\mathcal{S}'$ in such a way that $\{n(v'_{i-1})\}_{i=1}^{s}$ is nonincreasing.

Note that $\beta < 0$ implies that $w(\mathcal{A}'') < w(\mathcal{A}')$, so

$$w(\mathcal{A}'_s) - w(\mathcal{A}'_0) < w(\mathcal{A}''_s) - w(\mathcal{A}''_0).$$

So there exists $j \in \{1, \dots, s\}$ such that $n(v'_{j-1}) = 1$ and $n(v''_{j-1}) = 2$. In particular $n(v''_0) = 2$, so $\beta = -1$ and $\alpha < -1$. Note that the vertex u'' is still present in $\mathcal{A}''_j$ and that its weight there is $\alpha + j$, which implies that $\alpha + j < 0$. Consequently, u' is still present in $\mathcal{A}'_j$; since $n(v'_{j-1}) = 1$, this implies that one of X, Y contracts to the empty graph. □

Lemma 5.4. *Let $\mathcal{L} = (0, -m, \omega_1, \dots, \omega_n)$ be a linear weighted pair such that $m \geq 1$, $n \geq 0$, $\omega_i \leq -2$ and the distinguished vertex is the one of weight 0. Consider a tableau $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right)$ where $\binom{p}{c} \neq \binom{1}{1}$ and $a \geq 1$. Suppose that the weighted graph $\Gamma = \mathcal{L} \oplus T$ is equivalent to an admissible chain and that, for some $\beta < 0$, Γ is also equivalent to one of:*

$$\mathcal{C} = (-a, \beta, \omega_1, \dots, \omega_n), \quad \mathcal{C}' = (-a, \beta, \omega_n, \dots, \omega_1).$$

Then $m = 1$ and $\binom{p}{c} \in \mathcal{T}_k(\mathcal{L})$ for some $k \geq 0$.

Proof. Consider the admissible chain $\mathcal{A}$ which is equivalent to Γ. Since $\mathcal{A}$ is equivalent to $\mathcal{C}$ or $\mathcal{C}'$, we have $|\mathcal{A}| \leq |\mathcal{L}|$; so Lemma 4.4 implies that $m = 1$ and that $\binom{p}{c}$ belongs to either $\mathcal{T}_k(\mathcal{L})$ (some $k > 0$) or $\mathrm{Cont}(\mathcal{L})$. So we may assume that $\binom{p}{c} \in \mathrm{Cont}(\mathcal{L})$; then $\mathcal{L}\binom{p}{c}$ contracts to

$$(\omega_1, \dots, \omega_{i-1}, x, 0, y, \omega_{i+1}, \dots, \omega_n)$$

and Γ contracts to

$$(\omega_1, \dots, \omega_{i-1}, x, -a, y, \omega_{i+1}, \dots, \omega_n),$$

where $1 \leq i \leq n$, $x \leq -2$ and $x + y = \omega_i$. Using Lemma 5.3 and again the fact that $\mathcal{A}$ is equivalent to $\mathcal{C}$ or $\mathcal{C}'$, we conclude that one of $(\omega_1, \dots, \omega_{i-1}, x)$, $(y, \omega_{i+1}, \dots, \omega_n)$ shrinks to the empty graph. Since x and all ω_j are strictly less than -1, $(y, \omega_{i+1}, \dots, \omega_n)$ shrinks to the empty graph. Since $\mathcal{L}\binom{p}{c}$ contracts to $(\omega_1, \dots, \omega_{i-1}, x, 0, y, \omega_{i+1}, \dots, \omega_n)$, where the distinguished vertex is the one of weight 0, we conclude that $\mathcal{L}\binom{p}{c}$ contracts to a linear weighted pair, so $\binom{p}{c} \in \mathcal{T}_k(\mathcal{L})$ for some k. □

Proof of Proposition 5.1. Write $\tau = (m, T_1, T_2)$ with $T_1 = \left(\begin{smallmatrix} p \\ a_1 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right)$ (see 2.28). For $i = 0, 1, 2$, define $\mathcal{A}_i$ and G_i as in 2.26; then $\det(G_0) = a_0$ because $\tau \in \mathbb{T}_{\mathrm{II}.1}(a_0, a_1, a_2)$; also, a calculation using 2.12 and 1.11 gives $\det(G_0) = a_2 c_2 \Delta - a_1$, where we define $\Delta = mc_2 a_1 - c_2 p - a_1 p_2$. In particular, a_2 divides $a_0 + a_1$. Let us record

(13) $$T_1 = \left(\begin{smallmatrix} p \\ b \end{smallmatrix}\right), \quad T_2 = \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a \end{smallmatrix}\right) \quad \text{and} \quad a \mid b + c,$$

where we define a, b, c by $(c, b, a) = (a_0, a_1, a_2)$. Note that $m \geq 1$, $c_2 > p_2 \geq 1$, $a \geq 1$ and $b > p \geq 0$ are integers and $\gcd(p_2, c_2) = 1 = \gcd(p, b)$. Also, $\binom{p}{b}$ is subject to 2.29. We may write $G_0 = \mathcal{L} \oplus T_2$, where $\mathcal{L}$ is the weighted pair $\mathcal{G}_{(-m)} \oplus \binom{p}{b}$:

$$\mathcal{L}: \quad \overset{0}{\bullet} \text{———} \overset{-m}{\bullet} \text{———} \underbrace{\overset{\omega_1}{\bullet} \text{———} \underbrace{\cdots \text{———} \overset{\omega_n}{\bullet}}_{p}}_{b} \tag{14}$$

where the leftmost vertex is the distinguished one, and where we used 2.12 for computing the determinants.

Claim. There exists a linear chain $\mathcal{C}^*$ and an integer $\gamma > 0$ satisfying

$$\mathcal{C}^* \text{ contracts to } \mathcal{A}_0 \tag{15}$$

and

$$\mathcal{C}^* = (-a, -\gamma, \omega_1, \ldots, \omega_n) \quad \text{or} \quad \mathcal{C}^* = (-a, -\gamma, \omega_n, \ldots, \omega_1). \tag{16}$$

The proof of the Claim splits into two cases.

Case $b > 1$**.** By 2.12, the subdiscriminants of $G_1 = Z \oplus \binom{p}{b}$ are $b-p$ and $b-p'$, where p' is defined by $\binom{p''}{p'} = \binom{p}{b}^*$ (Z denotes the weighted pair consisting of a single vertex of weight zero). On the other hand, $\tau \in \mathbb{T}_{\mathrm{II}.1}(c, b, a)$ implies that G_1 has subdiscriminants $b'(a, c)$ and $b'(c, a)$, so $\{b-p, b-p'\} = \{b'(a, c), b'(c, a)\}$; for later use, we record:

$$b - b' \in \{p, p'\}, \quad \text{where } b' = b'(a, c). \tag{17}$$

Observe that $b + c - ab' \equiv c - ab' \equiv 0 \pmod b$ by definition of b', and $b + c - ab' \equiv b + c \equiv 0 \pmod a$ by (13); since a, b are relatively prime, $b + c - ab' = (\gamma - 1)ab$ (some $\gamma \in \mathbb{Z}$), so $c = (\gamma - 1)ab + ab' - b$. Since $c \geq 1$, we have $\gamma \geq 1$. Let us define

$$\bar{c} = \gamma b - (b - b')$$

then $\bar{c} > 0$ and we have equations (i) and (ii) in:

$$\begin{aligned}
&\text{(i)} && c = a\bar{c} - b \\
&\text{(ii)} && \bar{c} = \gamma b - (b - b') \\
&\text{(iii)} && b = q_1(b - b') - r_2 \\
& && b - b' = q_2 r_2 - r_3 \\
& && \vdots \\
&\text{(iv)} && r_{s-1} = q_s r_s - r_{s+1}
\end{aligned}$$

where equations (iii)–(iv) are the outer euclidean algorithm on $r_0 = b$ and $r_1 = b - b'$ ($r_i, q_i \in \mathbb{N}$, $r_{i-1} = q_i r_i - r_{i+1}$, $0 \leq r_{i+1} < r_i$, $r_{s+1} = 0$). The integers q_i are now used to define a linear chain

$$\mathcal{C}^* :\quad \underbrace{\overset{-a}{\bullet}\!-\!\!-\!\underbrace{\overset{-\gamma}{\bullet}\!-\!\!-\!\underbrace{\overset{-q_1}{\bullet}\!-\!\!-\!\underbrace{\cdots\!-\!\!-\!\overset{-q_s}{\bullet}}_{b-b'}}_{b}}_{\bar{c}}}_{c} \tag{18}$$

with determinants as indicated. Note that, in $\mathcal{C}^*$, all weights are negative and at most one is -1 ($q_i \geq 2$ for all i and if $a = 1 = \gamma$ then equations (i) and (ii) give $c = b' - b < 0$, a contradiction); this and $\det(\mathcal{C}^*) > 0$ imply that $\mathcal{C}^*$ shrinks to an admissible chain. Since $\mathcal{C}^*$ and $\mathcal{A}_0$ have the same discriminant c and, modulo c, have a subdiscriminant in common (Equation (i) gives $\bar{c} \equiv c'(a,b) \pmod{c}$), 1.15 implies that (15) holds. By (14), (18) and (17), we have that $(-q_1, \dots, -q_s)$ is $(\omega_1, \dots, \omega_n)$ or $(\omega_n, \dots, \omega_1)$, so (16) holds.

Case $b = 1$. Define $\gamma = (b + c)/a$ then, by (13), γ is a positive integer. Let $\mathcal{C}^*$ be the linear chain $(-a, -\gamma)$, then $\det(\mathcal{C}^*) = \gamma a - 1 = \gamma a - b = c > 0$ and it is easy to see that $\mathcal{C}^*$ shrinks to an admissible chain. Since $\mathcal{C}^*$ and $\mathcal{A}_0$ have the same discriminant c and, modulo c, have a subdiscriminant in common ($\gamma \equiv c'(a,b) \pmod{c}$), 1.15 implies that (15) holds. We have $n = 0$ in (14), so (16) holds and the above Claim is proved.

Now (15), (16) and Lemma 5.4 imply that $m = 1$ and that $\binom{p_2}{c_2} \in \mathcal{T}_k(\mathcal{L})$ for some $k \geq 0$. By 2.19, τ is non-minimal in $\mathbb{T}(\ddagger)$ and we may consider its immediate predecessor τ^-. By 2.31, $\tau^- \in \mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2)$. $\square$

6. Basic affine rulings of type III

Lemma 6.1. *Consider a linear weighted pair $\mathcal{L} = (0, -1, \omega_1, \dots, \omega_n)$, where $n \geq 0$, $\omega_j \leq -2$ for all j and where the distinguished vertex is the one of weight 0. Consider an element C of $\mathrm{Cont}(\mathcal{L})$ and its $\mathcal{L}$-dual $\tilde{C} \in \mathrm{Cont}(\mathcal{L}^t)$.*

(1) $\mathcal{L}C \approx \mathcal{L}^t\tilde{C}$ *(equivalence of weighted pairs).*
(2) $\mathcal{L} \mathbin{\bigcirc\!\!\!\!\!\!\uparrow} C \sim \mathcal{L}^t \mathbin{\bigcirc\!\!\!\!\!\!\downarrow} \tilde{C}$ *and* $\mathcal{L} \mathbin{\bigcirc\!\!\!\!\!\!\downarrow} C \sim \mathcal{L}^t \mathbin{\bigcirc\!\!\!\!\!\!\uparrow} \tilde{C}$ *(equivalences of weighted graphs).*
(3) *Write* $C = \binom{p}{c}$ *and* $\tilde{C} = \binom{\tilde{p}}{\tilde{c}}$*, using the convention of 2.29 if necessary. Then* $c = \det(\mathcal{L} \mathbin{\bigcirc\!\!\!\!\!\!\downarrow} C)$ *and* $\tilde{c} = \det(\mathcal{L} \mathbin{\bigcirc\!\!\!\!\!\!\uparrow} C)$*.*

Proof. We prove assertions (1) and (2) simultaneously. Let $i_1 < \cdots < i_h$ be as in 4.3 and write $z_j = \omega_{i_j}$ for $j = 1, \dots, h$. Then

$$\mathcal{L} = (0, -1, [n_0], z_1, [n_1], \dots, z_h, [n_h]) \quad \text{and} \quad \mathcal{L}^t = (0, -1, [n_h], z_h, \dots, z_1, [n_0])$$

for some integers $n_j \geq 0$. If $h = 0$ then $\mathcal{L} = \mathcal{L}^t$ and $C = \mathbf{1} = \tilde{C}$, so (1) is trivial in this case; also, $\mathcal{L} \mathbin{\bigcirc\!\!\!\!\!\!\uparrow} C = (-1, [n]) \sim \mathcal{L}^t \mathbin{\bigcirc\!\!\!\!\!\!\downarrow} \tilde{C}$, since $\mathcal{L}^t \mathbin{\bigcirc\!\!\!\!\!\!\downarrow} \tilde{C}$ is the empty graph; similarly, $\mathcal{L}^t \mathbin{\bigcirc\!\!\!\!\!\!\uparrow} \tilde{C} \sim \mathcal{L} \mathbin{\bigcirc\!\!\!\!\!\!\downarrow} C$, so (1) and (2) hold in this case. Assume $h > 0$.

If $C = \operatorname{cont}(\mathcal{L}, \nu; x, -1)$ then

$$\begin{aligned} \mathcal{L}C &\geq ([n_0], \dots, [n_{\nu-1}], z_\nu + 1, 0^*, -1, [n_\nu], z_{\nu+1}, [n_{\nu+1}], \dots, [n_h]) \\ &\geq ([n_0], \dots, [n_{\nu-1}], z_\nu + 1, (n_\nu + 1)^*, z_{\nu+1} + 1, [n_{\nu+1}], \dots, [n_h]); \qquad (19) \end{aligned}$$

since $\tilde{C} = \operatorname{cont}(\mathcal{L}^t, h - \nu; x', -1)$, we also have:

$$\begin{aligned} \mathcal{L}^t\tilde{C} &\geq ([n_h], \dots, [n_{\nu+1}], z_{\nu+1} + 1, 0^*, -1, [n_\nu], z_\nu, [n_{\nu-1}], \dots, [n_0]) \qquad (20) \\ &\geq ([n_h], \dots, [n_{\nu+1}], z_{\nu+1} + 1, (n_\nu + 1)^*, z_\nu + 1, [n_{\nu-1}], \dots, [n_0]). \qquad (21) \end{aligned}$$

Since the weighted pairs (19) and (21) are the same, $\mathcal{L}C \approx \mathcal{L}^t\tilde{C}$. This also shows that

$$\mathcal{L} \mathbin{\textcircled{$\uparrow$}} C \geq (-1, [n_\nu], z_{\nu+1}, [n_{\nu+1}], \dots, [n_h]) \geq (z_{\nu+1} + 1, [n_{\nu+1}], \dots, [n_h]) = \mathcal{L}^t \mathbin{\textcircled{$\downarrow$}} \tilde{C}$$

and

$$\mathcal{L}^t \mathbin{\textcircled{$\uparrow$}} \tilde{C} \geq (-1, [n_\nu], z_\nu, [n_{\nu-1}], \dots, [n_0]) \geq (z_\nu + 1, [n_{\nu-1}], \dots, [n_0]) = \mathcal{L} \mathbin{\textcircled{$\downarrow$}} C,$$

so (2) holds as well.

If $C = \operatorname{cont}(\mathcal{L}, \nu; x, y)$ with $y \leq -2$, then

$$\mathcal{L}C \geq ([n_0], \dots, [n_{\nu-1}], x, 0^*, y, [n_\nu], z_{\nu+1}, [n_{\nu+1}], \dots, [n_h]);$$

since $\tilde{C} = \operatorname{cont}(\mathcal{L}^t, h - \nu + 1; y, x)$, we also have:

$$\mathcal{L}^t\tilde{C} \geq ([n_h], \dots, [n_\nu], y, 0^*, x, [n_{\nu-1}], z_{\nu-1}, \dots, [n_0]).$$

So we have $\mathcal{L}C \approx \mathcal{L}^t\tilde{C}$,

$$\mathcal{L} \mathbin{\textcircled{$\uparrow$}} C \geq (y, [n_\nu], z_{\nu+1}, [n_{\nu+1}], \dots, [n_h]) = \mathcal{L}^t \mathbin{\textcircled{$\downarrow$}} \tilde{C}$$

and

$$\mathcal{L}^t \mathbin{\textcircled{$\uparrow$}} \tilde{C} \geq (x, [n_{\nu-1}], z_{\nu-1}, \dots, [n_0]) = \mathcal{L} \mathbin{\textcircled{$\downarrow$}} C,$$

so (1) and (2) hold in all cases.

We already know that $c = \det(\mathcal{L} \mathbin{\textcircled{$\downarrow$}} C)$: this follows from 2.12 and was observed at the beginning of the proof of Lemma 4.1 ($\det(\omega_1, \dots, \omega_{i-1}, x) = c$). Applying this fact to $\mathcal{L}^t$ gives $\tilde{c} = \det(\mathcal{L}^t \mathbin{\textcircled{$\downarrow$}} \tilde{C})$, and this is equal to $\det(\mathcal{L} \mathbin{\textcircled{$\uparrow$}} C)$ by part (2). $\square$

Lemma 6.2. *Let $\tau = (m, T_1, T_2)$ be a minimal element of $\mathbb{T}(\ddagger)$, where $T_i = \left(\begin{smallmatrix} p_i & 1 \\ c_i & a_i \end{smallmatrix}\right) \in \mathcal{T}$ $(i = 1, 2)$. Suppose that the weighted graph $(\mathcal{G}_{(-m)} \mathbin{\textcircled{$\uparrow$}} T_1) \mathbin{\textcircled{$\uparrow$}} T_2$ shrinks to a graph with at most $|\mathcal{G}_{(-m)} \mathbin{\textcircled{$\uparrow$}} T_1|$ vertices. Then $m = 1$ and if we write $\mathcal{L} = \mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} T_1$ then:*

(1) $\binom{p_2}{c_2} \in \operatorname{Cont}(\mathcal{L})$ *and its $\mathcal{L}$-dual is not the empty tableau.*

From now-on, let $\binom{\tilde{p}_2}{\tilde{c}_2} \in \operatorname{Cont}(\mathcal{L}^t)$ denote the $\mathcal{L}$-dual of $\binom{p_2}{c_2}$, define $\tilde{T}_2 = \left(\begin{smallmatrix} \tilde{p}_2 & 1 \\ \tilde{c}_2 & a_2 \end{smallmatrix}\right) \in \mathcal{T}$ and $\tilde{\tau} = (1, \check{T}_1, \tilde{T}_2)$. Then:

(2) *$\tau \equiv \tilde{\tau}$ and $\tilde{\tau}$ is a minimal element of $\mathbb{T}(\ddagger)$.*
(3) $(\mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} T_1) \mathbin{\textcircled{$\uparrow$}} T_2 \sim (\mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} \check{T}_1) \mathbin{\textcircled{$\uparrow$}} \tilde{T}_2$.
(4) *$c_2 + \tilde{c}_2 = a_1 c_1 \Delta(\tau)$, where $\Delta(\tau) = m c_1 c_2 - c_1 p_2 - c_2 p_1 = c_1 c_2 - c_1 p_2 - c_2 p_1$.*
(5) $\tilde{p}_2 = -c_2 + p_2 + a_1 p_1 \Delta(\tau)$.

(6) $\Delta(\tau) = \Delta(\tilde{\tau})$.

(7) *If* $\tau \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$, *for some pairwise relatively prime positive integers* a_0, a_1, a_2, *then* $\tilde{\tau} \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$.

Proof. Since $\tau \in \mathbb{T}(\ddagger)$, the intersection matrix of $\Gamma = \mathcal{L} \,\textcircled{\scriptsize$\uparrow$}\, T_2$ is negative definite; thus Γ contracts to an admissible chain $\mathcal{A}$, and $|\mathcal{A}| \leq |\mathcal{L}|$ by the assumption. By Lemma 4.4, $m = 1$ and $\binom{p_2}{c_2}$ belongs to either $\mathcal{T}_k(\mathcal{L})$ (for some $k > 0$) or $\mathrm{Cont}(\mathcal{L})$. By 2.19 and minimality of τ, we have in fact $\binom{p_2}{c_2} \notin \mathcal{T}_k(\mathcal{L})$ (for all $k \in \mathbb{N}$), so $\binom{p_2}{c_2} \in \mathrm{Cont}(\mathcal{L})$. If the $\mathcal{L}$-dual of $\binom{p_2}{c_2}$ is empty then $\mathcal{L}\binom{p_2}{c_2} \approx \mathcal{L}^t$ by Lemma 6.1, so $\mathcal{L}\binom{p_2}{c_2}$ contracts to a linear weighted pair, so $\binom{p_2}{c_2} \in \mathcal{T}_k(\mathcal{L})$ for some $k \in \mathbb{N}$ (by 2.15) and this contradicts an earlier observation. So assertion (1) holds.

If $\tilde{\tau}$ is non-minimal then (2.19) $\binom{\tilde{p}_2}{\tilde{c}_2} \in \mathcal{T}_k(\mathcal{L}^t)$ for some k, so (2.15) $\mathcal{L}^t\binom{\tilde{p}_2}{\tilde{c}_2}$ contracts to a linear weighted pair, so (6.1) $\mathcal{L}\binom{p_2}{c_2}$ has the same property, so (2.15) $\binom{p_2}{c_2} \in \mathcal{T}_k(\mathcal{L})$ for some k, a contradiction. Hence, $\tilde{\tau}$ is minimal. Lemma 6.1 implies

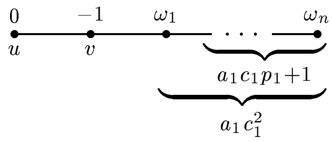
$$(\mathcal{G}_{(-1)} \,\textcircled{\scriptsize$\uparrow$}\, T_1)\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} = \mathcal{L}\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} \approx \mathcal{L}^t\begin{pmatrix} \tilde{p}_2 \\ \tilde{c}_2 \end{pmatrix} = (\mathcal{G}_{(-1)} \,\textcircled{\scriptsize$\uparrow$}\, \check{T}_1)\begin{pmatrix} \tilde{p}_2 \\ \tilde{c}_2 \end{pmatrix} \tag{22}$$

and $(\mathcal{G}_{(-1)} \ominus T_1)\binom{p_2}{c_2} \approx (\mathcal{G}_{(-1)} \ominus \check{T}_1)\binom{\tilde{p}_2}{\tilde{c}_2}$ easily follows; "multiplying" both sides by $\binom{1}{a_2}$ gives $(\mathcal{G}_{(-1)} \ominus T_1)T_2 \approx (\mathcal{G}_{(-1)} \ominus \check{T}_1)\tilde{T}_2$, i.e., assertion (2) holds.

If $\mathcal{P} \approx \mathcal{P}'$ are equivalent weighted pairs and T is a tableau, then $\mathcal{P} \,\textcircled{\scriptsize$\uparrow$}\, T \sim \mathcal{P}' \,\textcircled{\scriptsize$\uparrow$}\, T$. Applying this to (22) (with $T = \binom{1}{a_2}$) gives assertion (3).

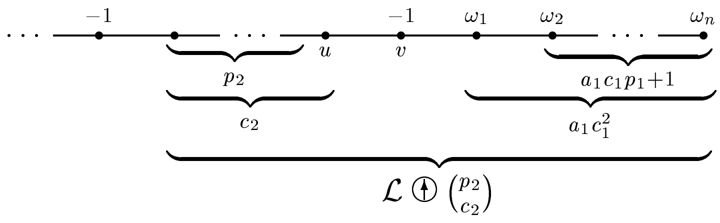
To prove assertion (4), note that $\mathcal{L} = \mathcal{G}_{(-1)} \,\textcircled{\scriptsize$\uparrow$}\, T_1$ is as follows:

$\mathcal{L}$:



and Lemma 2.12 gives:

$\mathcal{L}\binom{p_2}{c_2}$:



We have $\tilde{c}_2 = \det(\mathcal{L} \,\textcircled{\scriptsize$\uparrow$}\, \binom{p_2}{c_2})$ by Lemma 6.1, so 1.11 gives

$$\tilde{c}_2 = c_2 a_1 c_1^2 - c_2(a_1 c_1 p_1 + 1) - p_2 a_1 c_1^2 = -c_2 + a_1 c_1 (c_1 c_2 - c_1 p_2 - c_2 p_1)$$

and assertion (4) holds.

Observe that $\tilde{\tau}$ satisfies the hypothesis of the Lemma and that assertion (4) gives $\tilde{c}_2 + \tilde{\tilde{c}}_2 = a_1 c_1 \Delta(\tilde{\tau})$; since $\tilde{\tilde{c}}_2 = c_2$, we obtain $a_1 c_1 \Delta(\tau) = a_1 c_1 \Delta(\tilde{\tau})$, so assertion (6) holds. Then (6) gives:

$$\begin{aligned}c_1c_2 - c_1p_2 - c_2p_1 &= c_1\tilde{c}_2 - c_1\tilde{p}_2 - \tilde{c}_2(c_1 - p_1)\\ &= p_1\tilde{c}_2 - c_1\tilde{p}_2\\ &= p_1(-c_2 + a_1c_1\Delta(\tau)) - c_1\tilde{p}_2,\end{aligned}$$

so $c_1c_2 - c_1p_2 = p_1a_1c_1\Delta(\tau) - c_1\tilde{p}_2$ and (5) follows from this. In view of 2.30, (7) follows from (2) and (3). □

THE SET $\mathcal{E}$

We will now define a subset $\mathcal{E}$ of $\mathbb{T}(\ddagger)$ and show that its elements can be constructed from those which are not minimal in $(\mathbb{T}(\ddagger), <)$.

Definition 6.3. Let $\mathcal{E}$ be the set of triples $\tau = (m, T_1, T_2) \in \mathbb{T}(\ddagger)$ satisfying:

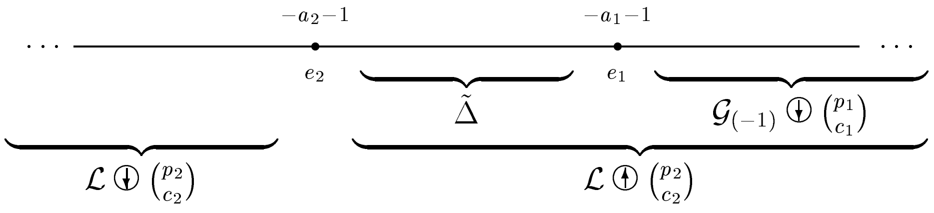
(1) For each $i = 1, 2$, T_i satisfies condition 2.16.3: $T_i = \left(\begin{smallmatrix} p_i & 1 \\ c_i & a_i \end{smallmatrix}\right)$;
(2) the weighted graph $(\mathcal{G}_{(-m)} \oplus T_1) \oplus T_2$ shrinks to an admissible chain containing at most four vertices;
(3) $\Delta(\tau) \neq 1$ or $\min(a_1, a_2) \neq 1$, where $\Delta(\tau) = mc_1c_2 - c_1p_2 - p_1c_2$.

6.4. Let $\tau = (m, T_1, T_2) \in \mathcal{E}$.

(1) $m = 1$, because $(\mathcal{G}_{(-m)} \oplus T_1) \oplus T_2$ contains at least 7 vertices and hence must contain a vertex of weight -1.
(2) If τ is minimal in $\mathbb{T}(\ddagger)$ then τ satisfies the hypothesis of 6.2. In particular, $\tilde{\tau}$ is defined and minimal, and we also have $\tilde{\tau} \in \mathcal{E}$ by parts (3) and (6) of 6.2.

6.5. Let $\tau = (1, T_1, T_2) \in \mathcal{E}$, with notation $T_i = \left(\begin{smallmatrix} p_i & 1 \\ c_i & a_i \end{smallmatrix}\right)$ as before. For $i = 1, 2$, consider the vertex e_i of $\Gamma = (\mathcal{G}_{(-1)} \oplus T_1) \oplus T_2$ which is the last vertex created by the blowing-up according to $\binom{p_i}{c_i}$:

(23) Γ:



where $\mathcal{L} = \mathcal{G}_{(-1)} \oplus T_1$, $\tilde{\Delta} = (\mathcal{G}_{(-1)} \oplus \binom{p_1}{c_1}) \oplus \binom{p_2}{c_2}$ and $\det(\tilde{\Delta}) = \Delta(\tau)$.

We claim that at least one of e_1, e_2 disappears in the shrinking process which transforms Γ into an admissible chain $\mathcal{A}$ such that $|\mathcal{A}| \leq 4$. Indeed, the subtrees $B_1 = \mathcal{G}_{(-1)} \oplus \binom{p_1}{c_1}$ and $B_2 = \mathcal{L} \oplus \binom{p_2}{c_2}$ are nonempty (because $\binom{p_i}{c_i} \neq \binom{1}{1}$) and at least one of them contains more than one vertex (otherwise $p_i = c_i - 1$ for each $i = 1, 2$, so $\Delta(\tau) = c_1c_2 - c_1(c_2 - 1) - (c_1 - 1)c_2 \leq 0$, which is absurd); since the shrinking is initiated in $\tilde{\Delta}$, if no e_i disappears then $|\mathcal{A}| \geq |B_1| + |B_2| + 2 \geq 5$, a contradiction. Note, also, that the shrinking process is unique, i.e., the order in which the vertices disappear is well-defined. This allows us to give:

Definition 6.6. We denote by $\mathcal{E}^+$ the set of $\tau \in \mathcal{E}$ for which e_1 disappears before e_2 (or e_1 disappears but e_2 does not). Given $\tau = (1, T_1, T_2) \in \mathcal{E}$, let $\tau^\times = (1, T_2, T_1)$. Then $\tau^\times \in \mathcal{E}$ and exactly one of τ, $\tau^\times$ is in $\mathcal{E}^+$.

Lemma and definition 6.7. *Let $\tau = (1, T_1, T_2) \in \mathcal{E}$, with notation $T_i = \left(\begin{smallmatrix} p_i & 1 \\ c_i & a_i \end{smallmatrix}\right)$.*

(1) *If τ is non-minimal in $(\mathbb{T}(\ddagger), <)$, then $\tau \in \mathcal{E}^+$.*
(2) *$\tau \in \mathcal{E}^+$ if and only if $c_1 < c_2$.*

Given $\tau \in \mathcal{E}$ minimal in $(\mathbb{T}(\ddagger), <)$, define $\tau^* = (\tilde{\tau})^\times$. By 6.4, τ^* is defined and belongs to $\mathcal{E}$.

(3) *If $\tau \in \mathcal{E}$ is minimal in $(\mathbb{T}(\ddagger), <)$ then $\tau \in \mathcal{E}^+ \iff \tau^* \in \mathcal{E}^+$. Moreover, if $\tau \in \mathcal{E}^+$ then $c_1^* < c_2^* = c_1 < c_2$, where we write $\tau^* = \Big(1, \left(\begin{smallmatrix} p_1^* & 1 \\ c_1^* & a_1^* \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2^* & 1 \\ c_2^* & a_2^* \end{smallmatrix}\right)\Big)$.*
(4) *If $\tau \in \mathcal{E}$ is minimal in $(\mathbb{T}(\ddagger), <)$, and if $\tau \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ for some pairwise relatively prime positive integers a_0, a_1, a_2, then $\tau^* \in \mathbb{T}_{\mathrm{III}}(a_0, a_2, a_1)$.*

Given $\tau \in \mathcal{E}^+$, define $\tau^\star = (\tau^\times)^\sim$.

(5) *If $\tau \in \mathcal{E}^+$ then $\tau^\star$ is defined, belongs to $\mathcal{E}^+$ and is minimal in $(\mathbb{T}(\ddagger), <)$.*
(6) *If $\tau \in \mathcal{E}^+$ then $(\tau^\star)^* = \tau$ and, if τ is minimal, $(\tau^*)^\star = \tau$.*

Proof. Let $\mathcal{L} = \mathcal{G}_{(-1)} \textcircled{\dagger} T_1$.

Suppose that τ is non-minimal. Then, by 2.19, $\binom{p_2}{c_2} \in \mathcal{T}_k(\mathcal{L})$ for some $k \in \mathbb{N}$, so (2.15) the weighted pair $\mathcal{L}\binom{p_2}{c_2}$ contracts to a linear weighted pair. Equivalently, the tree $\tilde{\Delta} \cup \{e_1\} \cup (\mathcal{G}_{(-1)} \textcircled{\dagger} \binom{p_1}{c_1})$ is equivalent to the empty graph (see the picture (23) in 6.5). In particular e_1 disappears before e_2, so $\tau \in \mathcal{E}^+$, which proves assertion (1). Let us continue and show that $c_1 < c_2$ in this case. By 5.38 of [2] we have $M(\mathcal{L}) = \left(\begin{smallmatrix} a_1p_1(c_1-p_1)-1 & a_1c_1^2 - a_1c_1p_1 - 1 \\ a_1c_1p_1 - 1 & a_1c_1^2 \end{smallmatrix}\right)$, so $\binom{p_2}{c_2} \in \mathcal{T}_k(\mathcal{L})$ implies:

$$\binom{p_2}{c_2} = M(\mathcal{L})\binom{1}{k} = \binom{a_1p_1(c_1-p_1)-1}{a_1c_1p_1-1} + k\binom{a_1c_1^2 - a_1c_1p_1 - 1}{a_1c_1^2}. \tag{24}$$

Consequently, if $c_1 \geq c_2$ then $k = 0$ and

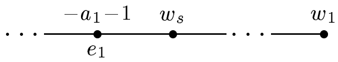
$$\binom{p_2}{c_2} = \binom{a_1p_1(c_1-p_1)-1}{a_1c_1p_1-1}, \tag{25}$$

so $c_1 \geq c_2 = a_1c_1p_1 - 1$, so $(a_1p_1 - 1)c_1 \leq 1$, so $a_1 = 1 = p_1$. Hence, $T_1 = \left(\begin{smallmatrix} 1 & 1 \\ c_1 & 1 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} c_1-2 & 1 \\ c_1-1 & a_2 \end{smallmatrix}\right)$ and it follows that $\Delta(\tau) = c_1(c_1-1) - c_1(c_1-2) - (c_1-1)1 = 1$. So, by assuming that $c_1 \geq c_2$, we derived that $\Delta(\tau) = 1 = a_1$, which contradicts the assumption that $\tau \in \mathcal{E}$. We conclude that $c_1 < c_2$ whenever $\tau \in \mathcal{E}$ is non-minimal in $(\mathbb{T}(\ddagger), <)$.

Assume that $\tau \in \mathcal{E}^+$ is minimal in $(\mathbb{T}(\ddagger), <)$. Then (6.4, 6.2) $\binom{p_2}{c_2} \in \mathrm{Cont}(\mathcal{L})$ and we may consider its $\mathcal{L}$-dual $\binom{\tilde{p}_2}{\tilde{c}_2} \in \mathrm{Cont}(\mathcal{L}^t)$. We claim that

$$\tilde{c}_2 < c_1. \tag{26}$$

For this argument, refer to the picture (23) in 6.5, but let the weights in $\mathcal{G}_{(-1)} \oplus \binom{p_1}{c_1}$ be as follows:



The shrinking of $\Gamma = \mathcal{L} \oplus T_2$ to an admissible chain $\mathcal{A}$ can be broken into two parts, $\Gamma \geq \Gamma' \geq \mathcal{A}$, where e_2 is still present in Γ' and either (i) e_2 has weight -1 in Γ' or (ii) $\Gamma' = \mathcal{A}$.

Since $\binom{p_2}{c_2} \in \mathrm{Cont}(\mathcal{L})$, we also have a contraction of weighted pairs

$$\mathcal{L}\binom{p_2}{c_2} \geq (\dots, [n_{i-1}], x, 0, y, [n_i], \dots) \tag{27}$$

(for some i, x, y) where e_2 is the vertex of weight 0 in the right hand side. Thus the contraction (27) increases the weight of e_2; consequently, the weight of e_2 is increased by the contraction $\Gamma \geq \Gamma'$. It follows that all vertices of $\tilde{\Delta} \cup \{e_1\}$ (see (23)) disappear in the contraction $\Gamma \geq \mathcal{A}$, because we know that e_1 disappears ($\tau \in \mathcal{E}^+$). Thus

$$\mathcal{L} \oplus \binom{p_2}{c_2} \geq (w_i', w_{i-1}, \dots, w_1)$$

for some $i \geq 1$, where $w_i' > w_i$ (note that $\mathcal{L} \oplus \binom{p_2}{c_2}$ cannot contract to the empty graph because τ is assumed to be minimal). Then 6.1 gives

$$\begin{aligned} \tilde{c}_2 = \det(\mathcal{L} \oplus \binom{p_2}{c_2}) &= \det(w_i', w_{i-1}, \dots, w_1) < \det(w_i, w_{i-1}, \dots, w_1) \\ &\leq \det(w_s, w_{s-1}, \dots, w_1) = \det(\mathcal{G}_{(-1)} \oplus \binom{p_1}{c_1}) = c_1, \end{aligned}$$

the last equality by 2.12. This proves (26).

Note that $\tau \in \mathcal{E}$ implies that $a_1\Delta(\tau) \geq 2$ so, by 6.2 and (26),

$$c_2 = a_1c_1\Delta(\tau) - \tilde{c}_2 \geq 2c_1 - \tilde{c}_2 > c_1.$$

This shows that $c_1 < c_2$ whenever $\tau \in \mathcal{E}^+$ is minimal in $\mathbb{T}(\ddagger)$. In view of the first part of the proof, we obtain the "only if" part of assertion (2), i.e., $\tau \in \mathcal{E}^+ \implies c_1 < c_2$. The converse is much easier: If $\tau \in \mathcal{E} \setminus \mathcal{E}^+$, applying the "only if" part of (2) to $\tau^\times \in \mathcal{E}^+$ gives $c_2 < c_1$; thus (given $\tau \in \mathcal{E}$) $c_1 \leq c_2 \implies \tau \in \mathcal{E}^+$ and (2) holds.

If $\tau \in \mathcal{E}$ is minimal in $\mathbb{T}(\ddagger)$ then (6.4) $\tilde{\tau}$ is defined and belongs to $\mathcal{E}$; thus $\tau^* = (\tilde{\tau})^\times$ is defined and belongs to $\mathcal{E}$. Observe that $(c_1^*, c_2^*) = (\tilde{c}_2, c_1)$.

Suppose that $\tau \in \mathcal{E}$ is minimal in $\mathbb{T}(\ddagger)$. If $\tau \in \mathcal{E}^+$ then (26) reads $c_1^* < c_2^*$, so $\tau^* \in \mathcal{E}^+$ by part (2). Conversely, if $\tau^* \in \mathcal{E}^+$ then part (2) gives $c_1^* < c_2^*$, or equivalently $\tilde{c}_2 < c_1$; since $c_2 + \tilde{c}_2 = a_1c_1\Delta(\tau) \geq 2c_1$, we get $c_1 < c_2$, so $\tau \in \mathcal{E}^+$ by part (2). Hence, $\tau \in \mathcal{E}^+ \iff \tau^* \in \mathcal{E}^+$ and (3) is proved.

Assertion (4) follows immediately from 6.2.

If $\tau \in \mathcal{E}^+$ then $\tau^\times \in \mathcal{E} \setminus \mathcal{E}^+$, so $\tau^\times$ is minimal in $\mathbb{T}(\ddagger)$ by part (1), so (6.4) $\tau^\star = (\tau^\times)^\sim$ is defined, minimal and belongs to $\mathcal{E}$. Clearly, $(\tau^\star)^* = \tau \in \mathcal{E}^+$, so $\tau^\star \in \mathcal{E}^+$ by part (3). This shows that (5) holds and (6) is obvious. $\square$

Corollary 6.8. *For each element τ of $\mathcal{E}_{NM} = \{\tau \in \mathcal{E} \mid \tau$ is not minimal in $(\mathbb{T}(\ddagger), <)\}$, define $[\tau] = \{\tau, \tau^\star, (\tau^\star)^\star, \dots\}$. Then $\{[\tau] \mid \tau \in \mathcal{E}_{NM}\}$ is a partition of $\mathcal{E}^+$.*

6.9. Suppose that $\tau \in \mathcal{E}^+$ is minimal in $(\mathbb{T}(\ddagger), <)$ and that, for some surface X satisfying ($\ddagger$), $\tau \in \mathbb{T}(X)$. Then $\tau^* \in \mathbb{T}(X)$. Indeed, $\tilde{\tau} \equiv \tau$ by part 2 of 6.2, so $\tilde{\tau} \in \mathbb{T}(X)$ by 2.25, and consequently $\tau^* = (\tilde{\tau})^\times \in \mathbb{T}(X)$.

Corollary 6.10. *If X is a surface satisfying ($\ddagger$) and such that $\mathbb{T}(X) \cap \mathcal{E} \neq \emptyset$, then X admits a basic affine ruling of type II.*

Proof. Choose $\tau_1 \in \mathbb{T}(X) \cap \mathcal{E}$; replacing τ_1 by $\tau_1^\times$ if necessary, we may arrange that $\tau_1 \in \mathcal{E}^+$. Then (6.8) $\tau_1 \in [\tau]$ for some $\tau \in \mathcal{E}_{NM}$ and, by iterating 6.9, we obtain $\tau \in \mathbb{T}(X)$. Since τ is non-minimal, we may consider $\tau' \in \mathbb{T}(\ddagger)$ such that $\tau > \tau'$; note that T_1' has two columns but T_2' has at most one, where $\tau' = (m, T_1', T_2')$. We have $\tau' \in \mathbb{T}(X)$ by 2.25, so $\tau' = \operatorname{disc}(X, \Lambda, F)$ for some affine ruling Λ of X and some $F \in \Lambda_*$. Since T_1' (resp. T_2') has two (resp. at most one) columns, Λ is basic and of type II. $\square$

Lemma 6.11. *If a_0, a_1, a_2 are pairwise relatively prime positive integers then*

$$\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2) \subset \mathcal{E}.$$

Proof. Let $\tau = (m, T_1, T_2) \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$; by 2.28, we may write $T_i = \left(\begin{smallmatrix} p_i & 1 \\ c_i & a_i \end{smallmatrix}\right)$ $(i = 1, 2)$. Define G_0, G_1, G_2 as in 2.26 and let us also write $\Gamma = G_0$; then $\det(G_i) = a_i$ (all $i = 0, 1, 2$) and a calculation using 2.12 and 1.11 gives $\det(G_0) = \Delta(\tau) a_1 a_2 c_1 c_2 - a_1 c_1^2 - a_2 c_2^2$.

By 2.12, G_1 has discriminant a_1 and subdiscriminants $a_1 - 1$ and $a_1 - 1$. Since $\tau \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$, this implies that $(-1)a_2 \equiv a_0 \pmod{a_1}$, so $a_0 + a_1 + a_2 \equiv 0 \pmod{a_1}$. Similarly, $a_0 + a_1 + a_2 \equiv 0 \pmod{a_2}$. Since $\gcd(a_1, a_2) = 1$, this implies

$$a_0 + a_1 + a_2 = \gamma a_1 a_2, \quad \text{for some } \gamma \in \mathbb{Z},\ \gamma \geq 1. \tag{28}$$

Note that G_0 shrinks to an admissible chain and has discriminant a_0. Since $(\gamma a_2 - 1) a_1 \equiv a_2$ and $(\gamma a_1 - 1) a_2 \equiv a_1 \pmod{a_0}$, the fact that τ belongs to $\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ implies that the subdiscriminants of G_0 are congruent to $\gamma a_1 - 1$ and $\gamma a_2 - 1$ modulo a_0. On the other hand, the linear chain

$$\Gamma' \colon \quad \overset{-a_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-\gamma}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-a_2}{\bullet}$$

shrinks to an admissible chain, has discriminant a_0 and subdiscriminants $\gamma a_1 - 1$ and $\gamma a_2 - 1$. So, by 1.15,

$$G_0 \text{ is equivalent to } \Gamma'. \tag{29}$$

In order to show that $\tau \in \mathcal{E}$, there remains to show that $\Delta(\tau) \neq 1$ or $\min(a_1, a_2) \neq 1$. Assume the contrary: $\Delta(\tau) = 1$ and $\min(a_1, a_2) = 1$. Replacing τ by $\tau^\times$ if necessary, we will assume from now-on:

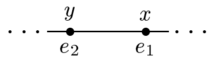
6.11.1. e_1 *disappears before* e_2.

(By 6.5, at least one of e_1, e_2 disappears in the shrinking process which transforms $\Gamma = G_0$ into an admissible chain—note that 6.5 is valid whenever τ satisfies conditions (1) and (2) of 6.3, which is the case here.) In particular we have $m = 1$, since G_0 is not a minimal weighted tree.

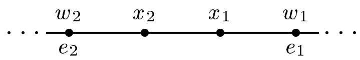
We will obtain a contradiction only after having established several facts. We begin with:

6.11.2. $a_1 = 1$, $a_2 \geq 5$, $c_1 > c_2$ *and the contraction of* $\tilde{\Delta}$ *increases the weight of* e_2 *by more than* 1.

To see this, consider the result of shrinking $\tilde{\Delta}$ in (23) (where $\Gamma = G_0$):

(30)


Since e_1 disappears before e_2, we must have $x = -1$ and $y < x$; thus $-1 - a_2 < y < -1$, so $a_2 \geq 2$ and consequently $a_1 = 1$. Let us be more precise. Since, in (23), $\tilde{\Delta}$ contains at least 3 vertices, we may consider the situation where there remains two vertices in $\tilde{\Delta}$:



(where $w_1 \geq -2$, since $a_1 = 1$). Since this contracts to (30), we must have $(x_1, x_2) = (-1, -2)$ or $(-2, -1)$; in fact we must have $(x_1, x_2) = (-2, -1)$ because the other possibility would give $x \geq 0$ in (30), which is absurd. So the contraction of $\tilde{\Delta}$ increases the weight of e_2 by more than 1. Recall that $a_0 = \Delta(\tau) a_1 a_2 c_1 c_2 - a_1 c_1^2 - a_2 c_2^2$ is strictly positive; with $\Delta(\tau) = 1 = a_1$, this implies that $a_2 c_2 (c_1 - c_2) > c_1^2$, so $c_1 > c_2$ and $a_2 \geq 5$, which proves 6.11.2.

6.11.3. G_0 *is equivalent to a tree with two vertices, one of which has weight* $-a_2$. *Moreover, if* Γ'' *is any tree with two vertices and equivalent to* G_0, *then one of the weights in* Γ'' *is* $-a_2$.

The first assertion is (29) with $a_1 = 1$; the second sentence follows easily from the first one. We also claim:

6.11.4. τ *is minimal in* $\mathbb{T}(\ddagger)$.

Assume the contrary then, arguing as in the proof of 6.7 (see (24) and (25)), we obtain $T_1 = \left(\begin{smallmatrix} 1 & 1 \\ c_1 & 1 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} c_1-2 & 1 \\ c_1-1 & a_2 \end{smallmatrix}\right)$. Then $G_0 = (-c_1 + 1, -1 - a_2, [c_1 - 2], -1, -c_1, [c_1]) \geq (-c_1 + 1, -a_2 + c_1 + 1)$; since $-a_2 + c_1 + 1 \neq -a_2$, 6.11.3 implies that $-c_1 + 1 = -a_2$, so the other weight is $-a_2 + c_1 + 1 = 2 \geq 0$, which is absurd.

Recall that $m = 1$ and let us use the notation:

$$\mathcal{L} = \mathcal{G}_{(-1)} \oplus T_1 = (0, -1, [n_0], z_1, \dots, [n_{h-1}], z_h, [n_h]) \tag{31}$$

where $n_j \geq 0$, $z_j \leq -3$ and where the distinguished vertex is the one of weight 0. Note that the hypothesis of 6.2 is satisfied, so $\binom{p_2}{c_2} \in \mathrm{Cont}(\mathcal{L})$ and $\tilde{\tau}$ is defined. In particular, $\mathrm{Cont}(\mathcal{L})$ contains a nonempty tableau, so $h \geq 1$.

6.11.5. *$h \geq 2$ and, for some $i \in \{1, \dots, h-1\}$,*

$$G_0 = \mathcal{L} \oplus T_2 \geq (\dots, z_{i-1}, [n_{i-1}], z_i + 1, -a_2, -1, [n_i], z_{i+1}, \dots). \tag{32}$$

Moreover, $n_i \geq 2$ and e_1 is either the leftmost or the rightmost vertex in $[n_i]$.

We have $\binom{p_2}{c_2} = \mathrm{cont}(\mathcal{L}, i; x, y)$ for some $i \in \{1, \dots, h\}$ (for suitable x, y); then

$$\mathcal{L}\binom{p_2}{c_2} \geq (\dots, z_{i-1}, [n_{i-1}], x, 0^*, y, [n_i], z_{i+1}, \dots), \tag{33}$$

or equivalently:

$$G_0 = \mathcal{L} \oplus T_2 \geq (\dots, z_{i-1}, [n_{i-1}], x, -a_2, y, [n_i], z_{i+1}, \dots), \tag{34}$$

where e_2 is the vertex of weight $-a_2$. Since the contraction (34) increases the weight of e_2 by only 1, 6.11.2 implies that some vertex of $\tilde{\Delta}$ is still present in the right hand side of (34). It follows that the vertex of weight y belongs to $\tilde{\Delta}$, so $x = z_i + 1$, $y = -1$ and (32) holds. Since e_1 disappears before e_2, e_1 is in $[n_i]$. If $i = h$ then the right hand side of (33) shrinks to a linear weighted pair, which contradicts 6.11.4 (2.15, 2.19); so $i < h$ and consequently $h \geq 2$. Since $c_1 > c_2$, we have in particular $c_1 > 2$; so e_1 has two neighbors in $\mathcal{L}$, one of them has weight -2 and the other has weight strictly less than -2. This proves 6.11.5.

Obseve that $\tilde{\tau}$ is defined and satisfies the hypothesis of the Lemma as well as $\Delta(\tilde{\tau}) = 1 = a_1$. We claim that $\tilde{\tau}$ also satisfies 6.11.1: if not, then $(\tilde{\tau})^\times$ does, so 6.11.2 applied to $(\tilde{\tau})^\times$ gives $\tilde{c}_2 > c_1$, which is not the case because we have $c_2 + \tilde{c}_2 = a_1 c_1 \Delta(\tau) = c_1$, so $c_1 > \tilde{c}_2$. So we may, if we want, replace τ by $\tilde{\tau}$. Note, however, that if (in 6.11.5) e_1 is the leftmost vertex of $[n_i]$, then the contrary claim holds for $\tilde{\tau}$. In other words, we may arrange that:

6.11.6. *e_1 is the rightmost vertex of $[n_i]$.*

Consider the weighted pair Z consisting of a single vertex of weight 0; then we may write $Z\binom{p_1}{c_1}$ in one of the following forms:

(a) $([x_h], y_{h-1}, \dots, y_4, [x_3], y_2, [x_1], -1^*, y_1, [x_2], y_3, [x_4] \dots, [x_{h-1}], y_h)$,
(b) $(y_h, [x_{h-1}], \dots, y_4, [x_3], y_2, [x_1], -1^*, y_1, [x_2], y_3, [x_4] \dots, y_{h-1}, [x_h])$,

where $y_j \leq -3$, $x_j \geq 0$ and $x_h > 0$; e_1 is the vertex of weight -1 and the unique vertex of Z is the leftmost vertex in (a) or (b). Note that, because of 6.11.6, we don't need to consider more cases than (a) and (b) (i.e., cases of the type $(\dots, y_1, -1^*, [x_1], \dots)$);

note, also, that h is odd in case (a) and even in case (b). The fact that (a) (resp. (b)) shrinks to a single vertex of weight 0 gives:

$$x_j + y_j = \begin{cases} -3, & \text{if } 1 < j < h, \\ -2, & \text{if } j = 1 \text{ or } j = h. \end{cases} \tag{35}$$

Note that $z_i = y_2$, $n_i = x_1 + 1$, $z_{i+1} = y_1$, etc., and rewrite (32) as
(36)

$$G_0 \geq \begin{cases} ([x_h], y_{h-1}, \dots, [x_3], y_2 + 1, -a_2, -1, [x_1], -2, y_1, [x_2], \dots, [x_{h-1}], y_h), & \text{or} \\ (y_h, [x_{h-1}], \dots, [x_3], y_2 + 1, -a_2, -1, [x_1], -2, y_1, [x_2], \dots, y_{h-1}, [x_h]), \end{cases}$$

in cases (a) and (b) respectively. Next we show:

6.11.7. *Case (a) is impossible.*

Assume that we are in case (a). By (36),

$$G_0 \geq ([x_h], y_{h-1}, \dots, [x_3], y_2 + 1, -a_2 + x_1 + 2, y_1 + 1, [x_2], \dots, [x_{h-1}], y_h), \tag{37}$$

where the right hand side contains at least 5 vertices ($h \geq 2$ by 6.11.5, so $h \geq 3$ since it is odd; also recall that $x_h > 0$). By 6.11.3, $-a_2 + x_1 + 2 = -1$, so:

$$a_2 = x_1 + 3; \tag{38}$$

together with (37), this gives

$$G_0 \geq ([x_h], y_{h-1}, \dots, [x_3], y_2 + 2, y_1 + 2, [x_2], \dots, [x_{h-1}], y_h), \tag{39}$$

which has at least 4 vertices. So $-1 \in \{y_2 + 2, y_1 + 2\}$. If $y_1 + 2 = -1$ then the right hand side of (39) shrinks to $(\dots, [x_3], y_2 + 2 + x_2 + 1, \dots) = (\dots, [x_3], 0, \dots)$ by (35); since there can't be a nonnegative weight in a tree equivalent to G_0, we conclude that $y_2 + 2 = -1$ and, by (39),

$$G_0 \geq ([x_h], y_{h-1}, \dots, y_4 + 1, x_3 + y_1 + 3, [x_2], \dots, [x_{h-1}], y_h). \tag{40}$$

Note that if $h = 3$ then (40) reads $G_0 \geq (x_3 + y_1 + 3, [x_2], y_3)$. More generally, we claim:

$$G_0 \geq (x_h + y_{h-2} + p, [x_{h-1}], y_h), \quad \text{where } p = \begin{cases} 3, & \text{if } h = 3, \\ 2, & \text{if } h > 3. \end{cases} \tag{41}$$

Indeed, if $h > 3$ then we can continue contracting (40) as long as we have more than 2 vertices. At each stage of the process, the next vertex to disappear is clearly identified and the contraction process inescapably leads to the right hand side of (41), unless contraction stops before that point; since the right hand side of (41) has at least 2 vertices, contraction doesn't stop before that point and (41) holds. Now (41) implies that, if $h > 3$,

$$\begin{aligned} &x_3 + y_1 = -4, && x_2 + y_4 = -3,\\ &x_5 + y_3 = -3, && x_4 + y_6 = -3,\\ &\quad\vdots && \quad\vdots\\ &x_{h-2} + y_{h-4} = -3, && x_{h-3} + y_{h-1} = -3. \end{aligned} \tag{42}$$

(These are obtained by writing down, at each stage of the contraction process, the equation which corresponds to the fact that the next vertex to disappear has weight -1.) Since G_0 contracts to an admissible chain, (41) implies that $x_h + y_{h-2} + p \le -1$, so $x_h + y_{h-2} \le -3$; together with the first column of (42), this gives:

$$x_j + y_{j-2} \le -3, \quad \text{for all odd } j \text{ such that } 3 \le j \le h. \tag{43}$$

(Note that, although the notation in (42) assumes that $h > 3$, (43) is valid when $h = 3$ as well.) We claim that:

$$y_j > -a_2 \text{ for all odd } j \text{ such that } 1 \le j \le h. \tag{44}$$

Indeed, $y_1 = -2 - x_1 > -3 - x_1 = -a_2$, by (35) and (38); if $j > 1$ then $y_j \ge -3 - x_j \ge y_{j-2}$, by (35) and (43), so (44) holds.

We may now obtain a contradiction from 6.11.3, (41) and (44): If $x_h + y_{h-2} + p = -1$ then $G_0 \ge (-1, y_h)$ by (41), so $y_h = -a_2$ by 6.11.3, and this contradicts (44). If $x_h + y_{h-2} + p < -1$, then the right hand side of (41) must be an admissible chain with exactly two vertices ($x_{h-1} = 0$); since $y_h \ne -a_2$ by (44), we have $x_h + y_{h-2} + p = -a_2$ by 6.11.3, but this is absurd because $y_{h-2} > -a_2$ and $x_h + p > 0$. This proves 6.11.7.

6.11.8. *Case (b) is impossible.*

This is very similar to 6.11.7 and we only sketch the argument. Assume that we are in case (b) (so h is even). By (36),

$$G_0 \ge (y_h, [x_{h-1}], \dots, [x_3], y_2 + 1, -a_2 + x_1 + 2, y_1 + 1, [x_2], \dots, y_{h-1}, [x_h]) \tag{37'}$$

and we deduce that $a_2 = x_1 + 3$ and $y_2 + 2 = -1$ (as before); we also find:

$$G_0 \ge (y_{h-1} + p, [x_h]), \quad \text{where} \quad p = \begin{cases} 3, & \text{if } h = 2,\\ 2, & \text{if } h > 2, \end{cases} \tag{41'}$$

and if $h > 2$:

$$\begin{aligned} &x_3 + y_1 = -4, && x_2 + y_4 = -3,\\ &x_5 + y_3 = -3, && x_4 + y_6 = -3,\\ &\quad\vdots && \quad\vdots\\ &x_{h-1} + y_{h-3} = -3, && x_{h-2} + y_h = -3. \end{aligned} \tag{42'}$$

Then (42') implies (43), and (44) follows; together with 6.11.3 and (41'), this gives a contradiction. So 6.11.8 holds and the Lemma is proved. ☐

Corollary 6.12. *Let X be a surface of type $[a, b, c]$ for some pairwise relatively prime integers $a, b, c \geq 1$. Then X is isomorphic to $\mathbb{P}(a, b, c)$.*

Proof. By 2.22.2, X admits a basic affine ruling Λ; if Λ is of type I or II then the assertion follows from 3.2 and 5.2.

Suppose that Λ is of type III, choose $F \in \Lambda_*$ and let $\tau = \mathrm{disc}(X, \Lambda, F)$; by 2.27, $\tau \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ for some permutation a_0, a_1, a_2 of a, b, c. Then 6.11 gives $\tau \in \mathcal{E}$ and, by 6.10, X admits a basic affine ruling of type II. □

Remark. Suppose X satisfies (‡). Then X has at most three singular points (1.8) and X admits a basic affine ruling (2.22.2). If X admits a basic ruling of type III (resp. II), then X has at most one (resp. two) singular points not a rational double point. Hence in case X has three singularities that are not rational double points, 3.2 gives a stronger statement than 6.12, namely:

> *Let X be a surface satisfying (‡). If the discriminants a_0, a_1, a_2 of its singular points are pairwise relatively prime, and if X has three singularities that are not rational double points, then $X = \mathbb{P}(a_0, a_1, a_2)$ and no a_i divides the sum of the other two.*

Corollary 6.13. *Let a_0, a_1, a_2 be pairwise relatively prime positive integers. Then the set $\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{III}}(a_0, a_2, a_1)$ is nonempty if and only if $a_1 a_2 \mid a_0 + a_1 + a_2$. Moreover, $\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{III}}(a_0, a_2, a_1)$ is equal to:*

$$\bigcup_{\tau \in E} \{\tau, \tau^\star, (\tau^\star)^\star, ((\tau^\star)^\star)^\star, \dots\} \cup \{\tau^\times, (\tau^\star)^\times, ((\tau^\star)^\star)^\times, (((\tau^\star)^\star)^\star)^\times, \dots\}$$

where E denotes the set of elements of $\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{III}}(a_0, a_2, a_1)$ which are non-minimal in $\mathbb{T}(‡)$.

Proof. Follows from 6.11 and 6.8. □

7. Explicit description of the set $\mathbb{T}_0(\mathbb{P})$

Let $\mathbb{P} = \mathbb{P}(a, b, c)$, where a, b, c are pairwise relatively prime positive integers. By [2], it is clear that the problem of describing all affine rulings of $\mathbb{P}$ reduces to that of describing the set $\mathbb{T}_0(\mathbb{P})$. Now we have:

Corollary 7.1. *$\mathbb{T}_0(\mathbb{P})$ is the union of the sets $\mathbb{T}_{\mathcal{P}}(a_0, a_1, a_2)$, for all $\mathcal{P} \in \{\mathrm{I}, \mathrm{II.1}, \mathrm{II.2}, \mathrm{III}\}$ and all permutations (a_0, a_1, a_2) of (a, b, c).*

Proof. By 2.27, $\mathbb{T}_0(\mathbb{P}) \subseteq \bigcup \mathbb{T}_{\mathcal{P}}(a_0, a_1, a_2)$. For the reverse inclusion, consider $\tau = (m, T_1, T_2) \in \mathbb{T}_{\mathcal{P}}(a_0, a_1, a_2)$; then $\tau = \mathrm{disc}[X, \Lambda, F]$ for some $[X, \Lambda, F] \in \mathbb{S}_0(‡)$, because $\mathrm{disc} : \mathbb{S}_0(‡) \to \mathbb{T}_0(‡)$ is surjective (2.24). Then (2.23) the resolution graph of X is equivalent to $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$, which is equivalent to $\mathcal{G}_{[a_0,a_1,a_2]} = \mathcal{G}_{[a,b,c]}$ by definition of $\mathbb{T}_{\mathcal{P}}(a_0, a_1, a_2)$. So X is a surface of type $[a, b, c]$ and 6.12 implies that $X = \mathbb{P}$. Consequently, $\tau \in \mathbb{T}_0(\mathbb{P})$. □

So our task is to describe the set $\mathbb{T}_{\mathcal{P}}(a_0, a_1, a_2)$ explicitely, for each permutation (a_0, a_1, a_2) of (a, b, c) and each $\mathcal{P} \in \{\mathrm{I}, \mathrm{II.1}, \mathrm{II.2}, \mathrm{III}\}$. We begin with an observation:

7.2. Given pairwise relatively prime positive integers a_0, a_1, a_2, it is clear that

$$\mathrm{Eq}(a_0, a_1, a_2)\textbf{:} \qquad a_0 = a_1 a_2 x_0 - a_2 x_1 - a_1 x_2$$

has a unique solution $(x_0, x_1, x_2) \in \mathbb{N}^3$ satisfying $0 \leq x_1 < a_1$ and $0 \leq x_2 < a_2$. Then $x_0 > 0$ and for $i = 1, 2$ we have $x_i = 0 \iff a_i = 1$ and $x_i \in \{0, 1\} \iff a_i \mid (a_0 + a_1 + a_2)$. For each $i = 1, 2$, there is a unique x_i' satisfying $x_i x_i' \equiv 1 \pmod{a_i}$ and $0 \leq x_i' < a_i$; and a unique $x_i'' \in \mathbb{Z}$ satisfying $x_i x_i' - x_i'' a_i = 1$.

Proposition 7.3. *Given pairwise relatively prime positive integers a_0, a_1, a_2, the set $\mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2)$ has exactly one element, namely*

$$\left(x_0, \begin{pmatrix} x_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ a_2 \end{pmatrix} \right),$$

where (x_0, x_1, x_2) is the unique solution of $\mathrm{Eq}(a_0, a_1, a_2)$.

Proof. Clear from the proof of 3.2. □

Proposition 7.4. *Given pairwise relatively prime positive integers a_0, a_1, a_2, the set $\mathbb{T}_{\mathrm{II.1}}(a_0, a_1, a_2)$ has at most one element, and is nonempty if and only if $\dfrac{a_0 + a_1 + a_2}{a_2}$ is a natural number strictly greater than 2. Moreover, if $\mathbb{T}_{\mathrm{II.1}}(a_0, a_1, a_2)$ is nonempty then let (x_0, x_1, x_2) be the unique solution to $\mathrm{Eq}(a_0, a_1, a_2)$, let x_1', x_1'' be as in 7.2 and define*

$$\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 - x_1 - x_1' + x_1'' \\ a_1 - x_1 \end{pmatrix} + (x_0 - x_2) \begin{pmatrix} a_1 - x_1' \\ a_1 \end{pmatrix}; \tag{45}$$

then the unique element of $\mathbb{T}_{\mathrm{II.1}}(a_0, a_1, a_2)$ is

$$\left(1, \begin{pmatrix} x_1' \\ a_1 \end{pmatrix}, \begin{pmatrix} p_2 & 1 \\ c_2 & a_2 \end{pmatrix} \right). \tag{46}$$

Remark. Since $\tau \mapsto \tau^{\times}$ is a bijection $\mathbb{T}_{\mathrm{II.1}}(a_0, a_2, a_1) \to \mathbb{T}_{\mathrm{II.2}}(a_0, a_1, a_2)$, a description of $\mathbb{T}_{\mathrm{II.2}}(a_0, a_1, a_2)$ is easily obtained from the above statement.

Proof. Suppose that $\tau = (1, T_1, T_2) \in \mathbb{T}_{\mathrm{II.1}}(a_0, a_1, a_2)$ and write $T_1 = \left(\begin{smallmatrix} p \\ a_1 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right)$ (see 2.28). We saw, at the beginning of the proof of 5.1, that $a_2 \mid a_0 + a_1$; so $(a_0 + a_1 + a_2)/a_2$ is a natural number at least 2 (we will see, below, that it is greater than 2). In particular, we have $x_2 \in \{0, 1\}$ by 7.2.

By 5.1, τ is not minimal and its immediate predecessor τ' belongs to $\mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2)$, so (by 7.3) $\tau' = (x_0, \binom{x_1}{a_1}, \binom{x_2}{a_2})$ where (x_0, x_1, x_2) is the unique solution of $\mathrm{Eq}(a_0, a_1, a_2)$. This implies that $\tau = (1, \binom{x_1'}{a_1}, T\binom{x_2}{a_2})$, where x_1' is defined in 7.2 and $T \in \mathcal{T}_{x_0 - 1}(\mathcal{L}^t)$, with $\mathcal{L} = \mathcal{G}_{(-1)} \textcircled{+} \binom{x_1}{a_1}$. Note that the first column of T must be $\binom{p_2}{c_2}$ and that we may write $T = \binom{p_2}{c_2}\binom{1}{1}^{1 - x_2}$ (recall that $x_2 \in \{0, 1\}$). So $\binom{p_2}{c_2}\binom{1}{1}^{1 - x_2} \in \mathcal{T}_{x_0 - 1}(\mathcal{L}^t)$, which implies $\binom{p_2}{c_2} \in \mathcal{T}_{x_0-x_2}(\mathcal{L}^t)$; thus $\binom{p_2}{c_2}$ is the matrix product $M(\mathcal{L}^t)\binom{1}{x_0-x_2}$, which is the same as $M(\mathcal{L})^t\binom{1}{x_0-x_2}$ by 2.14. By 5.38 of [2] we have $M(\mathcal{L}) = \left(\begin{smallmatrix} a_1-x_1-x_1'+x_1'' & a_1-x_1 \\ a_1-x_1' & a_1 \end{smallmatrix}\right)$, so (45) and (46) hold.

If $(a_0+a_1+a_2)/a_2 = 2$ then $a_0+a_1 = a_2$; feeding this in $\mathrm{Eq}(a_0,a_1,a_2)$ and manipulating gives $x_1 = a_1-1 = x_1'$, $x_1'' = a_2-2$ and $x_0 = x_2$; then (45) gives $p_2 = 0$, which is absurd. Hence, $(a_0+a_1+a_2)/a_2 > 2$.

Conversely, suppose that $(a_0+a_1+a_2)/a_2$ is a natural number greater than 2; in order to show that $\mathbb{T}_{\mathrm{II.1}}(a_0,a_1,a_2)$ is nonempty, consider the unique element $\tau' = (x_0, \binom{x_1}{a_1}, \binom{x_2}{a_2})$ of $\mathbb{T}_{\mathrm{I}}(a_0,a_1,a_2)$, let $\mathcal{L} = \mathcal{G}_{(-1)} \,\textcircled{\small$\uparrow$}\, \binom{x_1}{a_1}$ and define $\binom{p_2}{c_2} = M(\mathcal{L}^t)\binom{1}{x_0-x_2}$. Note that $x_0 > 0$ and $x_2 \in \{0,1\}$ imply $x_0 - x_2 \geq 0$. We claim:

$$\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} \in \mathcal{T}_{x_0-x_2}(\mathcal{L}^t). \tag{47}$$

If this is the case then it is easy to see that we may construct an element τ of $\mathbb{T}_{\mathrm{II.1}}(a_0,a_1,a_2)$ by reading the above argument backward. Observe that, by definition of $\mathcal{T}_{x_0-x_2}(\mathcal{L}^t)$, if (47) is false then we must have $x_0 - x_2 = 0$, so (i) $x_0 = 1 = x_2$; and (ii) $\mathcal{L}$ must satisfy the condition "$w_i = -2$ for all i" (see 2.13). Now (i) and $\mathrm{Eq}(a_0,a_1,a_2)$ give $a_0+a_1 = a_2(a_1-x_1)$ and (ii) gives $x_1 = a_1-1$, so $a_0+a_1 = a_2$, a contradiction. So (47) holds and the proof is complete. □

Proposition 7.5. *Let a_0, a_1, a_2 be pairwise relatively prime positive integers. Then at most one element of $\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)$ is non-minimal in $\mathbb{T}(\ddagger)$ and such an element exists if and only if $a_1a_2 \mid a_0+a_1+a_2$ and $a_0 > a_1 - a_2$. Moreover, if such an element τ exists then $\tau = (1, \left(\begin{smallmatrix} p_1 & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right))$, where*

$$\begin{pmatrix} p_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ a_2 - x_2 \end{pmatrix} + (x_0 - x_1)\begin{pmatrix} x_2 \\ a_2 \end{pmatrix}, \tag{48}$$

$$\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1p_1(c_1-p_1)-1 \\ a_1c_1p_1 - 1 \end{pmatrix} + (1-x_2)\begin{pmatrix} a_1c_1(c_1-p_1)-1 \\ a_1c_1^2 \end{pmatrix} \tag{49}$$

and where (x_0,x_1,x_2) is the solution to $\mathrm{Eq}(a_0,a_1,a_2)$.

Proof. Suppose that $\tau \in \mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)$ is a non-minimal element of $\mathbb{T}(\ddagger)$. By 2.31, the immediate predecessor τ_1 of τ belongs to $\mathbb{T}_{\mathrm{II.2}}(a_0,a_1,a_2)$, so $\tau_1^\times \in \mathbb{T}_{\mathrm{II.1}}(a_0,a_2,a_1)$. Now 7.4 describes $\tau_1^\times$ as follows: Let (x_0,x_2,x_1) be the unique solution to $\mathrm{Eq}(a_0,a_2,a_1)$ (equivalently, (x_0,x_1,x_2) solves $\mathrm{Eq}(a_0,a_1,a_2)$) and define x_2', x_2'' as in 7.2. By (28), a_1a_2 divides $a_0+a_1+a_2$; thus $x_1, x_2 \in \{0,1\}$, $x_2' = x_2$ and $x_2'' = x_2 - 1$. (Note, also, that $\mathbb{T}_{\mathrm{II.1}}(a_0,a_2,a_1) \neq \emptyset$ implies that $(a_0+a_1+a_2)/a_1 > 2$, so $a_0 > a_1 - a_2$.) Define

$$\begin{aligned} \begin{pmatrix} p_1' \\ c_1 \end{pmatrix} &= \begin{pmatrix} a_2 - x_2 - x_2' + x_2'' \\ a_2 - x_2 \end{pmatrix} + (x_0 - x_1)\begin{pmatrix} a_2 - x_2' \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} a_2 - x_2 - 1 \\ a_2 - x_2 \end{pmatrix} + (x_0 - x_1)\begin{pmatrix} a_2 - x_2 \\ a_2 \end{pmatrix}, \end{aligned} \tag{50}$$

then (7.4) $\tau_1^\times = \left(1, \binom{x_2'}{a_2}, \left(\begin{smallmatrix} p_1' & 1 \\ c_1 & a_1 \end{smallmatrix}\right)\right) = \left(1, \binom{x_2}{a_2}, \left(\begin{smallmatrix} p_1' & 1 \\ c_1 & a_1 \end{smallmatrix}\right)\right)$, so $\tau_1 = \left(1, \left(\begin{smallmatrix} p_1' & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \binom{x_2}{a_2}\right)$.

Let $\mathcal{L} = \mathcal{G}_{(-1)} \circledast \left(\begin{smallmatrix} p_1' & 1 \\ c_1 & a_1 \end{smallmatrix}\right)$. Then $\tau = (1, \left(\begin{smallmatrix} c_1 - p_1' & 1 \\ c_1 & a_1 \end{smallmatrix}\right), T\binom{x_2}{a_2})$, where $T \in \mathcal{T}_0(\mathcal{L}^t)$. Let $\binom{p_2}{c_2}$ be the first column of T, then $\binom{p_2}{c_2}\binom{1}{1}^{1-x_2} = T \in \mathcal{T}_0(\mathcal{L}^t)$, so $\binom{p_2}{c_2} \in \mathcal{T}_{1-x_2}(\mathcal{L}^t)$, so $\binom{p_2}{c_2} = M(\mathcal{L}^t)\binom{1}{1-x_2} = M(\mathcal{L})^t\binom{1}{1-x_2}$. Now 5.38 of [2] gives $M(\mathcal{L}) = \left(\begin{smallmatrix} a_1 p_1'(c_1 - p_1') - 1 & a_1 c_1^2 - a_1 c_1 p_1' - 1 \\ a_1 c_1 p_1' - 1 & a_1 c_1^2 \end{smallmatrix}\right)$, so

$$\binom{p_2}{c_2} = \begin{pmatrix} a_1 p_1'(c_1 - p_1') - 1 \\ a_1 c_1^2 - a_1 c_1 p_1' - 1 \end{pmatrix} + (1 - x_2)\begin{pmatrix} a_1 c_1 p_1' - 1 \\ a_1 c_1^2 \end{pmatrix}. \tag{51}$$

Now $\tau = \left(1, \left(\begin{smallmatrix} c_1 - p_1' & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right)\right) = (1, \left(\begin{smallmatrix} p_1 & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right))$, where $p_1 = c_1 - p_1'$. Formulas (48) and (49) are obtained from (50), (51) and $p_1 = c_1 - p_1'$.

We leave it to the reader to verify that, if $a_1 a_2 \mid (a_0 + a_1 + a_2)$ and $a_0 > a_1 - a_2$, then $\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ contains a non-minimal element of $\mathbb{T}(\ddagger)$ (there is a similar argument in the proof of 7.4). $\square$

Our next task is to make 6.13 more explicit; this is done in 7.7, below.

Definition 7.6. Let a_0, a_1, a_2 be pairwise relatively prime positive integers satisfying $a_1 a_2 \mid a_0 + a_1 + a_2$, and write $\gamma = \frac{a_0 + a_1 + a_2}{a_1 a_2}$. Then (a_0, a_1, a_2) determines two sets, $W_{(a_0, a_1, a_2)}$ and $W^{(a_0, a_1, a_2)}$, which we now proceed to define.

7.6.1. Each 2×2 matrix M (with entries in $\mathbb{Z}$) determines a pair of sequences

$$s(M) = (s_0, s_1, s_2, \dots), \qquad t(M) = (t_0, t_1, t_2, \dots)$$

defined by $\begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix} = M$ and $\begin{cases} s_{n-1} + s_{n+1} = a_2 \gamma t_n \\ t_{n-1} + t_{n+1} = a_1 \gamma s_n \end{cases}$.

7.6.2. Let $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and define $u_n = s_n(M)$ and $v_n = t_n(M)$. Note that the beginning terms of these two sequences are:

n	0	1	2	3	4	$\dots$
u_n	1	1	$a_2\gamma - 1$	$a_2\gamma(a_1\gamma - 1) - 1$	$a_2\gamma[a_1\gamma(a_2\gamma - 1) - 1] - (a_2\gamma - 1)$	$\dots$
v_n	1	1	$a_1\gamma - 1$	$a_1\gamma(a_2\gamma - 1) - 1$	$a_1\gamma[a_2\gamma(a_1\gamma - 1) - 1] - (a_1\gamma - 1)$	$\dots$

7.6.3. Let $M' = \left(\begin{smallmatrix} -\gamma - 1 + x_2 & x_1 - 1 \\ -\gamma - 1 + x_1 & x_2 - 1 \end{smallmatrix}\right)$, where (x_0, x_1, x_2) is the solution to $\mathrm{Eq}(a_0, a_1, a_2)$, and define $\xi_n = s_n(M')$ and $\eta_n = t_n(M')$. Note that, in each of the following cases:

(i) $a_1 > 1$ and $a_2 > 1$;
(ii) $1 = a_1 < a_2$;
(iii) $a_1 > a_2 = 1$;
(iv) $a_1 = 1 = a_2$,

the beginning terms of $\{\xi_n\}_{n=0}^{\infty}$ and $\{\eta_n\}_{n=0}^{\infty}$ are as follows:

	n	0	1	2	3	4	$\dots$
(i)	ξ_n	$-\gamma$	0	γ	$a_2\gamma^2$	$a_1a_2\gamma^3-\gamma$	$\dots$
	η_n	$-\gamma$	0	γ	$a_1\gamma^2$	$a_1a_2\gamma^3-\gamma$	$\dots$
(ii)	ξ_n	$-\gamma$	-1	γ	$a_2\gamma+1$	$a_2\gamma^3-\gamma$	$\dots$
	η_n	$-\gamma-1$	0	1	γ^2	$\gamma(a_2\gamma+1)-1$	$\dots$
(iii)	ξ_n	$-\gamma-1$	0	1	γ^2	$\gamma(a_1\gamma+1)-1$	$\dots$
	η_n	$-\gamma$	-1	γ	$a_1\gamma+1$	$a_1\gamma^3-\gamma$	$\dots$
(iv)	$\xi_n=\eta_n$	$-\gamma-1$	-1	1	$\gamma+1$	$\gamma(\gamma+1)-1$	$\dots$

7.6.4. For every $n\in\mathbb{N}$, define

$$f_n=\Big(1,\left(\begin{smallmatrix}\xi_n & 1\\ u_n & a_1\end{smallmatrix}\right),\left(\begin{smallmatrix}v_{n+1}-\eta_{n+1} & 1\\ v_{n+1} & a_2\end{smallmatrix}\right)\Big)\quad\text{and}\quad g_n=\Big(1,\left(\begin{smallmatrix}u_{n+1}-\xi_{n+1} & 1\\ u_{n+1} & a_1\end{smallmatrix}\right),\left(\begin{smallmatrix}\eta_n & 1\\ v_n & a_2\end{smallmatrix}\right)\Big)$$

(we are not claiming that these always belong to $\mathbb{T}(\ddagger)$). Then define

$$W_{(a_0,a_1,a_2)}=\begin{cases}\{f_2,g_3,f_4,g_5,\dots\}, & \text{if } a_0>a_1-a_2,\\ \emptyset, & \text{else,}\end{cases}$$

and

$$W^{(a_0,a_1,a_2)}=\begin{cases}\{g_2,f_3,g_4,f_5,\dots\}, & \text{if } a_0>a_2-a_1,\\ \emptyset, & \text{else.}\end{cases}$$

Proposition 7.7. *Let a_0,a_1,a_2 be pairwise relatively prime positive integers. Then $\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)$ is nonempty if and only if $a_1a_2\mid a_0+a_1+a_2$, in which case we have:*

$$\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)=W_{(a_0,a_1,a_2)}\cup W^{(a_0,a_1,a_2)}.$$

Remark. If $a_1a_2\mid a_0+a_1+a_2$ and $a_0>|a_1-a_2|$, then

$$\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)=\{f_2,f_3,f_4,\dots\}\cup\{g_2,g_3,g_4,\dots\}.$$

Observe, also, that $\mathbb{T}_{\mathrm{III}}(a_0,a_2,a_1)=\{\tau^\times\mid\tau\in\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)\}$ holds in all cases.

Proof of 7.7. The fact that $\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)$ is nonempty if and only if a_1a_2 divides $a_0+a_1+a_2$ is an immediate consequence of 6.13. Assume that $a_1a_2\mid a_0+a_1+a_2$. If $a_0>a_1-a_2$ then, by 7.5, $\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)$ has a unique element which is non-minimal in $\mathbb{T}(\ddagger)$, and a direct calculation shows that this element is f_2 (one verifies that, in each of the cases (i–iv) of 7.6.3, the element τ given by 7.5 is equal to f_2). Similarly, if $a_0>a_2-a_1$ then the unique element of $\mathbb{T}_{\mathrm{III}}(a_0,a_2,a_1)$ which is non-minimal in $\mathbb{T}(\ddagger)$ can be seen to be $g_2^\times$. Again by calculation, one checks that $\{f_2,f_2^\star,(f_2^\star)^\star,\dots\}=\{f_2,g_3^\times,f_4,g_5^\times,\dots\}$ and that $\{g_2^\times,(g_2^\times)^\star,((g_2^\times)^\star)^\star,\dots\}=\{g_2^\times,f_3,g_4^\times,f_5,\dots\}$ (one can use parts (4) and (5) of 6.2 to compute $\tau\mapsto\tau^\star$ explicitely). The desired result follows from this and 6.13. $\square$

Example 7.8. The following is a description of $\mathbb{T}_0(\mathbb{P}^2)$. First, 7.3 and 7.4 give:

- $\mathbb{T}_{\mathrm{I}}(1,1,1)=\{(1,\mathbf{1},\mathbf{1})\}$ (where $\mathbf{1}$ is the empty tableau);
- $\mathbb{T}_{\mathrm{II.1}}(1,1,1)=\{(1,\mathbf{1},\left(\begin{smallmatrix}1 & 1\\ 2 & 1\end{smallmatrix}\right))\}$;
- $\mathbb{T}_{\mathrm{II.2}}(1,1,1)=\{(1,\left(\begin{smallmatrix}1 & 1\\ 2 & 1\end{smallmatrix}\right),\mathbf{1})\}$.

We have $\mathbb{T}_{\mathrm{III}}(1,1,1) = \{f_2, f_3, f_4, \dots\} \cup \{g_2, g_3, g_4, \dots\}$ by 7.7; by 7.6.3 (case (iv), with $\gamma = 3$), we find that $u_n = v_n$ and $\xi_n = \eta_n$ for all n, and:

$$\begin{aligned} u_n &= 3u_{n-1} - u_{n-2}, \quad u_0 = 1, \quad u_1 = 1; \\ \xi_n &= 3\xi_{n-1} - \xi_{n-2}, \quad \xi_0 = -4, \ \xi_1 = -1. \end{aligned}$$

So,

$$\begin{aligned} \mathbb{T}_{\mathrm{III}}(1,1,1) = &\{(1, \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\5&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}4&1\\5&1\end{smallmatrix}\right), \left(\begin{smallmatrix}2&1\\13&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}11&1\\13&1\end{smallmatrix}\right), \left(\begin{smallmatrix}5&1\\34&1\end{smallmatrix}\right)), \dots\} \\ &\cup \{(1, \left(\begin{smallmatrix}1&1\\5&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}2&1\\13&1\end{smallmatrix}\right), \left(\begin{smallmatrix}4&1\\5&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}5&1\\34&1\end{smallmatrix}\right), \left(\begin{smallmatrix}11&1\\13&1\end{smallmatrix}\right)), \dots\}. \end{aligned}$$

Example 7.9. We now describe $\mathbb{T}_0(\mathbb{P}(2,3,5))$. By 7.3 and 7.4,

- $\mathbb{T}_{\mathrm{I}}(2,3,5) = \{(1, \binom{2}{3}, \binom{1}{5})\}$;
- $\mathbb{T}_{\mathrm{I}}(2,5,3) = \{(1, \binom{1}{5}, \binom{2}{3})\}$;
- $\mathbb{T}_{\mathrm{I}}(3,2,5) = \{(1, \binom{1}{2}, \binom{1}{5})\}$;
- $\mathbb{T}_{\mathrm{I}}(3,5,2) = \{(1, \binom{1}{5}, \binom{1}{2})\}$;
- $\mathbb{T}_{\mathrm{I}}(5,2,3) = \{(2, \binom{1}{2}, \binom{2}{3})\}$;
- $\mathbb{T}_{\mathrm{I}}(5,3,2) = \{(2, \binom{2}{3}, \binom{1}{2})\}$;
- $\mathbb{T}_{\mathrm{II.1}}(3,5,2) = \big\{\big(1, \binom{1}{5}, \left(\begin{smallmatrix}3&1\\4&2\end{smallmatrix}\right)\big)\big\}$;
- $\mathbb{T}_{\mathrm{II.2}}(3,2,5) = \big\{\big(1, \left(\begin{smallmatrix}3&1\\4&2\end{smallmatrix}\right), \binom{1}{5}\big)\big\}$;
- $\mathbb{T}_{\mathrm{II.1}}(5,3,2) = \big\{\big(1, \binom{2}{3}, \left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right)\big)\big\}$;
- $\mathbb{T}_{\mathrm{II.2}}(5,2,3) = \big\{\big(1, \left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right), \binom{2}{3}\big)\big\}$.

We have $\mathbb{T}_{\mathrm{III}}(3,5,2) = \{g_2, f_3, g_4, f_5, \dots\}$ by 7.7; by 7.6.3 (case (i), with $\gamma = 1$),

$$\begin{aligned} u_{n-2} + u_n &= 2v_{n-1}, \quad u_0 = 1, \quad u_1 = 1; \\ v_{n-2} + v_n &= 5u_{n-1}, \quad v_0 = 1, \quad v_1 = 1; \\ \xi_{n-2} + \xi_n &= 2\eta_{n-1}, \quad \xi_0 = -1, \ \xi_1 = 0; \\ \eta_{n-2} + \eta_n &= 5\xi_{n-1}, \quad \eta_0 = -1, \ \eta_1 = 0, \end{aligned}$$

so $\mathbb{T}_{\mathrm{III}}(3,5,2) = \{(1, \left(\begin{smallmatrix}5&1\\7&5\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}2&1\\7&5\end{smallmatrix}\right), \left(\begin{smallmatrix}22&1\\31&2\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}39&1\\55&5\end{smallmatrix}\right), \left(\begin{smallmatrix}9&1\\31&2\end{smallmatrix}\right)), \dots\}$. Also,

$$\begin{aligned} \mathbb{T}_{\mathrm{III}}(3,2,5) &= \{g_2^\times, f_3^\times, g_4^\times, f_5^\times, \dots\} \\ &= \{(1, \left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right), \left(\begin{smallmatrix}5&1\\7&5\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}22&1\\31&2\end{smallmatrix}\right), \left(\begin{smallmatrix}2&1\\7&5\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}9&1\\31&2\end{smallmatrix}\right), \left(\begin{smallmatrix}39&1\\55&5\end{smallmatrix}\right)), \dots\}. \end{aligned}$$

8. Further remarks

Corollary 8.1. *Let a_0, a_1, a_2 be pairwise relatively prime positive integers. Then*

$$\Delta(\tau) = \frac{a_0 + a_1 + a_2}{a_1 a_2}, \quad \textit{for all } \tau \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{III}}(a_0, a_2, a_1).$$

Proof. By 6.11, 6.7 and 6.8, there exists $\tau' \in \mathcal{E}_{NM} \cap \mathbb{T}_{\mathrm{III}}(a_0, a_i, a_j)$ such that $\Delta(\tau') = \Delta(\tau)$ (for a suitable permutation i, j of $1, 2$); thus we may assume that $\tau \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ is non-minimal in $\mathbb{T}(\ddagger)$. Write $\tau = (1, \left(\begin{smallmatrix}p_1&1\\c_1&a_1\end{smallmatrix}\right), \left(\begin{smallmatrix}p_2&1\\c_2&a_2\end{smallmatrix}\right))$, where $\binom{p_1}{c_1}$ and $\binom{p_2}{c_2}$ are given by 7.5. Then (by (49)) $\Delta(\tau) = c_1 c_2 - c_1 p_2 - c_2 p_1 = \left|\begin{smallmatrix}c_1 - p_1 & p_2\\ c_1 & c_2\end{smallmatrix}\right| = A + (1 - x_2)B$, where $A = \left|\begin{smallmatrix} c_1-p_1 & a_1p_1(c_1-p_1)-1 \\ c_1 & a_1c_1p_1-1 \end{smallmatrix}\right| = p_1$ and $B = \left|\begin{smallmatrix} c_1-p_1 & a_1c_1(c_1-p_1)-1 \\ c_1 & a_1c_1^2 \end{smallmatrix}\right| = c_1$. Thus $\Delta(\tau) = p_1 + (1-x_2)c_1 = 1 + (x_0 - x_1)x_2 + (1-x_2)(a_2 - x_2 + (x_0 - x_1)a_2) = x_0 - x_1 - x_2 + 2$, where the second equality follows from (48) and the third equality can be verified in each of the cases: $x_2 = 0$, $x_2 = 1$. On the other hand, $\mathrm{Eq}(a_0, a_1, a_2)$ gives

$$\frac{a_0 + a_1 + a_2}{a_1a_2} = x_0 + \frac{1-x_1}{a_1} + \frac{1-x_2}{a_2} = x_0 + (1-x_1) + (1-x_2) = \Delta(\tau).$$

□

Corollary 8.2. *Let a_0, a_1, a_2 be pairwise relatively prime positive integers satisfying $a_1a_2 \mid a_0 + a_1 + a_2$, and write $\gamma = \frac{a_0+a_1+a_2}{a_1a_2}$. Then the elements of $\mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ are the triples $\tau = (1, \left(\begin{smallmatrix} p_1 & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right))$ such that c_1, c_2, p_1, p_2 are positive integers satisfying*

$$\gamma a_1a_2c_1c_2 - a_1c_1^2 - a_2c_2^2 = a_0, \tag{52}$$

$$c_1c_2 - c_1p_2 - c_2p_1 = \gamma \quad (0 < p_i < c_i,\ i = 1, 2). \tag{53}$$

Proof. If $\tau = (1, \left(\begin{smallmatrix} p_1 & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right)) \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{III}}(a_0, a_2, a_1)$ then, as noted at the beginning of the proof of 6.11, we have $a_0 = \Delta(\tau)a_1a_2c_1c_2 - a_1c_1^2 - a_2c_2^2$; by 8.1, we get that (52) and (53) hold.

For the converse, we use the notations of 7.6. Observe: (i) *The set of positive solutions (c_1, c_2) to* (52) *is* $\{(u_n, v_{n+1}) \mid n \in \mathbb{N}\} \cup \{(u_{n+1}, v_n) \mid n \in \mathbb{N}\}$; it follows that $\gcd(c_1, c_2) = 1$ and consequently: (ii) *If we give ourselves a solution (c_1, c_2) of* (52), *then* (53) *has at most one solution (p_1, p_2).* (We leave (i) as an exercise for the reader; (ii) is obvious.)

Let $\tau = (1, \left(\begin{smallmatrix} p_1 & 1 \\ c_1 & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right))$ be such that (52) and (53) hold; by observation (i),

$$(c_1, c_2) = (u_n, v_{n+1}) \text{ or } (u_{n+1}, v_n) \tag{54}$$

for some $n \geq 0$. Let n be minimal such that (54) holds and note that $n \geq 2$ because (53) implies $c_1 > 1$ and $c_2 > 1$. Define $\tau' = f_n$ if $(c_1, c_2) = (u_n, v_{n+1})$ and $\tau' = g_n$ otherwise. Note that if

$$\tau' \in W_{(a_0,a_1,a_2)} \cup W^{(a_0,a_1,a_2)} \tag{55}$$

holds then $\tau' \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ by 7.7, so (52) and (53) hold for τ' by the first part of the proof, so observation (ii) implies that $\tau = \tau' \in \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$ and we are done.

Since τ' is f_n or g_n with $n \geq 2$, (55) is obvious if $a_0 > |a_1 - a_2|$; so we may assume that $a_0 = a_1 - a_2 > a_2 - a_1$ (the other case, $a_0 = a_2 - a_1$, has a similar proof). Now $a_0 = a_1 - a_2$ implies that $a_2\gamma = 2$, which implies that $u_m = u_{m+1}$ and $v_{m-1} = v_m$ for every odd $m > 0$. Since n is minimal such that (54) holds, we have $\tau' = f_n$ if n is odd and $\tau' = g_n$ if n is even. So $\tau' \in \{g_2, f_3, g_4, f_5, \dots\} = W^{(a_0,a_1,a_2)}$ and (55) holds. □

Special pairs

In the following, $\mathbb{A}^1_*$ denotes the affine line minus one point.

8.3. Let X be a surface satisfying (‡) and let Λ be an affine ruling of X.

(1) An ordered pair (F, G) of members of Λ $(F, G \in \Lambda)$ is called a *special pair* of Λ if (i) $F \neq G$, (ii) $F \in \Lambda_*$ and (iii) $\{F, G\}$ contains all multiple members of Λ.

Note the following facts (3 and 4 follow from 1.11 of [2]):

(2) *Λ admits a special pair: Λ_* is nonempty and, given $F \in \Lambda_*$, the definition of Λ_* guarantees that there exists $G \in \Lambda$ such that (F, G) is a special pair.*

(3) *If (F, G) is a special pair of Λ then $X \setminus \operatorname{supp}(F + G)$ is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1_*$, in such a way that the projection $\mathbb{A}^1 \times \mathbb{A}^1_* \to \mathbb{A}^1_*$ extends to a rational map $X \to \mathbb{P}^1$ which is compatible with the linear system Λ (i.e., the fibres of the map are members of Λ).*

(4) *Suppose that U is an open subset of X isomorphic to the product of $\mathbb{A}^1$ with some open subset of $\mathbb{P}^1$, in such a way that the so obtained rational map $X \to \mathbb{P}^1$ is compatible with Λ. If $X \setminus U$ contains at least two curves, then there exists a special pair (F, G) of Λ and members $M_1, \dots, M_n$ $(n \geq 0)$ of Λ such that $U = X \setminus \operatorname{supp}(F + G + M_1 + \cdots M_n)$.*

Given a tableau $T = \left(\begin{smallmatrix} p_1 & p_2 & \cdots & c_k \\ c_1 & c_2 & \cdots & c_k \end{smallmatrix}\right) \in \mathcal{T}$, we define (as in 5.35 of [2]) $\mu(T) = c_1 \cdots c_k$ (where $\mu(T) = 1$ if T is the empty tableau). The following is a special case of Corollary 5.37 of [2]:

8.4. *Let X be a surface satisfying (‡), let Λ be an affine ruling of X and let (F, G) be a special pair of Λ. If (m, T_1, T_2) is the discrete part of (X, Λ, F), then*

$$F = \mu(T_2) C_2 \quad \textit{and} \quad G = \mu(T_1) C_1,$$

where $C_1, C_2 \subset X$ are irreducible curves. Moreover, $\operatorname{Pic}(X_s) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(\mu(T_1), \mu(T_2))$.

Part (1) of the following result was also obtained in [1]:

Corollary 8.5. *Let the notation be as in 8.4 and suppose that $X = \mathbb{P}(a_0, a_1, a_2)$ where a_0, a_1, a_2 are pairwise relatively prime. Then*

(1) $\gcd(\deg(C_1), \deg(C_2)) = 1$*;*

(2) $\mu(T_1) = \deg(C_2)$ *and* $\mu(T_2) = \deg(C_1)$.

Proof. We have $\mu(T_2)\deg(C_2) = \mu(T_1)\deg(C_1)$ and $\gcd(\mu(T_1), \mu(T_2)) = 1$ by 8.4, so assertions (1) and (2) are equivalent. By part (2) of 2.25 together with the results of sections 5 and 6, there exists a sequence $(\tau_0, \dots, \tau_n)$ in $\mathbb{T}(‡)$ satisfying:

(a) $\tau_n = (m, T_1, T_2)$ is the discrete part of (X, Λ, F);

(b) $\tau_0 \in \mathbb{T}_{\mathrm{I}}(a, b, c)$, for some permutation a, b, c of a_0, a_1, a_2;

(c) for each i such that $1 \leq i \leq n$, the pair (τ_{i-1}, τ_i) satisfies one of the following conditions:

  (i) $\tau_i > \tau_{i-1}$,

(ii) $\tau_i \in \mathcal{E}$ is minimal in $\mathbb{T}(\ddagger)$ and $\tau_{i-1} = \tilde{\tau}_i$,
(iii) $\tau_i \in \mathcal{E}$ and $\tau_{i-1} = \tau_i^{\times}$,
(iv) $\tau_i \in \mathbb{T}_{\mathrm{II}.2}(a, b, c)$ and $\tau_{i-1} = \tau_i^{\times}$ (some permutation a, b, c of a_0, a_1, a_2).

We proceed by induction on n. If $n = 0$ then Λ is basic of type I, so $C_1 = R_i$ and $C_2 = R_j$ for some distinct $i, j \in \{0, 1, 2\}$ (notations as in sections 1 and 3). Since $\gcd(a_i, a_j) = 1$, (1) is clear in this case. Suppose that $n > 0$ and that (1) (or equivalently (2)) holds for smaller values of n.

If (τ_{n-1}, τ_n) satisfies (iii) or (iv) then Λ is basic, so (G, F) is also a special pair of Λ and $\tau_{n-1} = \operatorname{disc}(X, \Lambda, G)$; by the inductive hypothesis, (1) holds for Λ and (G, F); it follows immediately that (1) holds for Λ and (F, G).

If (τ_{n-1}, τ_n) satisfies (i) or (ii) then $\tau \equiv \tau_{n-1}$ (by 2.20 or 6.2), so, by 2.25, there exists an affine ruling Λ' of X and $F' \in \Lambda'_*$ such that $\operatorname{supp}(F) = \operatorname{supp}(F')$ and $\tau_{n-1} = \operatorname{disc}(X, \Lambda', F')$. Let G' be such that (F', G') is a special pair of Λ', write $\tau_{n-1} = (m', T_1', T_2')$ and note that 8.4 gives

$$F' = \mu(T_2')C_2' \quad \text{and} \quad G' = \mu(T_1')C_1',$$

where C_1' and C_2' are irreducible curves. Then

$$\deg(C_2) = \deg(C_2') = \mu(T_1') = \mu(T_1),$$

where the middle equality is the inductive hypothesis (i.e., (2) holds for Λ' and (F', G')), the first equality is $C_2 = \operatorname{supp}(F) = \operatorname{supp}(F') = C_2'$ and the last equality follows from $T_1' = T_1^{(\sim s)}$ for some $s \geq 1$. Consequently, (1) and (2) hold for Λ and (F, G). □

Remark. Given $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2)$, where a_0, a_1, a_2 are pairwise relatively prime positive integers, one may ask: *What are all pairs of irreducible curves $C_1, C_2 \subset \mathbb{P}$ with the property that $\mathbb{P} \setminus (C_1 \cup C_2)$ is isomorphic to the product of $\mathbb{A}^1$ with a curve?* As mentioned in 8.3, above, these are exactly the special pairs associated to affine rulings of $\mathbb{P}$; consequently, a description of these curves can be derived from this paper. In particular, one can give all pairs of integers $(\deg(C_1), \deg(C_2))$ by following these steps:

(1) Give all elements of $\mathbb{T}_0(\mathbb{P})$ (7.8 and 7.9 are two examples of this);
(2) for each $(m', T_1', T_2') \in \mathbb{T}_0(\mathbb{P})$, give all elements of

$$\{ (\mu(T_1), \mu(T_2)) \mid (1, T_1, T_2) \in \mathbb{T}(\mathbb{P}) \text{ and } (1, T_1, T_2) \geq (m', T_1', T_2') \}$$

(this step is computed explicitely in 5.40 of [2]). By 8.5, this set of pairs is the desired one.

For instance, if $X = \mathbb{P}^2 = \mathbb{P}(1, 1, 1)$ then one finds that the set of pairs $(\deg(C_1), \deg(C_2))$ is the union of the following four sets (where the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{\xi_n\}_{n=0}^{\infty}$ are defined in 7.8):

(1) $(1, n)$, with $n \geq 1$;
(2) $(2, 4n + 1)$, with $n \geq 0$;

(3) $(u_n, u_{n+1}P)$, where $n \geq 3$ and (for n fixed) P is any finite product of the form $P = \prod_{i=1}^{s}(\alpha_i + u_n^2\nu_i)$ where $s \geq 0$, $\nu_i \geq 0$ and $\alpha_i = \begin{cases} u_n(u_n-\xi_n)-1, & \text{if } i \text{ is odd,} \\ u_n\xi_n-1, & \text{if } i \text{ is even;} \end{cases}$

(4) (u_{n+1}, u_nQ), where $n \geq 2$ and (for n fixed) Q is any finite product of the form $Q = \prod_{i=1}^{s}(\alpha_i + u_{n+1}^2\nu_i)$ where $s \geq 0$, $\nu_i \geq 0$ and $\alpha_i = \begin{cases} u_{n+1}\xi_{n+1}-1, & \text{if } i \text{ is odd,} \\ u_{n+1}(u_{n+1}-\xi_{n+1})-1, & \text{if } i \text{ is even.} \end{cases}$

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Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada K1N 6N5

E-mail address: daniel@zenon.mathstat.uottawa.ca

Department of Mathematics and Statistics, McGill University, Montréal, Qc, Canada H3A 2K6

E-mail address: russell@math.mcgill.ca