TRIANGULAR DERIVATIONS OF $\mathbf{k}[X_1, X_2, X_3, X_4]$

DANIEL DAIGLE AND GENE FREUDENBURG

ABSTRACT. It is shown that, if $\mathbf{k}$ is a field of characteristic zero, then the kernel of any triangular $\mathbf{k}$-derivation of $\mathbf{k}[X_1, X_2, X_3, X_4]$ is finitely generated as a $\mathbf{k}$-algebra. This is obtained as a corollary of a more general result concerning triangular R-derivations of $R[X, Y, Z]$ for certain rings R.

1. INTRODUCTION

Let R be a commutative ring and $B = R[X_1, \ldots, X_n]$ a polynomial algebra in n variables over R (abbreviated $B \cong R^{[n]}$). An R-derivation $D : B \to B$ is called *triangular* if $DX_i \in R[X_1, \ldots, X_{i-1}]$ holds for all $i = 1, \ldots, n$ (in particular, $DX_1 \in R$). Note that such a derivation is necessarily *locally nilpotent*, i.e., for each $b \in B$ there exists $m > 0$ satisfying $D^m(b) = 0$. The phrase "A is $\mathbf{k}$-affine" means that A is a finitely generated $\mathbf{k}$-algebra.

Consider a triangular R-derivation $0 \neq D : R[X, Y, Z] \to R[X, Y, Z]$ where R is a domain containing $\mathbb{Q}$. If R is a field, then a well-known theorem of Miyanishi [7] asserts that $\ker D \cong R^{[2]}$ (even in the more general case when D is locally nilpotent). On the other hand, if $R \cong \mathbf{k}^{[2]}$ ($\mathbf{k}$ a field) then [1] gives an example where $\ker D$ is not finitely generated as an R-algebra. The aim of this paper is to prove:

1.1. **Theorem.** *Let $\mathbf{k}$ be a field of characteristic zero and let R be a $\mathbf{k}$-affine Dedekind domain or a localization of such a ring. If D is any triangular R-derivation of $R^{[3]}$, then the kernel of D is finitely generated as an R-algebra.*

Let us immediately derive:

1.2. **Corollary.** *If $\mathbf{k}$ is a field of characteristic 0, then the kernel of any triangular $\mathbf{k}$-derivation of $\mathbf{k}^{[4]}$ is finitely generated as a $\mathbf{k}$-algebra.*

Proof. Suppose δ is a triangular $\mathbf{k}$-derivation of $B = \mathbf{k}[W, X, Y, Z] = \mathbf{k}^{[4]}$ and write $\delta W = \lambda \in \mathbf{k}$ and $\delta X = f(W) \in \mathbf{k}[W]$. Choose $F(W) \in \mathbf{k}[W]$ such that $F'(W) = f(W)$, noting that $\delta(\lambda X - F(W)) = 0$.

If $\lambda = 0$, then δ is a triangular R-derivation of $B = R[X, Y, Z] \cong R^{[3]}$, where $R = \mathbf{k}[W] \cong \mathbf{k}^{[1]}$. If $\lambda \neq 0$, then δ is a triangular R-derivation of $B = R[W, Y, Z] \cong R^{[3]}$, where $R = \mathbf{k}[\lambda X - F(W)] \cong \mathbf{k}^{[1]}$. In either case, 1.1 implies the desired result. $\square$

Note the following related fact (see [2]): *For each positive integer n, there exists a triangular $\mathbf{k}$-derivation Δ_n of $\mathbf{k}^{[4]}$ whose kernel cannot be generated by n elements.* So, in 1.1 and 1.2, there is no a priori upper bound on the number of generators.

2000 *Mathematics Subject Classification.* Primary: 14R10. Secondary: 14R20, 13N15.

Key words and phrases. Derivations, Hilbert fourteenth problem, additive group actions, invariants.

Research of the first author supported by NSERC Canada.

We remind the reader that locally nilpotent derivations of $\mathbf{k}^{[4]}$ are equivalent to algebraic actions of $G_a = (\mathbf{k}, +)$ on $\mathbb{A}^4$ in such a way that the kernel of a derivation is *equal* to the ring of invariants of the corresponding G_a-action. So 1.2 asserts that the ring of invariants of any triangular G_a-action on $\mathbb{A}^4$ is finitely generated, whereas the results of [1] (*op. cit.*) imply that, for each $n \geq 5$, there is a triangular G_a-action on $\mathbb{A}^n$ with non-finitely generated ring of invariants.

Essentially, the proof of 1.1 is based on the well-known algorithm of van den Essen [5] for computing the kernel of a locally nilpotent derivation: We consider a certain increasing sequence of subalgebras of the kernel,

$$R[f_0, f_1] \subseteq R[f_0, f_1, f_2] \subseteq R[f_0, f_1, f_2, f_3] \subseteq \dots,$$

and we have to show that $R[f_0, \dots, f_n] = R[f_0, \dots, f_{n+1}]$ for some n. However, in our situation, that integer n can be arbitrarily large and the proof of its existence requires a delicate analysis of the sequence $(f_0, f_1, f_2, \dots)$. The tools needed for this analysis, *simple sequences* (2.2) and *simple pairs* (2.4), are developed in Section 2.

In particular, we need the fact that $(\mathbf{k}[[t]], t)$ is a simple pair whenever $\mathbf{k}$ is an algebraically closed field of characteristic zero and t is an indeterminate over $\mathbf{k}$. This fact (2.8) is an easy consequence of the following result:

1.3. **Theorem** (A. Sathaye). *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, let R be an affine $\mathbf{k}$-domain with one place at infinity, let t be an indeterminate over R and let $f_0, f_1 \in R[[t]]$. If[1] $f_1(0)$ does not belong to $\mathbf{k}$, then there exists a sequence $(u_i)_{i=0}^{\infty}$ in $R[[t]]$ satisfying:*

(1) *$u_0 = f_0$ and $u_1 = f_1$;*
(2) *for each $n \geq 1$, there exists a monic polynomial $\varphi_n(Y) \in \mathbf{k}[u_0, \dots, u_{n-1}][Y]$ such that[2]*
   (a) *$\bar{\varphi}_n(Y) \in \mathbf{k}[u_0(0), \dots, u_{n-1}(0)][Y]$ is the minimal polynomial of $u_n(0)$ over $\mathbf{k}(u_0(0), \dots, u_{n-1}(0))$,*
   (b) *$u_{n+1} = \varphi_n(u_n)/t$.*

Sathaye obtained this result in [9] as an application of his theory of Generalized Newton-Puiseux Expansions. We acknowledge that this paper would not exist without Sathaye's contribution.

2. Simple sequences

2.1. **Setup.** We consider triples (R, t, S) satisfying:

(1) R is a PID containing $\mathbb{Q}$ and t a prime element of R;
(2) S is an overdomain of R such that t is prime in S.

Given a triple (R, t, S) as above, we use the following notations:

- $\bar{S} = S/tS$ (an integral domain);
- given $s \in S$, write $\bar{s} = s + tS \in \bar{S}$;

---

[1]Given $f = \sum_{i=0}^{\infty} a_i t^i \in R[[t]]$ (where $a_i \in R$), we write $f(0) = a_0$.

[2]If $\varphi_n(Y) = \sum_i a_i Y^i$ ($a_i \in \mathbf{k}[u_0, \dots, u_{n-1}]$), then we write $\bar{\varphi}_n(Y) = \sum_i a_i(0) Y^i \in \mathbf{k}[u_0(0), \dots, u_{n-1}(0)][Y]$.

- given a ring A such that $R \subseteq A \subseteq S$, (i) let $\bar{A}$ be the image of the composite $A \hookrightarrow S \to \bar{S}$; (ii) let $A^+ = A[S \cap \frac{1}{t}A]$;
- the field R/tR is denoted κ (note that $\bar{R} = \kappa$).

2.2. **Definition.** Let (R, t, S) be as in 2.1. Let $f = (f_0, \dots, f_r)$ be a finite sequence in S and write $A_n = R[f_0, \dots, f_n]$ and $K_n = \mathrm{Frac}(\bar{A}_n)$ for $0 \le n \le r$. We say that f is a *simple sequence* of (R, t, S) if the following hold:

(1) $r \ge 1$;
(2) $\bar{f}_0 \in \bar{S}$ is transcendental over κ;
(3) for each n such that $0 < n < r$, $\bar{f}_n$ is algebraic over K_{n-1} and there exists a monic polynomial $\varphi_n \in A_{n-1}[T]$ satisfying: (i) $\bar{\varphi}_n \in \bar{A}_{n-1}[T]$ is the minimal polynomial of $\bar{f}_n$ over K_{n-1}; and (ii) $f_{n+1} = \varphi_n(f_n)/t$.

We distinguish three types of simple sequences:

(i) f is *transcendental* if $\bar{f}_r$ is transcendental over K_{r-1};
(ii) f is *extendable* if $\bar{f}_r$ is algebraic over K_{r-1} and its minimal polynomial is in $\bar{A}_{r-1}[T]$;
(iii) f is *obstructed* if it is neither transcendental nor extendable, i.e., if $\bar{f}_r$ is algebraic over K_{r-1} but its minimal polynomial fails to have all its coefficients in $\bar{A}_{r-1}$.

Remark. It is easy to see that a simple sequence $f = (f_0, \dots, f_r)$ of (R, t, S) is extendable if and only if $\exists f_{r+1} \in S$ such that $(f_0, \dots, f_r, f_{r+1})$ is a simple sequence of (R, t, S).

2.3. **Lemma.** *Let (R, t, S) be as in 2.1 and let $f = (f_0, \dots, f_r)$ be a simple sequence of (R, t, S). Then the rings $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r$ (where $A_n = R[f_0, \dots, f_n]$) have the following properties:*

(1) *$A_0^+ = A_0$;*
(2) *for all n such that $0 < n < r$, $A_n^+ = A_{n+1}$;*
(3) *if f is of transcendental type, then $A_r^+ = A_r$.*

Proof. Assertion (1) easily follows from the fact that $\bar{f}_0$ is transcendental over κ. We prove (2) by induction on n. Consider n such that $0 < n < r$ and note that $A_{n-1}^+ \subseteq A_n$ (by part (1) if $n = 1$, by inductive hypothesis if $n > 1$); also, $A_n = A_{n-1}[f_n]$. Let $\varphi_n \in A_{n-1}[T]$ be the monic polynomial given by 2.2. Since $f_{n+1} = \varphi_n(f_n)/t \in S \cap \frac{1}{t}A_n$, it is clear that $A_{n+1} \subseteq A_n^+$. For the reverse inclusion, it suffices to check that $S \cap \frac{1}{t}A_n \subseteq A_{n+1}$. Consider $x \in S \cap \frac{1}{t}A_n$; then $tx = \psi(f_n)$ for some $\psi \in A_{n-1}[T]$. Since φ_n is monic, the division algorithm gives

$$\psi = q\varphi_n + \rho \qquad (q, \rho \in A_{n-1}[T], \quad \deg(\rho) < \deg(\varphi_n)).$$

Now $\bar{\rho} \in \bar{A}_{n-1}[T]$ satisfies $\bar{\rho}(\bar{f}_n) = 0$ and $\deg(\bar{\rho}) < \deg(\bar{\varphi}_n)$, so $\bar{\rho} = 0$ because $\bar{\varphi}_n$ is the minimal polynomial of $\bar{f}_n$. So all coefficients of ρ belong to $A_{n-1} \cap tS \subseteq tA_{n-1}^+ \subseteq tA_n$, so $\frac{1}{t}\rho \in A_n[T]$ and

$$x = q(f_n)\frac{\varphi_n(f_n)}{t} + \frac{\rho(f_n)}{t} = q(f_n)f_{n+1} + (\tfrac{1}{t}\rho)(f_n) \in A_n[f_{n+1}].$$

This shows that $S \cap \frac{1}{t}A_n \subseteq A_{n+1}$, so $A_n^+ = A_{n+1}$ and (2) is proved.

We have $A_{r-1}^+ \subseteq A_r$ (by (1) and (2)), $A_r = A_{r-1}[f_r]$ and (assuming that f is transcendental) $\bar{f}_r$ is transcendental over K_{r-1}; it easily follows that $A_r^+ = A_r$, so (3) holds. $\square$

2.4. **Definition.** Suppose that (R,t) satisfies condition (1) of 2.1. We say that the pair (R,t) is *simple* if $(R,t,R^{[1]})$ does not have an obstructed simple sequence (i.e., if every simple sequence of $(R,t,R^{[1]})$ is extendable).

2.5. **Lemma.** *Let (R,t) be a simple pair and let $S=R[X,Y]$, where X is an indeterminate over R and Y is a family of indeterminates over $R[X]$. If $f=(f_0,\dots,f_r)$ is a simple sequence of (R,t,S) satisfying $\bar f_0\in\kappa[X]$, then f is not obstructed.*

Proof. The hypothesis is that $\bar f_0$ belongs to the subring $\kappa[X]$ of $\bar S=\kappa[X,Y]$. Assume that f is not transcendental; then

$$\bar f_n\in\kappa[X]\quad\text{for all } n=0,\dots,r,$$

because each $\bar f_n$ is algebraic over $\kappa(\bar f_0)$ and $\bar f_0\in\kappa[X]$. Define a sequence $e=(e_0,\dots,e_r)$ in $R[X]$ by $e_n=f_n(X,0)$. It is clear that $\bar e_n=\bar f_n(X,0)$, so $\bar f_n\in\kappa[X]$ implies that

$$\bar e_n=\bar f_n,\quad\text{for all } n=0,\dots,r.$$

We claim that e is a simple sequence of $(R,t,R[X])$ and that e and f are of the same type. For $n=0,\dots,r$, write $E_n=R[e_0,\dots,e_n]$, $A_n=R[f_0,\dots,f_n]$ and $K_n=\operatorname{Frac}(\bar A_n)$; clearly, $\bar E_n=\bar A_n$ and so $\operatorname{Frac}(\bar E_n)=K_n$. For each n such that $0<n<r$, let $\varphi_n\in A_{n-1}[T]$ be a monic polynomial satisfying: (i) $\bar\varphi_n\in\bar A_{n-1}[T]$ is the minimal polynomial of $\bar f_n$ over K_{n-1}; and (ii) $f_{n+1}=\varphi_n(f_n)/t$. Also, choose $\Phi_n\in R[T_0,\dots,T_{n-1},T]$ monic in T such that $\Phi_n(f_0,\dots,f_{n-1},T)=\varphi_n$.

For each n such that $0<n<r$, define $\psi_n=\Phi_n(e_0,\dots,e_{n-1},T)\in E_{n-1}[T]$. Then ψ_n is monic and $\bar\psi_n=\bar\varphi_n$ is the minimal polynomial of $\bar e_n=\bar f_n$ over K_{n-1}. Also, setting $Y=0$ in the equation $tf_{n+1}=\Phi_n(f_0,\dots,f_n)$ yields $te_{n+1}=\Phi_n(e_0,\dots,e_n)=\psi_n(e_n)$. Thus e is a simple sequence of $(R,t,R[X])$; since $\bar e_r=\bar f_r$ and $\bar E_{r-1}=\bar A_{r-1}$, e and f are of the same type, as claimed.

Since (R,t) is simple, e, and therefore f, must be extendable. □

2.6. **Lemma.** *Let (R,t,S) be as in 2.1, with $S=R[X,Y,Z]=R^{[3]}$. Let $f=(f_0,\dots,f_r)$ be a simple sequence of (R,t,S) satisfying $f_0\in R[X,Y]$, $\bar f_0\in\kappa[X]$ and $f_1\notin R[X,Y]$. Let k be the natural number such that $f_1=w_0+w_1Zt^k$, where $w_0\in R[X,Y]$ and $w_1\in S\setminus tS$. Then $r\le k+1$.*

Remark. 2.6 remains true if "Y" is interpreted as a family of indeterminates.

Proof. Define a map $\operatorname{ord}: S\to\mathbb N\cup\{\infty\}$ by setting $\operatorname{ord}(b)=\infty$ for $b\in R[X,Y]$ and $\operatorname{ord}(b)=s\in\mathbb N$ for $b\in S\setminus R[X,Y]$, where $b=b_0+b_1Zt^s$ is the unique expression of b satisfying $b_0\in R[X,Y]$, $b_1\in S\setminus tS$ and $s\in\mathbb N$. For a nonempty subset E of S, define $\operatorname{ord}(E)=\min_{b\in E}\operatorname{ord}(b)$.

Let $A_n=R[f_0,\dots,f_n]$ and $K_n=\operatorname{Frac}(\bar A_n)$ for $0\le n\le r$. We have $\operatorname{ord}(f_1)=k$, so $\operatorname{ord}(A_1)\le k$; then it suffices to show that $\{\operatorname{ord}(A_n)\}_{n=1}^r$ is a strictly decreasing sequence in $\mathbb N$. Fix n such that $0<n<r$; we shall show that $\operatorname{ord}(A_n)$ is a positive integer and that $\operatorname{ord}(A_n)>\operatorname{ord}(A_{n+1})$.

Since $n\ge1$, we have $A_1\subseteq A_n$, so $\operatorname{ord}(A_n)\le\operatorname{ord}(A_1)\le k$ and consequently $\operatorname{ord}(A_n)\in\mathbb N$. Since $n<r$, K_n is algebraic over $K_0\subseteq\kappa(X)$, so $\bar A_n\subseteq\kappa[X]$ and consequently $\operatorname{ord}(A_n)>0$. So $s=\operatorname{ord}(A_n)$ is a positive integer. Choose $a\in A_n$ such that $\operatorname{ord}(a)=s$ and write $a=a_0+a_1Zt^s$ with $a_0\in R[X,Y]$, $a_1\in S\setminus tS$. The element $\bar a=\bar a_0$ of $\kappa[X]$

is algebraic over K_0, so we may consider its minimal polynomial $\bar{\varphi} \in K_0[T]$. Since $\bar{a}$ is integral over $\bar{A}_0$ and $\bar{A}_0 = \kappa^{[1]}$ is a normal ring, $\bar{\varphi} \in \bar{A}_0[T]$ and we may lift $\bar{\varphi}$ to a monic polynomial $\varphi \in A_0[T]$. Then $\bar{\varphi}(\bar{a}) = 0$ implies that $\varphi(a)$ belongs to tS and hence to $tA_n^+ = tA_{n+1}$ (equality by 2.3). So the Taylor expansion of $\varphi(a)$ gives

$$tA_{n+1} \ni \varphi(a) = \varphi(a_0 + a_1 Z t^s) = \varphi(a_0) + \varphi'(a_0) a_1 Z t^s + \epsilon t^{s+1} \quad (\text{some } \epsilon \in S)$$

and if we set $\tilde{a} = t^{-1}\varphi(a) \in A_{n+1}$ then

$$\tilde{a} = t^{-1}\varphi(a_0) + \varphi'(a_0) a_1 Z t^{s-1} + \epsilon t^s.$$

We have $t^{-1}\varphi(a_0) \in A_0[a_0]^+ \subseteq R[X,Y]^+ = R[X,Y]$ and (by minimality of $\bar{\varphi}$) $\varphi'(a_0)a_1 \in S \setminus tS$; it follows that $\operatorname{ord}(\tilde{a}) = s-1$ and $\operatorname{ord}(A_{n+1}) < \operatorname{ord}(A_n)$. □

In the remainder of the section we prove that certain pairs (R,t) are simple. The main result of this type is 2.11.

2.7. **Lemma.** *Let (R,t,S) and (R,t,S') be as in 2.1, with $S \subseteq S'$ and $tS' \cap S = tS$. Let $f = (f_0, \dots, f_r)$ and $f' = (f'_0, \dots, f'_r)$ be simple sequences of (R,t,S) and (R,t,S') respectively, and suppose that $f_0 = f'_0$ and $f_1 = f'_1$. Then f and f' are simple sequences of the same type.*

Proof. Let $A_n = R[f_0, \dots, f_n]$ and $A'_n = R[f'_0, \dots, f'_n]$ for $0 \le n \le r$. By assumption,

$$(1) \qquad f_n - f'_n \in A_{n-1} \quad \text{and} \quad A_{n-1} = A'_{n-1}$$

holds for $n = 1$. Consider any n satisfying (1) and $0 < n < r$; then it is clear that $A_n = A'_n$, and we shall show that $f_{n+1} - f'_{n+1} \in A_n$.

Write $f_n - f'_n = a \in A_{n-1}$. Let $\varphi_n \in A_{n-1}[T]$ be a monic polynomial satisfying (i) $\varphi_n(f_n)/t = f_{n+1}$ and (ii) $\bar{\varphi}_n$ is the minimal polynomial of $\bar{f}_n$ over $K_{n-1} = \operatorname{Frac}(\bar{A}_{n-1})$; also, let $\psi_n \in A'_{n-1}[T] = A_{n-1}[T]$ be a monic polynomial satisfying (i$'$) $\psi_n(f'_n)/t = f'_{n+1}$ and (ii$'$) $\bar{\psi}_n$ is the minimal polynomial of $\bar{f}'_n$ over $\operatorname{Frac}(\bar{A}'_{n-1}) = K_{n-1}$. Then $\bar{\varphi}_n(T + \bar{a})$ is also the minimal polynomial of $\bar{f}'_n$ over K_{n-1}, so $\bar{\psi}_n = \bar{\varphi}_n(T+\bar{a})$, so the polynomial $\psi_n(T) - \varphi_n(T+a)$ has its coefficients in $A_{n-1} \cap tS' = A_{n-1} \cap tS \subseteq tA^+_{n-1} \subseteq tA_n$. So $\psi_n(T) - \varphi_n(T+a) = t\zeta(T)$, with $\zeta(T) \in A_n[T]$, and it follows that $t(f'_{n+1} - f_{n+1}) = \psi_n(f'_n) - \varphi_n(f_n) = \varphi_n(f'_n + a) + t\zeta(f'_n) - \varphi_n(f_n) = t\zeta(f'_n)$, so $f'_{n+1} - f_{n+1} = \zeta(f'_n) \in A_n$.

By induction, this proves that (1) holds for all n such that $0 < n \le r$. The case $n = r$ implies the desired assertion. □

The last lemma and Sathaye's Result (1.3) imply the following:

2.8. **Corollary.** *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and t an indeterminate over $\mathbf{k}$. Then $(\mathbf{k}[[t]], t)$ is a simple pair.*

Proof. Let $f = (f_0, \dots, f_r)$ be a simple sequence of $(\mathbf{k}[[t]], t, \mathbf{k}[[t]][x])$, where x is an indeterminate over $\mathbf{k}[[t]]$; it suffices to show that f is extendable.

Write $S = \mathbf{k}[[t]][x]$ and $S' = \mathbf{k}[x][[t]]$, then $S \subset S'$, so $f_0, f_1 \in S'$. Since $\mathbf{k}[x]$ has one place at infinity, we may consider the sequence $(u_i)_{i=0}^\infty \subset S'$ given by 1.3. By the properties of $(u_i)_{i=0}^\infty \subset S'$ stated in 1.3, it is clear that $(u_0, \dots, u_{r+1})$ is a simple sequence of $(\mathbf{k}[[t]], t, S')$; thus $f' = (u_0, \dots, u_r)$ is an *extendable* simple sequence of $(\mathbf{k}[[t]], t, S')$ whose first two terms are f_0, f_1. Thus f is extendable by 2.7. □

2.9. **Definition.** Let (R,t) and (R',t') be pairs satisfying the condition 1. of 2.1. We use the notation $(R,t) \prec (R',t')$ to indicate that the following conditions hold:

(1) R' is a DVR with maximal ideal $t'R'$;
(2) $R \subseteq R'$ and $R \cap t'R' = tR$;
(3) $R'/t'R'$ is an algebraic extension of R/tR.

2.10. **Lemma.** *Let (R,t) and (R',t') be pairs satisfying the condition 1. of 2.1. If $(R,t) \prec (R',t')$ and (R',t') is simple, then (R,t) is simple.*

Proof. Suppose that $(R,t) \prec (R',t')$ and that (R',t') is simple; let $f = (f_0, \dots, f_r)$ be a simple sequence of $(R, t, R[X])$; we show that f is extendable.

As usual, write $\kappa = R/tR$ and, for each n such that $0 \le n \le r$, $A_n = R[f_0, \dots, f_n]$, $\bar{A}_n = \kappa[\bar{f}_0, \dots, \bar{f}_n]$, $K_n = \mathrm{Frac}(\bar{A}_n) = \kappa(\bar{f}_0, \dots, \bar{f}_n)$; for each n such that $0 < n < r$, let $\varphi_n(T) \in A_{n-1}[T]$ be a monic polynomial satisfying (i) $\varphi_n(f_n)/t = f_{n+1}$ and (ii) $\bar{\varphi}_n$ is the minimal polynomial of $\bar{f}_n$ over K_{n-1}. Also, let $\kappa' = R'/t'R'$ and write $t = u(t')^d$ where u is a unit of R' and d is a positive integer.

Consider the following condition $\mathcal{P}(n)$ on an integer $n \in \{1, \dots, r\}$: *There exists a simple sequence $g = (g_0, \dots, g_m)$ of $(R', t', R'[X])$ and an integer $k \ge 0$ such that $g_m = u^k f_n$, $R'[g_0, \dots, g_{m-1}] \supseteq A_{n-1}$ and $\kappa'[\bar{g}_0, \dots, \bar{g}_{m-1}] = \kappa'[\bar{A}_{n-1}]$.*

Note that $\mathcal{P}(1)$ is true (with $g = (f_0, f_1)$ and $k = 0$). We will use an inductive argument to show that $\mathcal{P}(r)$ is true.

Consider any $n \in \{1, \dots, r\}$ such that $\mathcal{P}(n)$ is true and define the following notations: $A'_i = R'[g_0, \dots, g_i]$, $\bar{A}'_i = \kappa'[\bar{g}_0, \dots, \bar{g}_i]$ and $K'_i = \mathrm{Frac}(\bar{A}'_i) = \kappa'(\bar{g}_0, \dots, \bar{g}_i)$. Note that, for each i, K_i is a regular extension of κ (because $\kappa \subset K_i \subseteq \kappa(X)$). Together with the fact that κ' is algebraic over κ, this implies that $\kappa' \otimes_\kappa K_i$ is a field, and hence is equal to $\kappa'(\bar{f}_0, \dots, \bar{f}_i)$. In particular,

$$\kappa' \otimes_\kappa K_{n-1} = \kappa'(\bar{f}_0, \dots, \bar{f}_{n-1}) \overset{\mathcal{P}(n)}{=} \kappa'(\bar{g}_0, \dots, \bar{g}_{m-1}) = K'_{m-1}, \tag{2}$$

$$\kappa' \otimes_\kappa K_n = \kappa'(\bar{f}_0, \dots, \bar{f}_{n-1})(\bar{f}_n) \overset{(2)}{=} \kappa'(\bar{g}_0, \dots, \bar{g}_{m-1})(\bar{f}_n) = \kappa'(\bar{g}_0, \dots, \bar{g}_{m-1})(\bar{g}_m) = K'_m$$

and consequently

$$[K'_m : K'_{m-1}] = [K_n : K_{n-1}]. \tag{3}$$

Assume, now, that $n < r$; then we may consider the polynomial $\varphi_n(T) \in A_{n-1}[T]$. Let $e = \deg \varphi_n(T)$ and define $\psi_m(T) = u^{ek}\varphi_n(T/u^k)$. Then $\psi_m(T)$ is a monic polynomial with coefficients in $A_{n-1}[u] \subseteq A'_{m-1}$ and

$$\psi_m(g_m) = u^{ek}\varphi_n(f_n) = u^{ek} t f_{n+1} = u^{ek+1}(t')^d f_{n+1}.$$

This implies that $\bar{\psi}_m(\bar{g}_m) = 0$; since $\deg \bar{\psi}_m(T) = e = [K_n : K_{n-1}] = [K'_m : K'_{m-1}]$, $\bar{\psi}_m(T)$ is the minimal polynomial of $\bar{g}_m$ over K'_{m-1}. So $(g_0, \dots, g_m, g_{m+1})$ is a simple sequence of $(R', t', R'[X])$, where we define

$$g_{m+1} = \frac{\psi_m(g_m)}{t'} = u^{ek+1}(t')^{d-1} f_{n+1}.$$

Consequently, if we set $g_{m+i} = u^{ek+1}(t')^{d-i} f_{n+1}$ $(1 \le i \le d)$, then

$$g' = (g_0, \dots, g_m, g_{m+1}, \dots, g_{m+d})$$

is a simple sequence of $(R', t', R'[X])$ and it is easily verified that $\mathcal{P}(n+1)$ holds (with g' and $ek+1$ in place of g and k).

By induction, we obtain that $\mathcal{P}(r)$ is true, i.e., there exists a simple sequence $g = (g_0, \dots, g_m)$ of $(R', t', R'[X])$ and an integer $k \geq 0$ such that

$$g_m = u^k f_r, \quad A'_{m-1} = R'[g_0, \dots, g_{m-1}] \supseteq A_{r-1} \text{ and } \bar{A}'_{m-1} = \kappa'[\bar{g}_0, \dots, \bar{g}_{m-1}] = \kappa'[\bar{A}_{r-1}].$$

Let $P(T)$ be the minimal polynomial of $\bar{f}_r$ over K_{r-1}, let $e = \deg P(T)$, define $Q(T) = \bar{u}^{ke} P(\bar{u}^{-k} T)$ and note that $Q(T)$ is a monic polynomial with coefficients in K'_{m-1}. Since $Q(\bar{g}_m) = \bar{u}^{ke} P(\bar{f}_r) = 0$, and since $\deg Q(T) = e = [K_r : K_{r-1}] = [K'_m : K'_{m-1}]$ by (3), $Q(T)$ is the minimal polynomial of $\bar{g}_m$ over K'_{m-1}. Since (R', t') is assumed to be simple, g is extendable, so $Q(T)$ has all its coefficients in $\bar{A}'_{m-1}$; it follows that $P(T)$ has all its coefficients in $\bar{A}'_{m-1}$, hence in $\bar{A}'_{m-1} \cap K_{r-1}$. So, in order to show that f is extendable, it suffices to show that $\bar{A}'_{m-1} \cap K_{r-1} = \bar{A}_{r-1}$.

Let $\mathcal{B}$ be a basis of κ' over κ such that $1 \in \mathcal{B}$. Since $\bar{A}'_{m-1} = \kappa'[\bar{g}_0, \dots, \bar{g}_{m-1}] = \kappa'[\bar{f}_0, \dots, \bar{f}_{r-1}] = \kappa' \otimes_\kappa \bar{A}_{r-1}$, $\mathcal{B}$ is a basis of $\kappa' \otimes_\kappa \bar{A}_{r-1} = \bar{A}'_{m-1}$ as a module over $\kappa \otimes_\kappa \bar{A}_{r-1} = \bar{A}_{r-1}$. Because $1 \in \mathcal{B}$, this allows us to write $\bar{A}'_{m-1} = \bar{A}_{r-1} \oplus F$, where F is a free (hence torsion free) module over $\bar{A}_{r-1}$. It easily follows that $\bar{A}'_{m-1} \cap K_{r-1} = \bar{A}_{r-1}$, which completes the proof. □

2.11. **Corollary.** *Let R be a DVR containing a field $\mathbf{k}$ of characteristic zero and such that* $\mathrm{Frac}(R)$ *is a function field in one variable over $\mathbf{k}$, and let t be a uniformizing parameter of R. Then (R, t) is a simple pair.*

Proof. Write $K = \mathrm{Frac}(R) = \mathbf{k}(\alpha_1, \dots, \alpha_s)$ (for some $\alpha_1, \dots, \alpha_s$), let Ω be an algebraically closed field containing K, let $\bar{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$ in Ω and let $L = \bar{\mathbf{k}}(\alpha_1, \dots, \alpha_s)$; note that L is a function field in one variable over $\bar{\mathbf{k}}$. Choose a DVR R' of L satisfying $R' \supseteq R$ and $R \cap t'R' = tR$, where t' is a uniformizing parameter of R'; then we have $(R, t) \prec (R', t')$. Since R' is a DVR satisfying $\bar{\mathbf{k}} \subseteq R'$ and $R'/t'R' = \bar{\mathbf{k}}$, we also have $R' \subseteq \bar{\mathbf{k}}[[t']]$, hence $(R', t') \prec (\bar{\mathbf{k}}[[t']], t')$. Now $(\bar{\mathbf{k}}[[t']], t')$ is a simple pair by 2.8, so two applications of 2.10 give that (R, t) is simple. □

3. Proof of the main theorem

By combining results 2.14 and 2.20 of [8] with 2.1 of [6], we obtain:

3.1. *Let $R \subseteq A \subseteq B$ be domains, where R is noetherian and B is finitely generated as an R-algebra. Then the following are equivalent:*

(1) *A is finitely generated as an R-algebra;*
(2) *for every maximal ideal $\mathfrak{M}$ of R, $A_\mathfrak{M}$ is finitely generated as an $R_\mathfrak{M}$-algebra.*

Then it immediately follows:

3.2. **Lemma.** *Let R be a noetherian domain containing $\mathbb{Q}$, B a finitely generated overdomain of R and $D : B \to B$ a locally nilpotent R-derivation. Then the following conditions are equivalent:*

(1) $\ker D$ *is finitely generated as an R-algebra;*

(2) *for every maximal ideal $\mathfrak{M}$ of R, the kernel of $D_{\mathfrak{M}}$ is finitely generated as an $R_{\mathfrak{M}}$-algebra (where $D_{\mathfrak{M}} : B_{\mathfrak{M}} \to B_{\mathfrak{M}}$ is the $R_{\mathfrak{M}}$-derivation obtained by localizing D).*

Proof. Let $A = \ker D$ and recall that if $S \subseteq A \setminus \{0\}$ is multiplicatively closed, the derivation $S^{-1}D : S^{-1}B \to S^{-1}B$ has kernel $S^{-1}A$; the result follows from 3.1. □

In view of the above statement, it suffices to prove 1.1 in the special case where R is a DVR. By 2.11, it's enough to prove the following:

3.3. **Proposition.** *Let R be a DVR containing $\mathbb{Q}$, let $B = R[X,Y,Z] = R^{[3]}$ and let $D : B \to B$ be an R-derivation such that $DX \in R$, $DY \in R[X]$ and $DZ \in R[X,Y]$. Suppose that (R,t) is a simple pair, where t is a uniformizing parameter of R. Then* $\ker D$ *is a finitely generated R-algebra.*

Proof. If $DX = 0$ then D is an $R[X]$-derivation of $B = R[X]^{[2]}$, and is locally nilpotent, so $\ker D = R[X]^{[1]}$ (by 2.1 of [3], for instance). From now on, we assume that $DX \neq 0$. Consider the localization $D_t : B_t \to B_t$ of D at $\{1, t, t^2, \dots\}$. Then $s = X/DX \in B_t$ is a "slice" of D_t, i.e., $D_t(s) = 1$. In this situation, it is known (see for instance 2.1 of [4]) that the exponential of $-sD_t$ is a surjective homomorphism of R_t-algebras (which we denote ξ):

$$\xi : B_t \to \ker(D_t) = (\ker D)_t, \quad w \mapsto \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} D_t^n(w).$$

We have $(\ker D)_t = R_t[\xi(X), \xi(Y), \xi(Z)] = R_t[0, Y + F(X), Z + G(X,Y)]$, where $F(X) \in R_t[X]$ and $G(X,Y) \in R_t[X,Y]$. Multipliying the generators by suitable powers of t, we obtain polynomials

$$\begin{aligned} f_0 &= f_0^*(X) + t^m Y \qquad (m \in \mathbb{N},\ f_0^*(X) \in R[X]) \\ f_1 &= f_1^*(X,Y) + t^k Z \qquad (k \in \mathbb{N},\ f_1^*(X,Y) \in R[X,Y]) \end{aligned}$$

satisfying:

$$R[f_0, f_1] \subseteq \ker D \quad \text{and} \quad R[f_0, f_1]_t = (\ker D)_t,$$

and it is easy to arrange that the image $\bar{f}_0$ of f_0 in $\bar{B} = \kappa[X,Y,Z] = \kappa^{[3]}$ is nonconstant (where $\kappa = R/tR$ as usual).

If $m = 0$, set $R' = R[f_0] = R^{[1]}$; then D is an R'-derivation of $B = R'[X,Z]$, so $\ker D = (R')^{[1]}$, again by 2.1 of [3].

From now on, assume that $m > 0$, so $\bar{f}_0 \in \kappa[X]$. Let E be the set of simple sequences $f' = (f'_0, \dots, f'_r)$ of (R,t,B) satisfying $f'_0 = f_0$ and $f'_1 = f_1$. Note that E is nonempty, since $(f_0, f_1) \in E$. By 2.6, we have $r \leq k+1$ for all $f' = (f'_0, \dots, f'_r) \in E$; so there exists $f = (f_0, \dots, f_r) \in E$ with r maximal; this f cannot be extendable and, by 2.5, it cannot be obstructed either. So f is transcendental and, by 2.3,

$$A^+ = A, \quad \text{where } A = R[f_0, \dots, f_r].$$

Consequently, $A \cap tB = tA$. Together with $A \subseteq \ker D$ and $A_t = (\ker D)_t$, this gives $\ker D = A$. □

The proof of Theorem 1.1 is now complete.

Remark. If R is a DVR of characteristic zero obtained by localizing an affine domain at a prime ideal of height one, then R satisfies the hypothesis of Theorem 1.1, i.e., R is the localization of an affine Dedekind domain at a maximal ideal.

Indeed, suppose that $R = B_{\mathfrak{p}}$, where B is an affine domain over a field $\mathbf{k}_1$ of characteristic zero and $\mathfrak{p}$ is a height one prime ideal of B. By Noether's Normalization Lemma, there exists a subring $A = \mathbf{k}_1[x_1, \dots, x_d] \cong \mathbf{k}_1^{[d]}$ of B such that B is integral over A and $\mathfrak{p} \cap A = x_d A$. Then the field $\mathbf{k} = \mathbf{k}_1(x_1, \dots, x_{d-1})$ is contained in R and consequently $\mathbf{k}[B] \subseteq R$. The fact that $\mathbf{k}[B]$ is $\mathbf{k}$-affine and one-dimensional implies that its integral closure N in $\mathrm{Frac}(\mathbf{k}[B])$ is a $\mathbf{k}$-affine Dedekind domain. We have $B \subseteq N \subseteq R$, so R is the localization of N at $\mathfrak{M} \cap N$, where $\mathfrak{M}$ is the maximal ideal of R.

References

1. D. Daigle and G. Freudenburg, *A counterexample to Hilbert's Fourteenth Problem in dimension five*, J. Algebra **221** (1999), 528–535.
2. ______, *A note on triangular derivations of* $k[x_1, x_2, x_3, x_4]$, To appear in Proc. Amer. Math. Soc.
3. ______, *Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of* $k[X_1, \dots, X_n]$, J. of Algebra **204** (1998), 353–371.
4. J. K. Deveney and D. R. Finston, *Fields of G_a invariants are ruled*, Canad. Math. Bull. **37** (1994), 37–41.
5. A. van den Essen, *An algorithm to compute the invariant ring of a G_a-action on an affine variety*, J. Symbolic Computation **16** (1993), 551–555.
6. J.M. Giral, *Krull dimension, transcendence degree and subalgebras of finitely generated algebras*, Arch. Math. **36** (1981), 305–312.
7. M. Miyanishi, *Normal affine subalgebras of a polynomial ring*, Algebraic and Topological Theories—to the memory of Dr. Takehiko MIYATA, Kinokuniya, 1985, pp. 37–51.
8. N. Onoda, *Subrings of finitely generated rings over a pseudogeometric ring*, Japan J. Math. **10** (1984), 29–53.
9. A. Sathaye, *An application of generalized Newton Puiseux Expansions to a conjecture of D. Daigle and G. Freudenburg*, preprint, 2000.

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada K1N 6N5

E-mail address: daniel@mathstat.uottawa.ca

Department of Mathematics, University of Southern Indiana, Evansville, IN 47712, U.S.A.

E-mail address: freudenb@usi.edu