A NECESSARY AND SUFFICIENT CONDITION FOR TRIANGULABILITY OF DERIVATIONS OF $k[X, Y, Z]$

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ABSTRACT. We give a triangulability criterion for derivations of $\mathbf{k}[X, Y, Z]$. As an application, we give an example of two locally nilpotent derivations D and T of $\mathbf{k}[X, Y, Z]$, where T is triangulable but D is not, with equality of the ideals of fixed points: $(DX, DY, DZ) = (TX, TY, TZ)$.

Throughout this paper, $\mathbf{k}$ is a field of characteristic zero and $R = \mathbf{k}^{[n]}$, with $n = 3$ at some places (if B is an A-algebra then the notation $B = A^{[n]}$ means that B is A-isomorphic to the polynomial ring in n variables over A). A *coordinate system* of R is an ordered n-tuple $(X_1, \dots, X_n)$ of elements of R such that $\mathbf{k}[X_1, \dots, X_n] = R$.

A $\mathbf{k}$-derivation $D : R \to R$ is *locally nilpotent* if for each $x \in R$ there exists an integer $s > 0$ such that $D^s(x) = 0$; D is *triangulable* if there exists a coordinate system $(X_1, \dots, X_n)$ of R such that $D(X_1) \in \mathbf{k}$ and $D(X_i) \in \mathbf{k}[X_1, \dots, X_{i-1}]$ for $2 \le i \le n$. Note that all triangulable derivations are locally nilpotent.

As is well-known, to give a locally nilpotent derivation D of R is equivalent to giving an algebraic action α of $\mathrm{G}_a(\mathbf{k})$ on the affine space $\mathbf{A}^n_{\mathbf{k}}$ (given $t \in \mathbf{k}$, let $\alpha(t) : \mathbf{A}^n \to \mathbf{A}^n$ be the map determined by the automorphism $\exp(tD)$ of R). Under that correspondence, triangulable derivations correspond to triangulable actions and, moreover, the set of fixed points of α is the closed set defined by the ideal (DR) of R (i.e., the ideal generated by the image of D). This motivates:

Definition . Let B be an integral domain of characteristic zero and let $D : B \to B$ be a locally nilpotent derivation. Then the closed subset $V(DB)$ of $\operatorname{Spec} B$ will be denoted $\operatorname{Fix}(D)$, and will be called the *set of fixed points of D*.

The first example of a non-triangulable G_a-action on an affine space was given by Bass in [1]. Since then, some attempts have been made to find triangulability criteria for derivations, and in particular to relate triangulability to some properties of fixed points (see [5], [7], [4]). It was shown, for instance, that if D is a triangulable derivation then its set of fixed points must be cylindrical and satisfy other conditions. In fact, to prove that a given derivation D is non-triangulable, one usually shows that the fixed points of D can't be those of a triangulable derivation. It is therefore natural to ask: *Does the set of fixed points contain enough information to decide whether the derivation*

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is triangulable?[1] Our Example 3.5 answers this negatively when $n = 3$. Indeed, we give two derivations D and T of $\mathbf{k}[X, Y, Z]$, where T is triangulable but D is not, and where we have equality of the ideals $(TX, TY, TZ) = (DX, DY, DZ)$. We stress that this means that T and D have the same fixed points *in the strongest possible sense*: if $\mathrm{Fix}\,(D)$ and $\mathrm{Fix}\,(T)$ are regarded as closed subschemes of $\mathbf{A}^3$, i.e., the subschemes determined by the ideals (DR) and (TR) respectively, then these subschemes are equal.

Section 3 gives triangulability criteria for derivations of $\mathbf{k}[X, Y, Z]$, in the form of necessary and sufficient conditions. First, 3.3 reduces the general problem to the special case of irreducible derivations; then 3.4 solves that special case. Example 3.5 is obtained as an application of 3.4; the relative ease with which this example is settled supports the claim that our criteria are usable.

All results of section 3 are corollaries of Theorem 2.5, which asserts that all locally nilpotent derivations of $\mathbf{k}^{[3]}$ are rigid. The notion of rigidity is defined and investigated in the second section. Finally, we note that the proof of 2.5 and of the other results of this paper make substantial use of [3].

1. Preliminaries

We gather here some definitions and results that will be needed.

Throughout this paper, all rings are commutative and have a unit element. If A is a ring then A^* is the group of units of A; if $a \in A$ then $A_a = S^{-1}A$ with $S = \{1, a, a^2, \dots\}$, and if $\mathbf{p}$ is a prime ideal of A then $A_{\mathbf{p}} = S^{-1}A$ with $S = A \setminus \mathbf{p}$.

Let B be an integral domain of characteristic zero. By a derivation "of B" or "on B", we mean a derivation going from B to B. A derivation D of B is *irreducible* if the only principal ideal of B containing the image of D is B itself (or equivalently, if D is not of the form $\alpha D'$ with α a nonunit element of B and D' a derivation of B). If B is a UFD, then every nonzero derivation of B has the form αD_0 where $\alpha \in B$ and D_0 is an irreducible derivation of B; moreover, α and D_0 are unique, up to multiplication by units of B. A derivation D of B is *locally nilpotent* if for each $x \in B$ there exists an integer $s > 0$ such that $D^s(x) = 0$. If this is the case, then it can be shown that $B^* \subseteq \ker D$; in particular, if B is a $\mathbf{k}$-algebra then all locally nilpotent derivations of B are $\mathbf{k}$-derivations.

See [4] for 1.1 and 1.2:

Definition 1.1. Let D be a $\mathbf{k}$-derivation of $R = \mathbf{k}^{[n]}$. The *rank* of D is the least integer $r \geq 0$ for which there exists a coordinate system $(X_1, \dots, X_n)$ of R satisfying $\mathbf{k}[X_1, \dots, X_{n-r}] \subseteq \ker D$.

Remark . If $n > 2$, it is not known whether locally nilpotent derivations of rank n exist on $\mathbf{k}^{[n]}$.

1.2. *Let D be a $\mathbf{k}$-derivation of $R = \mathbf{k}^{[n]}$ of rank 1, and let $(X_1, \dots, X_n)$ be a coordinate system of R satisfying $\mathbf{k}[X_1, \dots, X_{n-1}] \subseteq \ker D$. Then:*

[1] This question is a variation on a comment made by Snow in [7], p. 169.

(1) $\ker D = \mathbf{k}[X_1, \dots, X_{n-1}]$ *;*
(2) *D is locally nilpotent if and only if $DX_n \in \ker D$.*

Locally nilpotent derivations of rank at most 2 are described in the third section of [3]. We summarize some of the results of that paper in paragraphs 1.3–1.6:

Definition 1.3. Given a coordinate system $\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ of $R = \mathbf{k}^{[n]}$ and an element $P \in R = \mathbf{k}[X_1, \dots, X_{n-2}, Y, Z]$, define a $\mathbf{k}$-derivation $\Delta_P^\gamma : R \to R$ by

$$\Delta_P^\gamma = -P_Z \frac{\partial}{\partial Y} + P_Y \frac{\partial}{\partial Z} \ .$$

1.4. *In 1.3, if P is a variable of $\mathbf{k}(X_1, \dots, X_{n-2})[Y, Z]$ and $\gcd_R(P_Y, P_Z) = 1$ then Δ_P^γ is an irreducible locally nilpotent derivation of R of rank 1 or 2.*

1.5. *Let $D \neq 0$ be a locally nilpotent derivation of $R = \mathbf{k}^{[n]}$ of rank at most 2, and let $\gamma = (X_1, \dots, X_{n-2}, Y, Z)$ be any coordinate system of R satisfying $\mathbf{k}[X_1, \dots, X_{n-2}] \subseteq \ker D$. Then:*

(1) $\ker D = \mathbf{k}[X_1, \dots, X_{n-2}, P]$*, for some $P \in R$.*
(2) *If $P \in R$ satisfies (1) then P is a variable of $\mathbf{k}(X_1, \dots, X_{n-2})[Y, Z]$, $\gcd_R(P_Y, P_Z) = 1$, and for some nonzero element α of $\ker D$ we have $D = \alpha\, \Delta_P^\gamma$ (where Δ_P^γ is irreducible and locally nilpotent by 1.4).*
(3) *If D is irreducible then, for some $P \in R$ satisfying (1), we have $D = \Delta_P^\gamma$.*

1.6. *For a locally nilpotent derivation D of $R = \mathbf{k}^{[n]}$ of rank at most 2, the following are equivalent:*

(1) $\operatorname{Fix}(D) = \emptyset$ *;*
(2) *the image of D contains a unit of R ;*
(3) *D is irreducible and of rank 1.*

We will also need the notion of residual variable:

Definition 1.7. Suppose $B = A^{[n]}$. A *variable of B over A* is an element $b \in B$ which satisfies $B = A[b, b_2, \dots, b_n]$ for some $b_2, \dots, b_n \in B$.[2] A *residual variable of B over A* is an element $b \in B$ such that, for each $\mathbf{p} \in \operatorname{Spec} A$, the image of b in $\kappa(\mathbf{p}) \otimes_A B$ is a variable of $\kappa(\mathbf{p}) \otimes_A B = \kappa(\mathbf{p})^{[n]}$. Here, $\kappa(\mathbf{p})$ denotes the field $A_{\mathbf{p}}/\mathbf{p}A_{\mathbf{p}}$.

The following is a part of Theorem B of [2], with a different notation:

1.8. *Let A be a noetherian ring such that either A contains the field of rationals or A_{red} is seminormal. Let $B = A^{[2]}$ and $b \in B$. Then:*

$$b \text{ is a variable of } B \text{ over } A \Leftrightarrow b \text{ is a residual variable of } B \text{ over } A.$$

[2] When A is a field we may simply speak of a variable of B, with no mention of A, because A is determined by B ($A = \{0\} \cup B^*$).

2. Rigidity of a derivation

We denote by $\Gamma(R)$ the set of coordinate systems of R. Given a $\mathbf{k}$-derivation D of R of rank r, let $\Gamma_D(R)$ be the set of $(X_1,\dots,X_n)\in\Gamma(R)$ satisfying $\mathbf{k}[X_1,\dots,X_{n-r}]\subseteq\ker D$. Then we may give:

Definition 2.1. Let D be a $\mathbf{k}$-derivation of $R=\mathbf{k}^{[n]}$ of rank r. We say that D is *rigid* if

$$\mathbf{k}[X_1,\dots,X_{n-r}]=\mathbf{k}[X'_1,\dots,X'_{n-r}]$$

holds whenever $(X_1,\dots,X_n)$ and $(X'_1,\dots,X'_n)$ are members of $\Gamma_D(R)$.

It is trivial that derivations of rank 0 or n are rigid, and 1.2 implies rigidity for derivations of rank 1. We record:

2.2. *If D is a $\mathbf{k}$-derivation of $\mathbf{k}^{[n]}$ of rank 0, 1 or n, then D is rigid.*

Definition 2.3. Let B be a domain of characteristic zero, $D:B\to B$ a locally nilpotent derivation of B, $A=\ker D$, and let $\pi:\operatorname{Spec}B\to\operatorname{Spec}A$ be the morphism determined by the inclusion map $A\hookrightarrow B$. Given a prime ideal $\alpha\in\operatorname{Spec}A$, let $\kappa(\alpha)$ denote the field $A_\alpha/\alpha A_\alpha$. Then the fibre $\pi^{-1}(\alpha)$ is, in a natural way, a scheme over $\kappa(\alpha)$. We define

$$\begin{aligned}\mathcal{R}(D)&=\{\alpha\in\operatorname{Spec}A\mid\pi^{-1}(\alpha)\neq\mathbf{A}^1_{\kappa(\alpha)}\}\\&=\{\alpha\in\operatorname{Spec}A\mid\kappa(\alpha)\otimes_A B\neq\kappa(\alpha)^{[1]}\}.\end{aligned}$$

We will also consider the closure $\overline{\mathcal{R}}(D)$ of $\mathcal{R}(D)$ in $\operatorname{Spec}A$.

Remark . In the above definition, if $D=0$ then $\mathcal{R}(D)=\overline{\mathcal{R}}(D)=\operatorname{Spec}A$. If $D\neq0$, and if we assume that B contains the field of rationals, then (as is well-known) there exists $a\in A\setminus\{0\}$ such that $B_a=(A_a)^{[1]}$; thus $\overline{\mathcal{R}}(D)$ is contained in the proper closed subset $V(a)$ of $\operatorname{Spec}A$.

Having given those general definitions, we now specialize to derivations of $\mathbf{k}^{[3]}$.

Lemma 2.4. *If D is a locally nilpotent derivation of $R=\mathbf{k}^{[3]}$ of rank 2, then*

1. $\dim\overline{\mathcal{R}}(D)=1$.
2. *Let ξ be an irreducible element of $A=\ker D$ such that $V(\xi)$ is an irreducible component of $\overline{\mathcal{R}}(D)$. Then, for every $(X,Y,Z)\in\Gamma_D(R)$, $\mathbf{k}[X]$ is the integral closure of $\mathbf{k}[\xi]$ in R. In particular, D is rigid.*

As an immediate corollary of 2.2 and 2.4, we obtain

Theorem 2.5. *All locally nilpotent derivations of $\mathbf{k}^{[3]}$ are rigid.*

Proof of 2.4. Let $\gamma=(X,Y,Z)\in\Gamma_D(R)$. Since D is locally nilpotent and $\operatorname{rank}D=2$, 1.5 implies that

$$A=\mathbf{k}[X,P]\tag{1}$$

for some $P\in R$ which also satisfies

$$P\text{ is a variable of }\mathbf{k}(X)[Y,Z].\tag{2}$$

Note, moreover, that if P were a variable of R over $\mathbf{k}[X]$ (which means that $R = (\mathbf{k}[X])[P,Q]$ for some Q) then (1) and the definition of rank would imply that $\operatorname{rank} D = 1$, contradicting one of our assumptions. Thus

$$P \text{ is not a variable of } R \text{ over } \mathbf{k}[X]. \tag{3}$$

By (2), there exist $h \in \mathbf{k}[X] \setminus \{0\}$ and $Q \in R$ such that $R_h = (\mathbf{k}[X]_h)[P,Q]$. In particular, $R_h = (A_h)^{[1]}$ and consequently $\mathcal{R}(D)$ is disjoint from the open subset $\operatorname{Spec} A_h$ of $\operatorname{Spec} A$. Hence

$$\overline{\mathcal{R}}(D) \subseteq V(h), \text{ for some } h \in \mathbf{k}[X] \setminus \{0\}, \tag{4}$$

which shows, in particular, that $\dim \overline{\mathcal{R}}(D) \leq 1$.

We now show that $\dim \overline{\mathcal{R}}(D) > 0$. From (3) and 1.8, we see that P is not a residual variable of R over $\mathbf{k}[X]$. In view of (2), this means that there exists a nonzero prime ideal $\mathbf{p}$ of $\mathbf{k}[X]$ satisfying the following condition: write

$$\Lambda = \mathbf{k}[X]_{\mathbf{p}}/\mathbf{p}\mathbf{k}[X]_{\mathbf{p}} = \mathbf{k}[X]/\mathbf{p} \quad \text{and} \quad \overline{R} = \Lambda \otimes_{\mathbf{k}[X]} R = R/\mathbf{p}R = \Lambda^{[2]}, \tag{5}$$

and given $r \in R$ let $\overline{r}$ denote its image in $\overline{R}$; then we require that $\mathbf{p}$ satisfies

$$\overline{P} \text{ is not a variable of } \overline{R} = \Lambda[\overline{Y}, \overline{Z}]. \tag{6}$$

We claim that the prime ideal $\alpha = \mathbf{p}A$ of A belongs to $\mathcal{R}(D)$. First, notice that the inclusions $\mathbf{k}[X] \hookrightarrow A \hookrightarrow R$ give rise to $\Lambda \hookrightarrow A/\alpha \hookrightarrow \overline{R}$, where $A/\alpha = \Lambda[\overline{P}] \subset \overline{R}$. Thus the residue field $\kappa(\alpha) = \operatorname{qt}(A/\alpha)$, which is used in the definition of $\mathcal{R}(D)$, may be identified with the subfield $\Lambda(\overline{P})$ of $\operatorname{qt}(\overline{R})$, and the $\kappa(\alpha)$-algebra $\kappa(\alpha) \otimes_A R$ may be identified with the $\Lambda(\overline{P})$-algebra $\Lambda(\overline{P})[\overline{Y}, \overline{Z}] \subset \operatorname{qt}(\overline{R})$.

So $\kappa(\alpha) \otimes_A R = \kappa(\alpha)^{[1]} \Leftrightarrow \overline{P}$ is "generically a line" (i.e., $\Lambda(\overline{P})[\overline{Y}, \overline{Z}] = \Lambda(\overline{P})^{[1]}$); as is well-known (see 2.4.2 of [6] for instance) $\overline{P}$ is generically a line if and only if it is a variable of $\Lambda[\overline{Y}, \overline{Z}]$, which is not the case by (6). Hence $\alpha \in \mathcal{R}(D)$ and assertion 1) follows from this.

Finally, if ξ is as in assertion 2) then $V(\xi) \subseteq V(h)$ by (4), so ξ is a prime factor of $h \in \mathbf{k}[X]$. It follows that $\mathbf{k}[\xi] \subseteq \mathbf{k}[X]$, and that $\mathbf{k}[X]$ is the integral closure of $\mathbf{k}[\xi]$ in R. $\square$

3. Triangulable derivations of $\mathbf{k}^{[3]}$

Throughout this section we let $R = \mathbf{k}^{[3]}$, where $\mathbf{k}$ continues to be an arbitrary field of characteristic zero.

Definition 3.1. Let D be a $\mathbf{k}$-derivation of $R = \mathbf{k}^{[3]}$.

(1) D is *triangular*, with respect to a coordinate system $\gamma = (X,Y,Z) \in \Gamma(R)$, if $DX \in \mathbf{k}$, $DY \in \mathbf{k}[X]$ and $DZ \in \mathbf{k}[X,Y]$.
(2) D is *triangulable* if it is triangular with respect to some $\gamma \in \Gamma(R)$.

The aim of this section is to give a triangulability criterion for $\mathbf{k}$-derivations of R. We point out that we may restrict ourselves to locally nilpotent derivations of rank 2. In fact, the following is well-known and easy:

3.2. *Let D be a $\mathbf{k}$-derivation of $R = \mathbf{k}^{[3]}$.*

1. *If D is triangulable, then it is locally nilpotent and has rank at most* 2.
2. *If D is locally nilpotent and* $\operatorname{rank} D \leq 1$, *then D is triangulable.*

The usual strategy, for constructing a non-triangulable locally nilpotent derivation of $\mathbf{k}^{[n]}$, is to start from a triangulable derivation D and multiply it by a cleverly chosen element α of $\ker D$. We make two comments about this strategy:

- Some non-triangulable locally nilpotent derivations cannot be obtained in this way. Indeed, 4.3 of [3] is an example of a non-triangulable locally nilpotent derivation of $\mathbf{k}^{[3]}$ which is also irreducible; 3.5, below, is another such example.
- When n=3, it is no longer needed to be clever when choosing α: 3.3, below, tells us exactly how to do it.

Corollary 3.3. *Let D be a $\mathbf{k}$-derivation of $R = \mathbf{k}^{[3]}$ of rank* 2.

1. *If D is not triangulable then, for all nonzero $\alpha \in \ker D$, αD is not triangulable.*
2. *Suppose that D is triangular with respect to $(X, Y, Z) \in \Gamma(R)$. Then for all $\alpha \in \ker D$ we have:*
$$\alpha D \text{ is triangulable } \Leftrightarrow \alpha \in \mathbf{k}[X].$$

Proof. Only "$\Rightarrow$" of part (2) requires a proof. Suppose that D is triangular with respect to $\gamma = (X, Y, Z)$, and that αD is triangulable. Note that we may assume that $\alpha \neq 0$. Let $\gamma' = (X', Y', Z') \in \Gamma(R)$ be such that αD is triangular with respect to γ'.

Since $\alpha D(X') \in \mathbf{k}$ and $\alpha \neq 0$, we have $D(X') \in \mathbf{k}$; since the condition $D(X') \in \mathbf{k}^*$ implies $\operatorname{rank} D = 1$ by 1.6, we actually have $D(X') = 0$. Similarly, $D(X) = 0$. Hence γ and γ' are two members of $\Gamma_D(R)$; since D is rigid by 2.5, we obtain $\mathbf{k}[X'] = \mathbf{k}[X]$.

Note that $D(Y') \neq 0$, for otherwise $D(X') = 0 = D(Y')$ would imply that $\operatorname{rank} D = 1$, which is false. Thus the condition $\alpha D(Y') \in \mathbf{k}[X']$ implies that $\alpha \in \mathbf{k}[X'] = \mathbf{k}[X]$. □

In view of the above result, the real question is: *Let D be a locally nilpotent derivation of R, and assume that D has rank 2 and is irreducible. How does one decide whether D is triangulable?* Corollary 3.4, below, answers this question.

Remark . The following terminology is used, in the second condition of 3.4. Let A be a ring, let $B = A^{[2]}$, and let $D : B \to B$ be an A-derivation. We say that D is *triangulable over A* if there exist $b_1, b_2 \in B$ such that $B = A[b_1, b_2]$, $Db_1 \in A$ and $Db_2 \in A[b_1]$.

Corollary 3.4. *Let D be an irreducible, locally nilpotent derivation of $R = \mathbf{k}^{[3]}$, of rank at most 2. Let $\gamma = (X, Y, Z) \in \Gamma(R)$ be such that $DX = 0$ and write $D = \Delta_P^\gamma$ as in 1.5. Then the following are equivalent:*

1. *D is triangulable;*
2. *D is triangulable over $\mathbf{k}[X]$;*

(3) *there exists a variable* Q *of* R *over* $\mathbf{k}[X]$ *such that* $\mathbf{k}(X)[P,Q] = \mathbf{k}(X)[Y,Z]$.

Proof. It is obvious that (2) implies (1), and the equivalence of (2) and (3) is a special case of Lemma 2.8 of [3]. We prove that (1) implies (2).

Suppose that D is triangulable. If $\operatorname{rank} D = 1$ then, by 1.2, $R = \mathbf{k}[X,P,Q]$ for some Q, and $DQ \in \ker D = \mathbf{k}[X,P]$. Thus D is triangular with respect to (X,P,Q) and, consequently, triangulable over $\mathbf{k}[X]$.

So we may assume that $\operatorname{rank} D = 2$. Consider $(X',Y',Z') \in \Gamma(R)$ such that $DX' \in \mathbf{k}$, $DY' \in \mathbf{k}[X']$ and $DZ' \in \mathbf{k}[X',Y']$. As in the proof of 3.3, we observe that $DX' = 0$. Thus (X',Y',Z') and (X,Y,Z) are two members of $\Gamma_D(R)$. Since D is rigid by 2.5, we obtain $\mathbf{k}[X'] = \mathbf{k}[X]$; since D is triangulable over $\mathbf{k}[X']$, (2) holds. □

Example 3.5. Choose $\gamma = (X,Y,Z) \in \Gamma(R)$ and define $P = Y + \frac{1}{4}(XZ+Y^2)^2$. Then P is a variable of $\mathbf{k}(X)[Y,Z]$ and

$$P_Y = 1 + Y(XZ+Y^2), \qquad P_Z = \frac{1}{2}X(XZ+Y^2)$$

are relatively prime in $\mathbf{k}[X,Y,Z]$. By 1.4, the derivation $D = \Delta_P^\gamma$ is locally nilpotent, irreducible, and has rank at most 2. Using 3.4, we now show that D is not triangulable.

Suppose that D is triangulable. Then, by 3.4, there exists a variable Q of R over $\mathbf{k}[X]$ which satisfies $\mathbf{k}(X)[P,Q] = \mathbf{k}(X)[Y,Z]$. Thus $\mathbf{k}(X)[P,Q] = \mathbf{k}(X)[P, XZ+Y^2]$ and consequently $Q = \lambda(XZ+Y^2) + g(P)$ for some $\lambda \in \mathbf{k}(X)^*$ and $g(W) \in \mathbf{k}(X)[W]$, where W is an indeterminate. Multiplying by a suitable element of $\mathbf{k}(X)^*$ yields

$$a_0 Q = a_1(XZ+Y^2) + G(P), \tag{7}$$

where $a_0,\ a_1 \in \mathbf{k}[X] \setminus \{0\}$, $G(W) = \sum_{i=0}^{d} r_i W^i$ $\ (r_i \in \mathbf{k}[X])$ and

$$\gcd(a_0, a_1, r_0, \dots, r_d) = 1 \ \text{ in } \mathbf{k}[X]. \tag{8}$$

Since Q is a variable of R over $\mathbf{k}[X]$, it follows that $\overline{Q} = Q(0,Y,Z)$ is a variable of $\mathbf{k}[Y,Z]$. Moreover, evaluating (7) at $X = 0$ yields

$$a_0(0)\overline{Q} = a_1(0)Y^2 + \overline{G}(Y + \tfrac{1}{4}Y^4), \tag{9}$$

where we write $\overline{G} = \sum r_i(0)\, W^i \in \mathbf{k}[W]$.

If $a_0(0) = 0$ then (9) yields $a_1(0) = 0$ and $\overline{G} = 0$, contradicting (8). So $a_0(0) \neq 0$ and $a_1(0)Y^2 + \overline{G}(Y + \frac{1}{4}Y^4)$ is a variable of $\mathbf{k}[Y,Z]$, which is absurd.

Hence D is not triangulable.

An interesting observation is that there exists a triangulable derivation T of R such that $(DR) = (TR)$. Indeed, define T by $TX = 0$, $TY = X$ and $TZ = 1 + Y^3$, then $(TR) = (X, 1+Y^3) = (DR)$.

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