AFFINE RULINGS OF NORMAL RATIONAL SURFACES

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Given an algebraic surface X satisfying:

($\dagger$) X is a complete normal rational surface, X is affine ruled and $\text{rank}(\text{Pic } X_s) = 1$,

where X_s denotes the smooth locus of X, consider:

Problem 1. Find all affine rulings of X.

Problem 2. Find all pairs of curves C_1, C_2 on X such that $X \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{P}^2$ minus two lines.

Problem 3. Find all curves C in X such that $\bar{\kappa}(X_s \setminus C) = -\infty$.

This paper investigates Problem 1 for an arbitrary X satisfying ($\dagger$). We define (1.14) the notion of a "basic" affine ruling of X and our main results describe how to construct all affine rulings of X, assuming that the basic ones are known. In the case where X is a weighted projective plane, the basic affine rulings of X are given in [6]; the present paper and [6] therefore constitute a solution to Problem 1 in that case.

Problem 3 (with $X = \mathbb{P}^2$) has been considered by several authors ([8], [9], [18], [19], [14]). In his review of [14] (see MR 82k:14013), M. H. Gizatullin mentions some unpublished examples found by V. I. Danilov and himself, and which seem to correspond to the list of basic affine rulings of $\mathbb{P}^2$. The case $X = \mathbb{P}^2$ was finally solved in [10]. Our generalization to weighted projective planes seems to be new, as well as our method—valid for any X satisfying ($\dagger$)—which reduces the general problem to the determination of the *basic* affine rulings.

Let us briefly indicate how problems 1–3 are related to each other. Consider the stronger condition ($\ddagger$) on a surface X:

($\ddagger$) X satisfies ($\dagger$) and every singular point of X is a cyclic quotient singularity.

As an example, note that the weighted projective planes satisfy ($\ddagger$) (they even satisfy $\text{Pic}(X_s) = \mathbb{Z}$; see [6] for these claims). Also note the following by-product of section 1: *A surface satisfying* ($\ddagger$) *cannot have more than* 3 *singular points* (see Corollary 1.16).

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It is clear that any solution (C_1, C_2) to Problem 2 gives rise to an affine ruling of X; by Theorem 1.15, the converse holds if X satisfies (‡), so:

For any surface X satisfying (‡), Problems 1 and 2 are equivalent.

The exact relation between Problem 3 and the other two is given by the following statement, which will be proved in 1.17, below: *Given X satisfying (‡) and a curve C on X, the following are equivalent:*

(i) $\bar{\kappa}(X_s \setminus C) = -\infty$;

(ii) *there exists at least one affine ruling*[1] Λ *of* X *such that* $nC \in \Lambda$ *for some* $n > 0$.

For instance, if $C \subset \mathbb{P}^2$ is Yoshihara's rational quintic ([19], Proposition 3, case $N = 1$), then infinitely many affine rulings Λ of $\mathbb{P}^2$ contain multiples of C.

By way of motivation, we now explain the connection between problems 1–3 and locally nilpotent derivations. Consider the polynomial ring $B = \mathbf{k}[X_1, X_2, X_3]$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. It is known ([13], [3]) that describing the locally nilpotent derivations $D : B \to B$ is equivalent to answering: *Which pairs of polynomials $f, g \in B$ have the property that $\mathbf{k}[f, g]$ is the kernel of a locally nilpotent derivation of B?* If we restrict ourselves to the case where D is (or equivalently f and g are) homogeneous with respect to weights $w(X_i) = a_i$, where a_1, a_2, a_3 are relatively prime positive integers, then we can think of f and g as defining curves in the weighted projective plane $\mathbb{P}(a_1, a_2, a_3) = \operatorname{Proj} B$; then [4] gives the following result:

Theorem. *For w-homogeneous elements $f, g \in B$ satisfying $\gcd(w(f), w(g)) = 1$, the following are equivalent:*

(1) *There exists a w-homogeneous locally nilpotent derivation D of B such that* $\ker D = \mathbf{k}[f, g]$;

(2) *f and g are irreducible elements of B and the algebraic surface* $\operatorname{Proj} B \setminus V(fg)$ *is isomorphic to $\mathbb{P}^2$ minus two lines.*

Note that the case where $\gcd(w(f), w(g)) \neq 1$ turns out to be very special, and is completely described in [4]. Hence, solving Problem 2 for $X = \mathbb{P}(a_1, a_2, a_3)$ is equivalent to describing homogeneous locally nilpotent derivations of B. Since that class of derivations is not well understood, and corresponds to a class of G_a-actions on $\mathbb{A}^3$ which ought to be understood, there is ample reason to study affine rulings.

Organization of the text

Fix a surface X satisfying (†).

Section 1 contains generalities about affine rulings of X.

Section 2 defines a process which is used to modify affine rulings of X (i.e., applying it to an affine ruling of X produces a different affine ruling of X). The process makes its first appearance in the proof of Theorem 2.1, where it is shown that every non-basic

[1]According to the definition of "affine ruling" adopted in 1.1, below, Λ is a linear system of X, so it makes sense to write $nC \in \Lambda$.

affine ruling of X can be "reduced" to a simpler one; this reduction process is in fact a special case of the modification process.

Some preparation is necessary before defining the modification process: 2.2 defines the notion of an "X-immersion"; then 2.3–2.8 show that each X-immersion determines an affine ruling of X, that each affine ruling can be obtained in this way, and that this can be turned into a bijective correspondence, modulo appropriate adjustments.

Given an X-immersion I, 2.9 defines a set $\Pi(I)$ and a new X-immersion $I * \pi$ for each $\pi \in \Pi(I)$. This operation $*$ is the modification process which was announced; it acts on X-immersions, so it indirectly modifies affine rulings via the correspondence mentioned in the preceding paragraph. Discussion 2.14 summarizes the results of section 2. In particular, it states that all affine rulings of X can be constructed from the basic ones by using the $*$ operation; and consequently the solution of Problem 1 consists of two parts:

(1a) Make a list of all basic affine rulings of X.
(1b) For each X-immersion I, describe the set $\Pi(I)$.

Problem (1b) is essentially a problem in the theory of weighted graphs, independent of the surface, and is completely solved in sections 3 and 4: section 3 does the graph theory and section 4 states the consequences for $\Pi(I)$. This paper does not solve Problem (1a), which is highly dependent on the surface X; [6] solves it for the weighted projective planes.

In contrast with sections 2 and 4, where rulings are described by saying that they can be constructed from basic ones by using the modification process, section 5 gives direct information on affine rulings. The main result of that section is Theorem 5.13; it is complemented by several other (more practical) statements, notably 5.17, 5.22, 5.23, 5.25, 5.34, 5.40.

Conventions

All curves and surfaces considered in this paper are assumed to be algebraic varieties over an algebraically closed field $\mathbf{k}$ of characteristic zero. In particular, curves and surfaces are irreducible and reduced.

If $f : X \to Y$ is a birational morphism of surfaces then the *center* of f (denoted center(f)) is the set of points $y \in Y$ such that $f^{-1}(y)$ contains more than one point.

Let S be a smooth complete surface. If D is a divisor of S then, by a *component* of D, we always mean an irreducible (or prime) component of D. If D and D' are divisors of S then $D \cdot D'$ denotes their intersection number and $D^2 = D \cdot D$. If $C \subset S$ is a smooth rational curve and $C^2 = r$, we call C an *r-curve*; by an *r-component* of a divisor D, we mean a component of D which is an r-curve. A reduced effective divisor D of S has *strong normal crossings* if: (i) each component of D is a smooth curve; (ii) if D_i and D_j are distinct components of D then $D_i \cdot D_j \leq 1$; and (iii) if D_i, D_j and D_k are distinct components of D then $D_i \cap D_j \cap D_k$ is empty.

Except for the graph $\mathbb{L}(X)$ of 2.14, every graph considered in this paper is a weighted graph, i.e., a graph in which each vertex is assigned an integer (called its weight). Every weighted graph in this paper is a finite undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices.

If S is a smooth complete surface and D a divisor of S with strong normal crossings, the *dual graph* of (D, S) is the weighted graph $\mathcal{G} = \mathcal{G}(D, S)$ whose vertices are the components of D; distinct vertices D_i and D_j are joined by an edge if $D_i \cap D_j \neq \emptyset$; and the weight of a vertex D_i is D_i^2. We assume familiarity with this idea, as well as with the basic theory of weighted graphs (their blowing-up, blowing-down and equivalence); the relevant definitions can be found in various sources, for instance [17], [16], or the appendix of [2] (see also the beginning of section 3, in this paper). Let $D_1, \dots, D_n$ be the distinct components of D. We say that D_j is a *neighbor* of D_i if $i \neq j$ and $D_i \cap D_j \neq \emptyset$ (i.e., if the vertices D_i, D_j of $\mathcal{G}$ are neighbors); the number of neighbors of D_i is called its *branching number*; if this number is greater than or equal to 3, we say that D_i is a *branching component* of D (or that the vertex D_i is a *branch point* of $\mathcal{G}$). We say that $\mathcal{G}$ is a *linear chain* (or a *linear tree*) if it is a tree without branch points; an *admissible chain* is a linear chain in which every weight is strictly less than -1; note that the empty graph is an admissible chain. We say that D is a tree (or a linear chain, or an admissible chain, etc) if $\mathcal{G}$ has the corresponding property.

Let X and X^* be complete normal surfaces, β a birational isomorphism between them (either $X \xrightarrow{\beta} X^*$ or $X \xleftarrow{\beta} X^*$) and Λ a one-dimensional linear system on X without fixed components. In this situation, we will often use the fact that Λ and β determine, in a natural way, a one-dimensional linear system Λ^* on X^* without fixed components. The tacit understanding is that, for suitably chosen rational maps $X \xrightarrow{\lambda} \mathbb{P}^1$ and $X^* \xrightarrow{\lambda^*} \mathbb{P}^1$ determining Λ and Λ^* respectively, β, λ and λ^* form a commutative diagram.

The set of nonnegative (resp. positive) integers is denoted $\mathbb{N}$ (resp. $\mathbb{Z}^+$).

1. Preliminaries on affine rulings

1.1. Let X be a complete normal rational surface. An "affine ruling" of X is usually defined to be a morphism $p : U \to \Gamma$ where Γ is a curve, U is a nonempty open subset of X isomorphic to $\Gamma \times \mathbb{A}^1$ and p is the projection $\Gamma \times \mathbb{A}^1 \to \Gamma$. Since $\Gamma \times \mathbb{A}^1$ is normal and rational, Γ is an open subset of $\mathbb{P}^1$ and U is contained in the smooth locus of X. The morphism p extends to a rational map $X \to \mathbb{P}^1$ which, in turn, determines a unique linear system Λ on X without fixed components. Since we do not want to distinguish between rulings which determine the same linear system Λ, *we adopt the viewpoint that* Λ *itself is the affine ruling:*

Definition. Let Λ be a one-dimensional linear system on X without fixed components. We say that Λ is an *affine ruling* of X if there exist nonempty open subsets $U \subset X$ and $\Gamma \subseteq \mathbb{P}^1$ such that $U \cong \Gamma \times \mathbb{A}^1$ and such that the projection morphism $\Gamma \times \mathbb{A}^1 \to \Gamma$ determines Λ.

If Λ is an affine ruling of X then the general member C of Λ satisfies $C \cap U \cong \mathbb{A}^1$; it follows:

- the general member of Λ is irreducible and reduced;
- Λ has at most one base point on X.

In the special case where X is smooth and $\mathrm{Bs}(\Lambda) = \emptyset$, the general member C of Λ satisfies $C \cong \mathbb{P}^1$ and $C^2 = 0$; so 1.2 applies to this situation.

1.2. Let X be a smooth, complete rational surface and C a curve on X satisfying $C \cong \mathbb{P}^1$ and $C^2 = 0$. Then the following facts are well-known (see 2.7.1 of [11] or Lemma 2.2 of [12], p. 115):

(1) The Riemann-Roch Theorem for X implies that the complete linear system $\Lambda = |C|$ has dimension one; since $\mathrm{Bs}(\Lambda) = \emptyset$, Λ gives rise to a morphism $\lambda : X \to \mathbb{P}^1$.
(2) There exists an open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^1$ such that $\lambda^{-1}(\Gamma) \cong \Gamma \times \mathbb{P}^1$ and such that the composition $\lambda^{-1}(\Gamma) \cong \Gamma \times \mathbb{P}^1 \to \Gamma$ is the restriction of λ (i.e., Λ is a $\mathbb{P}^1$-ruling of X).
(3) There exists an irreducible curve $H \subset X$ such that $H \cdot \Lambda = 1$; such a curve H is called a *section* of Λ (or λ). If H is a section then $H \cong \mathbb{P}^1$ and, given Γ satisfying (2) and $\Gamma \neq \mathbb{P}^1$, we have $\lambda^{-1}(\Gamma) \setminus H \cong \Gamma \times \mathbb{A}^1$ and the composition $\lambda^{-1}(\Gamma) \setminus H \cong \Gamma \times \mathbb{A}^1 \to \Gamma$ is the restriction of λ (so Λ is also an affine ruling of X).
(4) If U is any open subset of X isomorphic to $\Gamma \times \mathbb{A}^1$ for some open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^1$, and if the composition $U \cong \Gamma \times \mathbb{A}^1 \to \Gamma$ is compatible with Λ, then $U = X \setminus \mathrm{supp}(H + C_1 + \cdots + C_r)$ for some section H of Λ and for some curves $C_1, \dots, C_r$ where each C_i is contained in some member of Λ.

Let H be a section of Λ, let $m = -H^2$ and, for each reducible[2] member F of Λ, let F° be the unique irreducible component of F which meets H (F° is an integral curve and occurs in F with multiplicity one).

(5) For each reducible member F of Λ, if $F^\sharp$ denotes the reduced effective divisor such that $\mathrm{supp}(F) = \mathrm{supp}(F^\sharp)$ then $F^\sharp$ has strong normal crossings and is a tree of projective lines. Moreover, $F^\sharp$ can be shrunk until only F° remains (F° itself is not shrunk) and, after that contraction, $(F^\circ)^2 = 0$. Note the following consequence: if C is an irreducible component of F and C is branching in $\mathrm{supp}(F + H)$ then $C^2 < -1$.
(6) If all reducible members are shrunk as described in (5), then one obtains the ruled surface $\mathbb{F}_m$. This shrinking process is a birational morphism $\sigma : X \to \mathbb{F}_m$ which maps the members of Λ (resp. H) to the members (resp. the negative section) of the ruling of $\mathbb{F}_m$.

[2]Note that if $F \in \Lambda$ has irreducible support then the condition $F \cdot H = 1$ implies that it is also reduced (i.e., F is an integral curve).

The shrinking processes described in (5) and (6) are uniquely determined by the choice of a section H.

Notation 1.3. If X is a complete normal rational surface and Λ is an affine ruling of X, let X_s be the smooth locus of X and $X' = X_s \setminus \mathrm{Bs}(\Lambda)$. We write $(\bar{X}, \bar{\Lambda}) \succ (X, \Lambda)$ to indicate that $\bar{X}$ is a smooth and complete surface containing X' as an open subset, the complement of X' in $\bar{X}$ is the support of a reduced effective divisor with strong normal crossings, $\bar{\Lambda}$ is a base point free affine ruling of $\bar{X}$ and $\bar{\Lambda}|_{X'}$ is equal to $\Lambda|_{X'}$.

Lemma 1.4. *Let X be a complete normal rational surface and Λ an affine ruling of X and suppose that $(\bar{X}, \bar{\Lambda}) \succ (X, \Lambda)$. Let D be the divisor of $\bar{X}$ with strong normal crossings and whose complement is X'. Then:*

1. *Each connected component of D is a tree of projective lines.*
2. *At most one irreducible component of D is a section of $\bar{\Lambda}$.*
3. *Every irreducible component of D which is not a section of $\bar{\Lambda}$ is contained in a member of $\bar{\Lambda}$.*

Proof. Consider an open subset $U \subset X$ isomorphic to $\Gamma \times \mathbb{A}^1$ (for some open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^1$) and such that the composition $U \cong \Gamma \times \mathbb{A}^1 \to \Gamma$ is compatible with Λ; note that $U \subseteq X'$. Since the complement of $\Gamma \times \mathbb{A}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is a tree of projective lines, and since $\mathrm{supp}(D)$ is contained in $\bar{X} \setminus U$, it easily follows that assertion (1) holds. By part (4) of 1.2 we have $\bar{X} \setminus U = \mathrm{supp}(H + C_1 + \cdots + C_r)$, for some section H of $\bar{\Lambda}$ and for some curves $C_1, \dots, C_r$, where each C_i is contained in some member of $\bar{\Lambda}$; since $\mathrm{supp}(D) \subseteq \mathrm{supp}(H + C_1 + \cdots + C_r)$, (2) and (3) hold. □

Proposition 1.5. *Let X be a complete normal rational surface and Λ an affine ruling of X. Let $X' = X_s \setminus \mathrm{Bs}(\Lambda)$.*

1. *There exists a unique pair $(X, \Lambda)^\sim = (\tilde{X}, \tilde{\Lambda})$ satisfying $(\tilde{X}, \tilde{\Lambda}) \succ (X, \Lambda)$ and the following condition:*
   (*) *Every irreducible component C of $\tilde{X} \setminus X'$ satisfies $C^2 \leq -1$, and if equality holds then C is a section of $\tilde{\Lambda}$.*
2. *Every irreducible component of $\tilde{X} \setminus X'$ which is not a section of $\tilde{\Lambda}$ is contained in some* reducible *member of $\tilde{\Lambda}$.*
3. *Every member of $\tilde{\Lambda}$ meets X'.*
4. *If $(\bar{X}, \bar{\Lambda})$ is any pair satisfying $(\bar{X}, \bar{\Lambda}) \succ (X, \Lambda)$, then there exists a birational morphism $\bar{X} \to \tilde{X}$ which restricts to an isomorphism from $X' \subset \bar{X}$ to $X' \subset \tilde{X}$.*

Proof. We begin by proving (4): assume that $(\tilde{X}, \tilde{\Lambda})$ is any pair satisfying $(\tilde{X}, \tilde{\Lambda}) \succ (X, \Lambda)$ and condition (*), and let $(\bar{X}, \bar{\Lambda})$ be as in assertion (4). There exists a smooth complete surface S and two birational morphisms, $\tilde{\pi} : S \to \tilde{X}$ and $\bar{\pi} : S \to \bar{X}$, such that if we regard $\tilde{\pi}$ (resp. $\bar{\pi}$) as a composition of monoidal transformations then each one of these is centered at a point infinitely near $\tilde{X} \setminus X'$ (resp. $\bar{X} \setminus X'$). We also assume that $(S, \tilde{\pi}, \bar{\pi})$ is minimal, i.e., that the total number of monoidal transformations in $\tilde{\pi}$ and $\bar{\pi}$ is minimal. It suffices to show that $\bar{\pi}$ is an isomorphism.

Assume that $\bar{\pi}$ is not an isomorphism and consider a curve $\Gamma \subset S$ which is first to shrink, in the contraction process going from S to $\bar{X}$. By minimality of $(S, \tilde{\pi}, \bar{\pi})$, Γ is not in the exceptional locus of $\tilde{\pi}$; thus it is the strict transform of some component C of $\tilde{X} \setminus X'$, where $C^2 \geq -1$. Since $(\tilde{X}, \tilde{\Lambda})$ satisfies (*), C must be a section of $\tilde{\Lambda}$. Since $\bar{\pi}(\Gamma)$ is a point, it follows that $\bar{\Lambda}$ has a base point, contradicting $(\bar{X}, \bar{\Lambda}) \succ (X, \Lambda)$. Hence, $\bar{\pi}$ is an isomorphism and (4) is proved.

Note that (4) implies, in particular, that if $(\tilde{X}, \tilde{\Lambda})$ exists then it is unique (up to isomorphism). So, to finish the proof, there remains to construct a pair $(\tilde{X}, \tilde{\Lambda})$ satisfying (1–3).

Consider the minimal resolution of singularities $\hat{X} \to X$ of X and let $\hat{E}$ be the inverse image of the singular points. Then $\hat{X}$ is a smooth complete surface, $\hat{E}$ is a reduced effective divisor of $\hat{X}$ with strong normal crossings and $\hat{X} \setminus \operatorname{supp}(\hat{E}) \to X_s$ is an isomorphism. Arguing as in the proof of 1.4, we see that each connected component of $\hat{E}$ is a tree of projective lines. Moreover, every irreducible component E of $\hat{E}$ satisfies $E^2 \leq -1$, and if $E^2 = -1$ then E is branching in $\hat{E}$.

Then Λ determines an affine ruling $\hat{\Lambda}$ of $\hat{X}$. Let $\rho : \tilde{X} \to \hat{X}$ be the minimal resolution of the base points of $\hat{\Lambda}$ and $\tilde{\Lambda}$ the corresponding base point free linear system on $\tilde{X}$. It is clear that $(\tilde{X}, \tilde{\Lambda}) \succ (X, \Lambda)$; we shall now argue that (*) holds. Let D be the divisor of $\tilde{X}$ with strong normal crossings such that $\tilde{X} \setminus X' = \operatorname{supp}(D)$ and consider a component C of D.

Since D is the union of the strict transform of $\hat{E}$ and of the exceptional locus of ρ, it is clear that $C^2 \leq -1$.

Assume that C is not a section of $\tilde{\Lambda}$. Then Lemma 1.4 implies that C is contained in some member F of $\tilde{\Lambda}$; since $F^2 = 0$ and $C^2 < 0$, F must have reducible support, which proves assertion (2) of the Proposition. There remains to show that $C^2 < -1$. Assume the contrary; then $C^2 = -1$ and, by 1.2, C is not branching in $\operatorname{supp}(H + F)$ for any section H of $\tilde{\Lambda}$.

Suppose that C is the strict transform of some component E of $\hat{E}$. Then $E^2 \geq -1$ in $\hat{X}$; by the properties of $\hat{E}$, $E^2 = -1$ and E is branching in $\hat{E}$. Consider three distinct neighbors E_i ($i = 1, 2, 3$) of E in $\hat{E}$. Since $E^2 = C^2$, we see that the strict transform C_i of E_i meets C in $\tilde{X}$ (for all $i = 1, 2, 3$). Since C_i is a component of $\tilde{X} \setminus X'$, Lemma 1.4 implies that C_i is a section of $\tilde{\Lambda}$ or is contained in some member F_i of $\tilde{\Lambda}$; in the latter case, $C_i \cap C \neq \emptyset$ implies that $F_i = F$. Since at most one C_i can be a section of $\tilde{\Lambda}$ it follows that, for a suitable section H, C_1, C_2, C_3 and C are all contained in $\operatorname{supp}(H + F)$. This contradicts the fact that C is not branching in $\operatorname{supp}(H + F)$, so C is not the strict transform of a component of $\hat{E}$.

Thus C is in the exceptional locus of ρ (and $\hat{\Lambda}$ has a base point). Write $\rho = \rho_r \circ \cdots \circ \rho_1$, where $\rho_i : X_i \to X_{i-1}$ is a monoidal transformation ($r \geq 1$, $X_0 = \hat{X}$, $X_r = \tilde{X}$), and note that the exceptional curve $H \subset \tilde{X}$ of ρ_r is a section of $\tilde{\Lambda}$. By (1.1), there is a unique base point on X_{i-1} ($1 \leq i \leq r$); it follows that the center of ρ_i lies on the exceptional curve of ρ_{i-1} for each $i > 1$, and consequently $C^2 = -1$ implies $C = H$. This contradicts our assumption that C is not a section of $\tilde{\Lambda}$, so we proved that $C^2 < -1$.

To prove (3), suppose that $F \in \tilde{\Lambda}$ satisfies $\operatorname{supp}(F) \subseteq \operatorname{supp}(D)$. Then each component C of F satisfies $C^2 < -1$ because $C \subseteq \operatorname{supp}(D)$ and C is not a section. This contradicts the fact (1.2) that F contracts to a 0-curve (or is a 0-curve). $\square$

Definition 1.6. Suppose that X is a complete normal rational surface and that Λ is an affine ruling of X. Let $X' = X_s \setminus \operatorname{Bs}(\Lambda)$ and consider $(X, \Lambda)^{\sim} = (\tilde{X}, \tilde{\Lambda})$.

For each member F of Λ, let $\tilde{F}$ be the unique element of $\tilde{\Lambda}$ such that $\tilde{F} \cap X' = F \cap X'$; then $F \mapsto \tilde{F}$ defines a bijection $\Lambda \to \tilde{\Lambda}$ (because $\tilde{\Lambda}|_{X'} = \Lambda|_{X'}$ and, by 1.5, each member of $\tilde{\Lambda}$ meets X').

1.7. Let X be a complete normal rational surface and Λ an affine ruling of X. In this paragraph, we relate the rank of $\operatorname{Pic}(X_s)$ to some numbers determined by the pair $(\tilde{X}, \tilde{\Lambda})$ of Proposition 1.5.

Let D be the divisor of $\tilde{X}$ with strong normal crossings such that $\tilde{X} \setminus X' = \operatorname{supp}(D)$. Proposition 1.5 implies that, for a suitable choice of a section H of $\tilde{\Lambda}$, every component C of D satisfies

(i) $C^2 \leq -1$

and one of

(ii) $C = H$

(iii) C is contained in some reducible member of $\tilde{\Lambda}$ and $C^2 < -1$.

Let $m = -H^2$ and let $F_1, \dots, F_s$ be the reducible members of $\tilde{\Lambda}$. For each i, we can write $F_i = F_i^\circ + F_i^\star$ where F_i° is an integral curve, $F_i^\circ \cdot H = 1$, $F_i^\star$ is effective and $F_i^\star \cdot H = 0$. By 1.2, $F_i^\star$ can be shrunk to a point and, if we do this for all $i = 1, \dots, s$, we obtain the ruled surface $\mathbb{F}_m$. Since $\operatorname{Pic}(\mathbb{F}_m)$ is freely generated by a section and a fibre, it follows that

(1) $\operatorname{Pic}(\tilde{X})$ is freely generated by H, a general member F of $\tilde{\Lambda}$ and all components of $F_1^\star, \dots, F_s^\star$.

We write $F_i^\star = F_i' + F_i''$, where F_i' and F_i'' are effective, F_i' contains the components of $F_i^\star$ which meet X' and F_i'' contains those included in $\tilde{X} \setminus X'$. We claim that $F_i' \neq 0$. In fact, consider a component C of $F_i^\star$ satisfying $C^2 = -1$ (such a C exists, since $F_i^\star$ is nonzero and shrinks to a point). Since C satisfies neither (ii) nor (iii), it is not a component of D, so C is contained in F_i'. Hence, $F_i' \neq 0$ for all i.

Observe that

$$\tilde{X} \setminus X' = \operatorname{supp}\left(\delta H + \sum_{i=1}^{s} (F_i'' + \delta_i F_i^\circ)\right), \tag{2}$$

where

$$\delta = \begin{cases} 1, & \text{if } H \cap X' = \emptyset, \\ 0, & \text{if } H \cap X' \neq \emptyset, \end{cases} \quad \text{and} \quad \delta_i = \begin{cases} 1, & \text{if } F_i^\circ \cap X' = \emptyset, \\ 0, & \text{if } F_i^\circ \cap X' \neq \emptyset. \end{cases}$$

In view of (1), (2) and the fact that, for each i, F is linearly equivalent to $F_i = F_i^\circ + F_i' + F_i''$, we obtain that $\mathrm{Pic}(X')$ is the abelian group generated by H, F and all components of $F_1', \dots, F_s'$, with relations:

$$(3) \qquad \begin{aligned} F &= F_i' \qquad \text{(for each i such that } \delta_i = 1\text{) and} \\ H &= 0 \qquad \text{(if } \delta = 1\text{).} \end{aligned}$$

Note that $\mathrm{Pic}(X_s) = \mathrm{Pic}(X')$ and let $k_i \geq 1$ be the number of components of F_i'. We conclude that

$$(4) \qquad \mathrm{rank}(\mathrm{Pic}\ X_s) = (2 - \delta) + \sum_{i=1}^{s} (k_i - \delta_i),$$

where $1 \leq 2 - \delta \leq 2$ and, for all i, $k_i - \delta_i \geq 0$.

Surfaces satisfying the condition (†)

From now-on, we restrict ourselves to the case where X satisfies the condition (†) defined in the introduction.

Proposition 1.8. *Suppose that X satisfies (†), let Λ be an affine ruling of X and consider the pair $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$.*

1. *Λ has one base point on X and exactly one irreducible component H of $\tilde{X} \setminus X'$ is a section of $\tilde{\Lambda}$.*
2. *Every member F of $\tilde{\Lambda}$ has a unique irreducible component C_F which meets X'. Consequently, every member of Λ has irreducible support.*
3. *If F is reducible then $C_F^2 = -1$ and C_F is the only component of F with this property. Moreover, C_F does not meet H, is not branching in* $\mathrm{supp}(F + H)$ *and the multiplicity of C_F in F is strictly greater than* 1.
4. *Under the bijection $\Lambda \to \tilde{\Lambda}$ defined in* 1.6, *the multiple members of Λ correspond to the reducible members of $\tilde{\Lambda}$. If $M = \nu C \in \Lambda$, where $C \subset X$ is a curve and $\nu \in \mathbb{N}$, then ν is equal to the multiplicity of $C_{\tilde{M}}$ in $\tilde{M}$.*
5. *Let $M_i = \nu_i C_i$ $(1 \leq i \leq s)$ be the multiple members of Λ, where $C_i \subset X$ is a curve and $\nu_i > 1$ is an integer, and let M be any member of Λ. Then* $\mathrm{Pic}(X_s)$ *is the abelian group given by $s+1$ generators $M, C_1, \dots, C_s$ and relations $\nu_i C_i = M$ for $i = 1, \dots, s$. In particular,* $\mathrm{Pic}(X_s) = \mathbb{Z}$ *if and only if $s < 2$ or $\nu_1, \dots, \nu_s$ are pairwise relatively prime.*

Proof. Let H be a section of $\tilde{\Lambda}$ satisfying conditions (i–iii) of 1.7; let the notations F_i, F_i°, $F_i^\star$, F_i' and F_i'' be as in 1.7.

We have $1 = (2 - \delta) + \sum_{i=1}^{s} (k_i - \delta_i)$ by equation (4), where $2 - \delta \geq 1$ and $k_i - \delta_i \geq 0$ for all i; thus $\delta = 1$ and $k_i = 1 = \delta_i$ for all $i = 1, \dots, s$. Since $\delta = 1$, $H \cap X' = \emptyset$ and assertion (1) is proved.

Let $F \in \tilde{\Lambda}$. If F is irreducible then $F^2 = 0$ implies that $F \cap X' \neq \emptyset$, by condition (i) of 1.7. If $F = F_i$ for some i, then $F_i^\circ \cap X' = \emptyset$ (because $\delta_i = 1$) and F_i' has irreducible support (because $k_i = 1$). Assertion (2) follows.

We have $F_i' = \nu_i C_{F_i}$ for some $\nu_i \geq 1$. In 1.7, when we proved that $F_i' \neq 0$, we actually showed that at least one component C of F_i' satisfies $C^2 = -1$; thus $C_{F_i}^2 = -1$. Conversely, if C is any component of F_i such that $C^2 = -1$, then $C \cap X' \neq \emptyset$ (otherwise conditions (i–iii) of 1.7 would be violated), so $C = C_{F_i}$. Since F_i' does not meet H, C_{F_i} does not meet H; C_{F_i} is not branching in $\operatorname{supp}(F + H)$ because, in the contraction of F_i to a 0-curve, C_{F_i} is the first component to shrink. By part (6) of Lemma 2.2 of [12], C_{F_i} must be a multiple component of F_i. So (3) holds.

In part (4), the assertion about ν is trivial and the correspondence between multiple members of Λ and reducible members of $\tilde{\Lambda}$ is essentially the fact that C_{F_i} is a multiple component of F_i (preceding paragraph).

Since $\delta = 1$ and $\delta_i = 1$ for all i, and in view of (3) of 1.7, $\operatorname{Pic}(X_s)$ is generated by F, C_{F_1}, $\dots$, C_{F_s}, with relations $\nu_i C_{F_i} = F$ for $i = 1, \dots, s$. This, together with (4), implies (5). $\square$

1.9. Suppose that X satisfies (†), let Λ be an affine ruling of X and consider the pair $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$. By 1.2, each reducible member of $\tilde{\Lambda}$ can be shrunk to a 0-curve and the shrinking is uniquely determined by the choice of a section of $\tilde{\Lambda}$. From now-on, whenever we shrink reducible members of $\tilde{\Lambda}$ to 0-curves, we tacitely assume that the shrinking is the one which is determined by the unique section of $\tilde{\Lambda}$ contained in $\tilde{X} \setminus X'$ (see Proposition 1.8).

1.10. The following notations and remarks are useful. Suppose that X satisfies (†), let Λ be an affine ruling of X, consider $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$ and let D be the divisor of $\tilde{X}$ with strong normal crossings such that $\tilde{X} \setminus X' = \operatorname{supp}(D)$.

By Proposition 1.8, $\tilde{\Lambda}$ has a unique section H contained in D and, if F is a reducible member of $\tilde{\Lambda}$, F has a unique component C_F which meets X' and the multiplicity ν of C_F in F is strictly greater than 1; moreover, $\operatorname{supp}(F - \nu C_F)$ has either one or two connected components and exactly one of those components meets H. Let us denote those connected components by F^u and F^ℓ, where F^u is the one which meets H and F^ℓ is allowed to be empty. We regard F^u and F^ℓ either as sets or as reduced effective divisors; we have $F^u \neq \emptyset$ and, recalling how the morphism $\tilde{X} \to \mathbb{F}_m$ contracts F (see 1.2, 1.9), we see that F^ℓ *is either empty or an admissible chain.* Finally, let D_0 denote the connected component of D which contains H; thus $D_0 = H + F_1^u + \dots + F_s^u$, where $F_1, \dots, F_s$ are the reducible members of $\tilde{\Lambda}$, and $D = D_0 + F_1^\ell + \dots + F_s^\ell$.

As explained in 1.1, our definition of "affine ruling" is slightly different from the standard one. The following gives the exact relation between the two definitions:

Proposition 1.11. *Suppose that X satisfies (†) and that Λ is an affine ruling of X. For an open subset U of X, the following are equivalent:*

1. *There exists an isomorphism $U \cong \Gamma \times \mathbb{A}^1$, for some open subset $\Gamma \neq \emptyset$ of $\mathbb{P}^1$, such that the composition $U \cong \Gamma \times \mathbb{A}^1 \to \Gamma$ is compatible with Λ.*
2. *$U = X \setminus \operatorname{supp}(M_1 + \dots + M_p)$, for some nonempty subset $\{M_1, \dots, M_p\}$ of Λ containing in particular all multiple members.*

Moreover, if these conditions hold (and $M_1, \dots, M_p$ are distinct) then U is isomorphic to $(\mathbb{P}^1 - p$ points$) \times \mathbb{A}^1$ (or equivalently to $\mathbb{P}^2$ minus p lines meeting at a point).

Some graph theory is needed for proving the above result. Given $q \in \mathbb{N}$, let $\mathcal{S}_q$ be the weighted tree consisting of $q+1$ vertices $v_0, v_1, \dots, v_q$, all of weight 0, and of the q edges $\{v_0, v_i\}$, $i = 1, \dots, q$. Note that $\det(\mathcal{S}_1) = -1$ and that $\det(\mathcal{S}_q) = 0$ for all $q \neq 1$ (see 3.15 for the determinant of a weighted graph). Note that if $q \geq 1$ and $\mathcal{S}$ is identical to $\mathcal{S}_q$ except for the weight of v_0, then $\mathcal{S}$ is equivalent to $\mathcal{S}_q$. Note, also, that if $\mathcal{S}_p$ and $\mathcal{S}_q$ are equivalent then $p = q$.

Lemma 1.12. *Let $p \geq 1$ and $r \geq 0$ be integers, $\mathcal{G}$ a weighted tree, v a vertex of $\mathcal{G}$, $\mathcal{A}_1, \dots, \mathcal{A}_p, \mathcal{B}_1, \dots, \mathcal{B}_r$ the branches of $\mathcal{G}$ at v, where each $\mathcal{A}_i$ consists of a single vertex of weight 0 and, in each $\mathcal{B}_i$, every weight is strictly less than -1.*

1. *If $\mathcal{G}$ is equivalent to $\mathcal{S}_q$ for some $q \in \mathbb{N}$, then $r = 0$ and $p = q$.*
2. *If $\mathcal{G}$ is equivalent to a linear chain Γ of the form*

$$(*) \qquad \overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{x}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\omega_1}{\bullet}\!-\cdots\;-\!\!\!-\!\!\overset{\omega_q}{\bullet} \qquad (q \geq 0,\ \omega_i \leq -2 \text{ and } x \in \mathbb{Z}),$$

then $\mathcal{G}$ itself has the form $(*)$.

Proof. Let us say, temporarily, that a weighted tree $\mathcal{T}$ *satisfies the condition* (NN) if it has a branch point b such that: (i) at least one branch of $\mathcal{T}$ at b has all its weights strictly less than -1; and (ii) every branch of $\mathcal{T}$ at b containing a weight ≥ -1 contains a nonnegative weight. Then we leave it to the reader to verify the following fact:

(5) If a weighted tree $\mathcal{T}$ satisfies (NN) then so does every minimal weighted tree equivalent to $\mathcal{T}$.

Note that $\mathcal{S}_q$ is minimal and does not satisfy (NN); also, Γ contracts to a minimal chain which does not satisfy (NN). Since $\mathcal{G}$ is equivalent to $\mathcal{S}_q$ or Γ, it follows from (5) that $\mathcal{G}$ does not satisfy (NN). We claim:

(6) If $r \neq 0$ then $\mathcal{G}$ is of the form $(*)$ and $\det(\mathcal{G}) \leq -2$.

Indeed, suppose that $r \neq 0$; if either v or some vertex of some $\mathcal{B}_i$ is a branch point of $\mathcal{G}$, then $\mathcal{G}$ satisfies (NN), a contradiction. So $\mathcal{G}$ is a linear chain. In particular, $p + r \leq 2$, so $p = 1 = r$ and $\mathcal{G}$ is of the form $(*)$. We have $\det(\mathcal{G}) = -\det(\mathcal{B}_1)$ by 3.18, and $\det(\mathcal{B}_1) \geq 2$ by 3.19; so (6) holds.

To prove assertion (1), suppose that $\mathcal{G}$ is equivalent to $\mathcal{S}_q$. Then $\det(\mathcal{G}) = \det(\mathcal{S}_q) \geq -1$, so $r = 0$ by (6). Since $r = 0$ and $p > 0$, $\mathcal{G}$ is equivalent to $\mathcal{S}_p$, so $p = q$.

To prove (2), suppose that $\mathcal{G}$ is equivalent to Γ. By (6), we may assume that $r = 0$. Then $\mathcal{G}$ is equivalent to $\mathcal{S}_p$, so $\det(\mathcal{S}_p) = \det(\Gamma) = -\det(\Gamma') \leq -1$ by 3.18 and 3.19, where Γ' is the admissible chain with weights $\omega_1, \dots, \omega_q$. So $p = 1$ (and $r = 0$) and consequently $\mathcal{G}$ is of the form $(*)$ (with $q = 0$). □

Proof of Proposition 1.11. We shall prove that (1) implies (2) and leave the rest to the reader. Suppose that U satisfies condition (1) and let $q \in \mathbb{N}$ be such that $\Gamma = \mathbb{P}^1 - q$ points. Regard U as an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$; then the complement of U

is a divisor W with strong normal crossings and whose dual graph is $\mathcal{S}_q$. Note that U is connected at infinity. We also observe that $U \subset X'$, where $X' = X_s \setminus \mathrm{Bs}(\Lambda)$; the inclusion is strict because the complement of U in X has pure dimension one (the intersection matrix of W is not negative definite, so W cannot be shrunk to a normal point).

Consider $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^{\sim}$ and recall that the open subset X' of X can be embedded in $\tilde{X}$ as the complement of a divisor D of $\tilde{X}$ with strong normal crossings. Since $U \subset X'$ (strictly) and $\tilde{X} \setminus U$ has pure dimension one,

$$\tilde{X} \setminus U = \mathrm{supp}(D + C_1 + \cdots + C_p) \qquad (p > 0) \tag{7}$$

for some distinct curves $C_1, \ldots, C_p$ not contained in D. By Proposition 1.8, some component H of D is a section of $\tilde{\Lambda}$; thus part 4 of 1.2 implies that each C_i is contained in a member of $\tilde{\Lambda}$. Since every member F of $\tilde{\Lambda}$ has a unique component C_F which meets X' (1.8), we have $C_i = C_{G_i}$ $(1 \le i \le p)$ for some distinct $G_1, \ldots, G_p \in \tilde{\Lambda}$. Using (7), we have (in X) $U = X' \setminus \mathrm{supp}(M_1 + \cdots + M_p)$ where $M_i \in \Lambda$ corresponds to $G_i \in \tilde{\Lambda}$ under the bijection $\Lambda \to \tilde{\Lambda}$ defined in 1.6. Since $X \setminus U$ has pure dimension one,

$$U = X \setminus \mathrm{supp}(M_1 + \cdots + M_p). \tag{8}$$

Suppose that the reducible members $F_1, \ldots, F_s$ of $\tilde{\Lambda}$ have been labeled in such a way that

$$\{F_1, \ldots, F_s\} \setminus \{G_1, \ldots, G_p\} = \{F_1, \ldots, F_r\} \qquad (\text{where } 0 \le r \le s).$$

Since U is connected at infinity, we may write (using (7) and 1.10):

$$\tilde{X} \setminus U = \mathrm{supp}(H + F_1^u + \cdots + F_r^u + G_1 + \cdots + G_p). \tag{9}$$

Let $\gamma : \tilde{X} \to S$ (where S is smooth) be the shrinking of $G_1, \ldots, G_p$ to 0-curves (see 1.2 and 1.9). Then $U \xrightarrow{\gamma} \gamma(U)$ is an isomorphism and $S \setminus \gamma(U) = \mathrm{supp}(D')$, where D' is a divisor of S with strong normal crossings. By (9), the dual graph $\mathcal{G}$ of (S, D') is a tree with $p+r$ branches at $\gamma(H)$: p branches $\gamma(G_i)$ consisting of a single vertex of weight zero and r branches $\gamma(F_i^u)$ in which every weight is strictly less than -1. Thus $\mathcal{G}$ satisfies the hypothesis of Lemma 1.12 and part 1 of that result gives $r = 0$, so $\{F_1, \ldots, F_s\} \subseteq \{G_1, \ldots, G_p\}$. From this and (8), it follows that U satisfies condition (2). $\square$

1.13. Let X be a complete normal rational surface.

Given an affine ruling Λ of X, consider $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^{\sim}$ and the divisor D of $\tilde{X}$ with strong normal crossings such that $\mathrm{supp}\, D = \tilde{X} \setminus X'$; let $\mathcal{G}(\Lambda) = \mathcal{G}(D, \tilde{X})$ (the dual graph of D in $\tilde{X}$).

Then the equivalence class of the weighted graph $\mathcal{G}(\Lambda)$ depends only on X and has a unique minimal element, say $\mathcal{E}_X$. Indeed, let $\hat{X}$ and $\hat{E}$ be as in the proof of Proposition 1.5, and let $\mathcal{E}_X$ be the dual graph of $\hat{E}$ in $\hat{X}$; then the weighted graph $\mathcal{E}_X$ is the only minimal element of its equivalence class and $\mathcal{G}(\Lambda)$ contracts to $\mathcal{E}_X$.

Definition 1.14. Let X be a complete normal rational surface and Λ an affine ruling of X. Define $\beta(\Lambda) \in \mathbb{N}$ by:

$$\beta(\Lambda) = \text{number of branch points of } \mathcal{G}(\Lambda) - \text{number of branch points of } \mathcal{E}_X,$$

where $\mathcal{G}(\Lambda)$ and $\mathcal{E}_X$ are as in 1.13. If $\beta(\Lambda) = 0$, we say that Λ is *basic*.

Remark. In 1.13 and 1.14, if X satisfies (‡) (which includes the case where X is smooth), then $\mathcal{E}_X$ has no branch point and, consequently, Λ is basic if and only if the divisor D has no branching component.

Theorem 1.15. *Suppose that X satisfies (†). Then:*

(1) *at most one singular point of X is not a cyclic quotient singularity.*

Let Λ be an affine ruling of X and assume that at least one of the following conditions holds:

(i) *X satisfies (‡); or*

(ii) *$\beta(\Lambda) > 0$.*

Let $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^{\sim}$. Then the following hold:

(2) *$\tilde{\Lambda}$ has at most two reducible members and one of them contains all branching components of $\tilde{X} \setminus X'$.*

(3) $\operatorname{Sing}(X) \cup \operatorname{Bs}(\Lambda)$ *contains at most three points.*

(4) *Λ has at most two multiple members.*

Moreover, if $\{F_1, F_2\}$ is a subset of Λ containing all multiple members (where $F_1 \neq F_2$), and if $F_i = \nu_i C_i$ (where C_i is a curve and $\nu_i \geq 1$, $i = 1, 2$), then:

(5) *$X \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{P}^2$ minus two lines.*

(6) $\operatorname{Pic}(X_s) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$, *where $d = \gcd(\nu_1, \nu_2)$.*

Proof. Let $\hat{X}$, $\hat{E}$, $\hat{\Lambda}$ and $\rho : \tilde{X} \to \hat{X}$ be as in the proof of Proposition 1.5; let the notation be as in 1.10.

First, it is clear that the connected components of $\tilde{X} \setminus X'$ are D_0 and the nonempty F_i^ℓ; in particular, there are at most $s + 1$ such components and, taking images under $\tilde{X} \to \hat{X} \to X$, we get that $\operatorname{Sing}(X) \cup \operatorname{Bs}(\Lambda)$ contains at most $s + 1$ points.

If $\hat{\Lambda}$ has a base point, denote it by $P \in \hat{X}$ and observe that $\rho^{-1}(P)$ is connected and that $H \subseteq \rho^{-1}(P) \subseteq \operatorname{supp}(D)$; thus $\rho^{-1}(P) \subseteq D_0$ and consequently the restriction of ρ to the open set $\tilde{X} \setminus D_0$ is an isomorphism. Of course, this is also the case if $\hat{\Lambda}$ does not have a base point (ρ is the identity map). Since F_i^ℓ is contained in that open set, 1.10 implies:

$$\rho(F_i^\ell) \text{ is either empty or an admissible chain (for each } i = 1, \dots, s).$$

On the other hand, the connected components of $\hat{E}$ are among those of $\hat{X} \setminus X'$, and these are $\rho(D_0)$ and the nonempty $\rho(F_i^\ell)$. So at most one connected component of $\hat{E}$ is not a $\rho(F_i^\ell)$; consequently, at most one connected component of $\hat{E}$ is not an admissible chain, i.e., (1) holds.

Recall that F_i^u meets H for all $i = 1, \dots, s$, so the branching number of H in D_0 is precisely s. Assuming that (i) or (ii) holds, we will now show that $s \le 2$ and that assertion (2) of the Theorem holds. For this, we may assume that D (or equivalently D_0) has a branching component.

Note that $\rho(D_0)$ is either a point or a connected component of $\hat{E}$. Thus, under assumption (i), D_0 contracts to an admissible chain or to a single point; since we assumed that D_0 has a branching component, it follows that $\beta(\Lambda) > 0$. Hence, we may assume that (ii) holds.

Then D is not minimal, i.e., it has a component C which is not branching in D and which satisfies $C^2 = -1$; since $C \neq H$ implies $C^2 < -1$, we must have $C = H$, so H is not branching in D and $s \le 2$. In particular, $\operatorname{supp}(F_1 + F_2)$ contains every branching component of D. If each of F_1, F_2 contains a branching component of D then (since $C^2 < -1$ for all components C of $F_1 + F_2$) no contraction of D can decrease the number of branching components—contradicting the assumption that $\beta(\Lambda) > 0$. This proves assertion (2) of the Theorem. The other assertions easily follow from (2) and results 1.8 and 1.11. □

Corollary 1.16. *If X satisfies (†) then at most one singular point of X is not a cyclic quotient singularity. If X satisfies (‡) then X has at most three singular points.*

1.17. We prove the following statement, which was claimed without proof in the introduction: *Given X satisfying (‡) and a curve C on X, the following are equivalent:*

(i) $\bar{\kappa}(X_s \setminus C) = -\infty$;

(ii) *there exists at least one affine ruling Λ of X such that $nC \in \Lambda$ for some $n > 0$.*

Condition (ii) clearly implies (i). If (i) is satisfied then we have to show that $U = X_s \setminus C$ is affine-ruled (then 1.11 implies (ii)). Consider $\tilde{X} \to \hat{X} \to X$, where $\hat{X} \to X$ is the minimal resolution of singularities of X and $\tilde{X} \to X$ is further blowing-up so that the inverse image $\tilde{C}$ of C has normal crossings. Then the complement of U in $\tilde{X}$ is a divisor D with normal crossings and every connected component of D other than $\tilde{C}$ is a linear chain. Since the divisor class group of X ($\cong \operatorname{Pic} X_s$) has rank 1, any two curves on X meet. Hence, if $E \subset \tilde{X}$ is a curve meeting U which is shrunk in making $(\tilde{X}, D)$ almost minimal, E meets $\tilde{C}$. Hence, on the almost minimal model, the boundary divisor again has at most one non-linear component. Since the connected component of the boundary containing C is not contractible, [15] implies that U is affine-ruled.

2. Modification of affine rulings

In the proof of the following theorem, we consider an arbitrary affine ruling Λ satisfying $\beta(\Lambda) > 0$ and "reduce" it to an affine ruling Λ' such that $\beta(\Lambda') < \beta(\Lambda)$ (see Definition 1.14 for β). We will see later that this reduction process is an instance of a more general modification process.

Theorem 2.1. *If X satisfies (†) then it admits a basic affine ruling.*

Proof. Suppose that Λ is an affine ruling of X satisfying $\beta(\Lambda) > 0$. Consider $X' = X_s \setminus \mathrm{Bs}(\Lambda)$, $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^{\sim}$, let D be the reduced effective divisor of $\tilde{X}$ such that $\tilde{X} \setminus X' = \mathrm{supp}(D)$, H the unique section of $\tilde{\Lambda}$ contained in D and $\tilde{X} \xrightarrow{\rho} \hat{X} \to X$ as in the proof of Proposition 1.5.

Since $\beta(\Lambda) > 0$, at least one component C of D satisfies:

(10) C is branching in D and $\rho(C)$ is either a point of $\hat{X}$ or a curve not branching in $\rho(D)$.

By Theorem 1.15, $\tilde{\Lambda}$ has at most two reducible members and one of them, say $\tilde{F}$, contains all branching components of D; in particular, $C \subset \mathrm{supp}(\tilde{F})$. Recall from Proposition 1.8 that $\tilde{F}$ has a unique component $C_{\tilde{F}}$ which meets X'. Consider the connected components Γ_1 and Γ_2 of $\mathrm{supp}(D + C_{\tilde{F}})$, where Γ_1 contains H and Γ_2 is either empty or an admissible chain of projective lines. Explicitely, if $\tilde{F}$ is the only reducible member of $\tilde{\Lambda}$ then $\Gamma_1 = \mathrm{supp}(H + \tilde{F})$ and $\Gamma_2 = \emptyset$; if $\tilde{\Lambda}$ has two reducible members, say $G_1 = \tilde{F}$ and G_2, then $\Gamma_1 = \mathrm{supp}(H + G_1 + G_2^u)$ and $\Gamma_2 = G_2^\ell$ (see 1.10 for the notations G_2^u and G_2^ℓ and note that G_2^ℓ may be empty).

Consider the birational morphism $m : \tilde{X} \to S$ which shrinks $\tilde{F}$ to a 0-curve (see 1.2, 1.9) and regard it as a composition $\tilde{X} = S_n \xrightarrow{m_n} \cdots \xrightarrow{m_1} S_0 = S$ of monoidal transformations. Since the exceptional locus of m has only one (-1)-component (namely, $C_{\tilde{F}}$), the center of m_i is on the exceptional curve of m_{i-1} for each $i > 1$. It follows, in particular, that the unique component of $\tilde{F}$ which meets H is not branching in D, so C is not that component and $m(C)$ is a point. Another consequence is that Γ_1 has precisely three branches at C, say $\mathcal{B}$, $\mathcal{B}^u$ and $\mathcal{B}^\ell$, where $\mathcal{B}$ contains $C_{\tilde{F}}$, $\mathcal{B}^u$ contains H and every component of $\mathcal{B}^\ell$ has self-intersection strictly less than -1.

Since $m(C)$ is a point, we may factor m as $\tilde{X} \to \tilde{S} \to S$, where the image of C in $\tilde{S}$ is a (-1)-curve and is the first curve to be shrunk by $\tilde{S} \to S$. Then it is easy to see that $\tilde{X} \to \tilde{S}$ is the shrinking of $\mathcal{B}$.

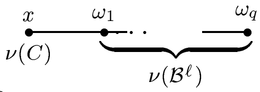
On the other hand, our choice of C (condition (10), above) allows us to factor ρ as $\tilde{X} \xrightarrow{\alpha} U \to \hat{X}$, where $\bar{C} = \alpha(C)$ is a curve, but is not branching in $\bar{D} = \alpha(D)$; then one sees that α is the shrinking of $\mathcal{B}^u$. So we may consider a commutative diagram of smooth complete surfaces and birational morphisms:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\alpha", "(\mathcal{B}^u)"'] \arrow[d, "(\mathcal{B})"'] & U \arrow[d, "(\mathcal{B})"] & \Omega \arrow[l, "\beta"'] \arrow[d, "(\mathcal{B})"] \\ \tilde{S} \arrow[r, "\gamma", "(\mathcal{B}^u)"'] & S'_+ & S' \arrow[l, "\sigma"'] \end{tikzcd} \tag{11}$$

where the labels ("$\mathcal{B}$" or "$\mathcal{B}^u$") indicate what set is shrunk by each morphism—only the left square is being defined at this time. Let $\nu : \tilde{X} \to S'_+$ be the composition of these maps.

Let x be the self-intersection number of $\nu(C)$ in S'_+. Since the image of C in $\tilde{S}$ has self-intersection -1 and $\tilde{S} \xrightarrow{\gamma} S'_+$ increases that number by at least one, we have $x \geq 0$. The dual graph of $\nu(\Gamma_1)$ in S'_+ is:

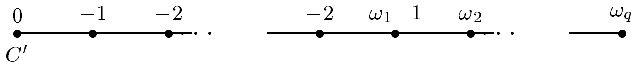
(12)



where $q > 0$, $\omega_i \leq -2$ and $x \geq 0$.

Let $P'_+ \in S'_+$ be the unique point of $\nu(C)$ which also belongs to another component of $\nu(\Gamma_1)$ and consider the birational morphism $S' \xrightarrow{\sigma} S'_+$ obtained by blowing-up x times at P'_+, in such a way that the dual graph of $\sigma^{-1}(\nu(\Gamma_1))$ is:

(13)



where the 0-curve C' is the strict transform of $\nu(C)$. Since the morphism $U \to S'_+$ is isomorphic in a neighborhood of P'_+, the same sequence of x blowings-up can be performed at the level of U; this defines a birational morphism $\Omega \xrightarrow{\beta} U$ and completes the definition of the above commutative diagram (11).

Note that each surface Y considered in this argument comes equipped with a birational transformation, say $\tau_Y : S' \to Y$; consequently, the complete linear system $|C'|$ on S' (a $\mathbb{P}^1$-ruling of S', by 1.2) determines a linear system (without fixed components) on each one of these surfaces. In particular, we will consider the linear systems Λ' on X and Λ^* on Ω defined in this way. Clearly, Λ^* is a $\mathbb{P}^1$-ruling of Ω.

We claim that Λ' is an affine ruling of X. For $i = 1, 2$, let $\Gamma'_i = \sigma^{-1}(\nu(\Gamma_i)) \subset S'$. Then the birational transformation $\tau_X : S' \to X$ restricts to an isomorphism μ' going from the open subset $W' = S' \setminus (\Gamma'_1 \cup \Gamma'_2)$ of S' to the open subset $X_s \setminus \operatorname{supp}(F)$ of X (where F is the member of Λ which corresponds to $\tilde{F}$ under the bijection $\Lambda \to \tilde{\Lambda}$ defined in (1.6)). Then Λ' is the affine ruling of X determined by the X-immersion (S', μ') (see 2.3 for details).

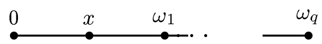
We now argue that Λ and Λ' have the same base point on X. Let $D^* = \beta^{-1}(\bar{D})$ and let C^* be the strict transform of $\bar{C}$ with respect to β. Note that $\alpha(H)$ is a point of $\bar{C}$ and that the image of $\bar{C}$ under $U \to \hat{X} \to X$ is a point (because the image of D under $\tilde{X} \to \hat{X} \to X$ is a finite set of points); thus the base point of Λ is the image of $\bar{C}$ under $U \to \hat{X} \to X$. On the other hand, consider the component H' of Γ'_1 which is a section of $|C'|$ (if $x > 0$ (resp. $x = 0$) then H' is the neighbor of the vertex of weight 0 in the graph (13) (resp. (12))); then the strict transform H^* of H' (with respect to $\Omega \to S'$) is a section of Λ^* and satisfies $H^* \cap C^* \neq \emptyset$. Since H^* and C^* are components of D^* and, under $\Omega \to U \to \hat{X} \to X$, D^* is mapped to a finite set of points, we deduce that the image of H^* in X coincides with that of $\bar{C}$; so Λ and Λ' have the same base point.

The morphisms $\tilde{X} \xrightarrow{\alpha} U \xleftarrow{\beta} \Omega$ give an isomorphism $\tilde{X} \setminus \operatorname{supp}(D) \cong \Omega \setminus \operatorname{supp}(D^*)$; it follows that the birational morphism $\Omega \xrightarrow{\beta} U \to \hat{X} \to X$ restricts to an isomorphism from $\Omega \setminus \operatorname{supp}(D^*)$ to $X_s \setminus \operatorname{Bs}(\Lambda)$, which is equal to $X_s \setminus \operatorname{Bs}(\Lambda')$ by the preceding paragraph. Since D^* is a reduced effective divisor with s.n.c., $(\Omega, \Lambda^*) \succ (X, \Lambda')$. Noting that the number of branching components of D^* is strictly less than that of D, and taking into account assertion (4) of Proposition 1.5, we conclude that $\beta(\Lambda') < \beta(\Lambda)$. $\square$

Formalization of the reduction process

Definition 2.2. Suppose that X is a complete normal rational surface. An X-*immersion* is a pair (S, μ) where:

(1) S is a smooth complete surface and μ is an isomorphism from an open subset W of S to an open subset of X.
(2) $S \setminus W$ is nonempty and is the support of a divisor (of S) with strong normal crossings.
(3) Exactly one of the connected components of $S \setminus W$ is a linear chain of projective lines with dual graph:



where $q \geq 0$, $\omega_i \leq -2$ and x is any integer. We call this connected component the *main component* of (S, μ) and often denote it by Γ. We stress that Γ has at least two irreducible components, corresponding to the vertices of weights 0 and x in the above picture.
(4) If C is an irreducible component of $S \setminus W$ which is not in the main component Γ, then $C^2 \leq -1$ and if equality holds then C is branching in $S \setminus W$.

By $\operatorname{dom} \mu$ we mean the open set W; by the *zero-component* of (S, μ), we mean the component of Γ which corresponds to the pending vertex of weight 0 in (3).[3] The neighbor of the zero-component (neighbor in the graph (3)) is called the *section* of (S, μ). If $x = -1$ in (3), we say that (S, μ) is in *standard form.*

Remark. Let the assumptions and notations as in Definition 2.2. Then $C \cong \mathbb{P}^1$ for every irreducible component C of $S \setminus W$. This follows from 2.3, below: $C \subset \operatorname{supp}(\Sigma + Z_1 + \cdots + Z_n)$.

2.3. Let X be a complete normal rational surface. We claim that each X-immersion determines an affine ruling of X.

To see this, let (S, μ) be an X-immersion; let $W = \operatorname{dom} \mu$ and let Γ, Z and Σ be the main component, zero-component and section of (S, μ) respectively. By 1.2, the complete linear system $|Z|$ is a $\mathbb{P}^1$-ruling of S; also, Σ is a section of $|Z|$. Every irreducible component C of $S \setminus W$ other than Σ satisfies $C \cdot Z = 0$, so is contained in some member of $|Z|$. Consequently, we can choose a finite subset $\{Z_1, \ldots, Z_n\}$ of $|Z|$ such that the open set

$$W_0 = S \setminus \operatorname{supp}(\Sigma + Z_1 + \cdots + Z_n)$$

[3]Note that the pending vertex of weight 0 is not unique when $q = 0$ and $x = 0$; let us agree that an X-immersion always comes equipped with a choice of a zero-component.

is contained in W. Enlarging the set $\{Z_1, \dots, Z_n\}$ if necessary, we may arrange that the morphism $S \to \mathbb{P}^1$ induced by $|Z|$ restricts to a projection map $W_0 = \Gamma \times \mathbb{A}^1 \to \Gamma$. It follows that, if we let Λ denote the linear system on X (without fixed components) determined by $|Z|$ via μ, then Λ is an affine ruling of X.

We will describe the image of μ in the case where X satisfies (†); to do it, we need:

Definition 2.4. Suppose that X satisfies (†), let Λ be an affine ruling of X and consider the pair $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$. Let $\tilde{\Lambda}_*$ be the set of members $\tilde{F}$ of $\tilde{\Lambda}$ which satisfy:

(1) at most one member of $\tilde{\Lambda} \setminus \{\tilde{F}\}$ is reducible; and
(2) all branching components of $\tilde{X} \setminus X'$ are in $\tilde{F}$.

We also define $\Lambda_* = \left\{F \in \Lambda \mid \tilde{F} \in \tilde{\Lambda}_*\right\}$, where $F \mapsto \tilde{F}$ is the bijection $\Lambda \to \tilde{\Lambda}$ of Definition 1.6.

2.5. Note that, in Definition 2.4, $\Lambda_* \neq \emptyset$ if and only if $\tilde{\Lambda}_* \neq \emptyset$, if and only if $\tilde{\Lambda}$ has at most two reducible members and some member contains all branching components of $\tilde{X} \setminus X'$. In particular, Theorem 1.15 implies:

(1) If X satisfies (‡) then Λ_* is nonempty.
(2) If $\beta(\Lambda) > 0$, then Λ_* has exactly one element.

Lemma and definition 2.6. *Suppose that X satisfies (†), let (S, μ) be an X-immersion and let Z and Γ be the zero-component and the main component of (S, μ) respectively.*

(1) *The complete linear system $|Z|$ on S determines (via μ) an affine ruling Λ of X. Moreover, there is a unique $F \in \Lambda_*$ such that* $\operatorname{im} \mu = X_s \setminus \operatorname{supp}(F)$. *In this context, we say that (S, μ)* determines (Λ, F).
(2) $S \setminus \operatorname{dom}(\mu)$ *has at most two connected components and, if it has two, the component other than Γ is an admissible chain.*

Proof. In view of 2.3, the proof of assertion (1) will be complete if we can show that $\operatorname{im} \mu = X_s \setminus \operatorname{supp}(F)$ for some $F \in \Lambda_*$.

Let $W = \operatorname{dom} \mu$ and $X' = X_s \setminus \operatorname{Bs}(\Lambda)$; since W is smooth and $|Z|$ is base point free, $\mu(W) \subseteq X'$. If $\mu(W) = X'$ then $(S, |Z|) \succ (X, \Lambda)$ and, by part (4) of Proposition 1.5, there exists a birational morphism $S \to \tilde{X}$ which restricts to an isomorphism $W \to X'$. Let $C \subset \tilde{X}$ be the image of Z under $S \to \tilde{X}$; then C is a component of D satisfying $C^2 \geq 0$, which is absurd. Hence $\mu(W) \subset X'$ (strictly).

Consider $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$ and let the notations of 1.10 be in effect (in particular D, H and, given $F \in \tilde{\Lambda}$, C_F, F^u and F^ℓ). Regard $\mu(W)$ and X' as open subsets of $\tilde{X}$. Observe that, in S, no connected component of $S \setminus W$ can be shrunk to a smooth point; thus $\tilde{X} \setminus \mu(W)$ has pure dimension one, so

$$\tilde{X} \setminus \mu(W) = \operatorname{supp}(D + C_1 + \cdots + C_p) \qquad (p > 0)$$

where the C_i are distinct curves not contained in D.

In 2.3 we noted that the morphism $S \to \mathbb{P}^1$ determined by $|Z|$ restricts to a projection map $W_0 = \Gamma \times \mathbb{A}^1 \to \Gamma$, where $W_0 \subseteq W$. Then, as in the proof of 1.11, 1.2 implies that $C_i = C_{G_i}$ $(1 \le i \le p)$ for some distinct $G_1, \dots, G_p \in \tilde{\Lambda}$. Let $F_1, \dots, F_r$ $(r \ge 0)$ denote the reducible members of $\tilde{\Lambda} \setminus \{G_1, \dots, G_p\}$. Define

$$G^\circ = \operatorname{supp}(H + G_1 + \cdots + G_p + F_1^u + \cdots + F_r^u), \tag{14}$$

$$G = \tilde{X} \setminus \mu(W) = \operatorname{supp}(G^\circ + F_1^\ell + \cdots + F_r^\ell) \tag{15}$$

and note that G° and the nonempty F_i^ℓ are the connected components of G.

Let $\mathcal{G}$ be the dual graph of G and $\mathcal{G}^\circ$ the dual graph of G° in $\tilde{X}$ (so $\mathcal{G}^\circ$ is a connected component of $\mathcal{G}$); let $\mathcal{Q}$ be the dual graph of $S \setminus W$ in S and let $\mathcal{Q}^\circ$ be the dual graph of Γ (so $\mathcal{Q}^\circ$ is a connected component of $\mathcal{Q}$). Clearly, $\mathcal{Q}$ and $\mathcal{G}$ are equivalent weighted graphs. Because no connected component of $\mathcal{G}$ or $\mathcal{Q}$ is equivalent to the empty graph, the connected components of $\mathcal{Q}$ correspond bijectively to those of $\mathcal{G}$, in such a way that each component of $\mathcal{Q}$ is equivalent to the corresponding component of $\mathcal{G}$. We claim that $\mathcal{Q}^\circ$ corresponds to $\mathcal{G}^\circ$ under that bijection. Indeed, $\mathcal{Q}^\circ$ corresponds (and so is equivalent) to some connected component $\mathcal{G}'$ of $\mathcal{G}$; if $\mathcal{G}' \neq \mathcal{G}^\circ$ then $\mathcal{G}'$ must be the dual graph of F_i^ℓ for some i, so every weight in $\mathcal{G}'$ is strictly less than -1 and $\mathcal{G}'$ is the unique minimal element of its equivalence class; consequently, $\mathcal{Q}^\circ$ contracts to $\mathcal{G}'$. This is absurd, because any contraction of $\mathcal{Q}^\circ$ contains a nonnegative weight. So $\mathcal{G}^\circ$ is equivalent to $\mathcal{Q}^\circ$, which is of the form $(*)$ described in Lemma 1.12.

By 1.2 (and 1.9), each G_j can be contracted to a 0-curve. Let $\overline{\mathcal{G}^\circ}$ be the weighted graph obtained from $\mathcal{G}^\circ$ by contracting all G_j to 0-curves; in view of (14), $\overline{\mathcal{G}^\circ}$ has $p+r$ branches at H: p branches consisting of a single vertex of weight zero and r branches in which every vertex has weight strictly less than -1. Thus part (2) of Lemma 1.12 implies that $\overline{\mathcal{G}^\circ}$ is of the form $(*)$. So $p = 1$, $r \le 1$ and, if $r = 1$, $H + F_1^u$ is a linear chain. So (15) simplifies to:

$$\tilde{X} \setminus \mu(W) = \begin{cases} \operatorname{supp}(H + G_1), & \text{if } r = 0; \\ \operatorname{supp}(H + G_1 + F_1^u + F_1^\ell), & \text{if } r = 1. \end{cases} \tag{16}$$

Since r is the number of reducible members of $\tilde{\Lambda} \setminus \{G_1, \dots, G_p\} = \tilde{\Lambda} \setminus \{G_1\}$, we have:

(17) At most one member of $\tilde{\Lambda} \setminus \{G_1\}$ is reducible.

Regarding (16), we observe: H is not branching in $\tilde{X} \setminus \mu(W)$; if F_1^ℓ is nonempty, then it is an admissible chain and a connected component of $\tilde{X} \setminus \mu(W)$; if $r = 1$ then $H + F_1^u$ is a linear chain. Thus all branching components of $\tilde{X} \setminus \mu(W)$ are in G_1 and in particular:

(18) All branching components of $\tilde{X} \setminus X'$ are in G_1.

By (17) and (18), we obtain $G_1 \in \tilde{\Lambda}_*$ and consequently $M_1 \in \Lambda_*$.

In $\tilde{X}$ we have $\mu(W) = \tilde{X} \setminus \operatorname{supp}(D + C_1) = \tilde{X} \setminus \operatorname{supp}(D + G_1) = X' \setminus \operatorname{supp}(G_1)$ so, in X, $\mu(W) = X' \setminus \operatorname{supp}(M_1) = X_s \setminus \operatorname{supp}(M_1)$. This proves assertion (1). Assertion (2) follows from (16) and the argument concerning the connected components of $\mathcal{G}$ and $\mathcal{Q}$. □

Definition 2.7. Let X be a complete normal rational surface.

(1) Let (S, μ) be an X-immersion, with zero-component Z and section Σ, and let $W = \operatorname{dom} \mu$. If P is a point of Z, we define an X-immersion $(S', \mu') = \operatorname{elm}_P(S, \mu)$ as follows: let $\pi : \tilde{S} \to S$ be the blowing-up of S at P, $\tilde{Z}$ the strict transform of Z on $\tilde{S}$ and $\pi' : \tilde{S} \to S'$ the contraction of $\tilde{Z}$. Let $W' = \pi'(\pi^{-1}(W))$, consider the isomorphism $\theta : W' \to W$ obtained by restricting $\pi \circ (\pi')^{-1}$ and define $\mu' = \mu \circ \theta$. We say that (S', μ') is obtained from (S, μ) by an *elementary transformation.* We distinguish two types of elementary transformations: elm_P is of *sprouting type* (resp. of *subdivisional type*) if $P \in Z \setminus \Sigma$ (resp. $\{P\} = Z \cap \Sigma$). Note that, if $(S', \mu') = \operatorname{elm}_P(S, \mu)$, then $(S, \mu) = \operatorname{elm}_Q(S', \mu')$ for a suitable choice of a point Q; here, elm_P and elm_Q are of distinct types.

(2) Two X-immersions are *equivalent* if one can be obtained from the other by a sequence of elementary transformations.

(3) Given X-immersions (S, μ) and (S', μ'), we write $(S', \mu') \leq (S, \mu)$ to indicate that (S', μ') is produced by performing on (S, μ) a sequence of elementary transformations of subdivisional type.

Proposition 2.8. *Suppose that X satisfies (†).*

(1) *If Λ is an affine ruling of X and $F \in \Lambda_*$ then there exists an X-immersion (S, μ) which determines (Λ, F) (as in 2.6).*

(2) *Let (S, μ) and (S', μ') be X-immersions determining pairs (Λ, F) and (Λ', F') respectively. Then:*

$$(\Lambda, F) = (\Lambda', F') \text{ if and only if } (S, \mu) \text{ is equivalent to } (S', \mu').$$

Proof. Let Λ be an affine ruling of X and $F \in \Lambda_*$; let $V = X_s \setminus \operatorname{supp}(F)$. Consider $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^{\sim}$ and $\tilde{F} \in \tilde{\Lambda}_*$ (recall the bijection $\Lambda \to \tilde{\Lambda}$, $F \mapsto \tilde{F}$, defined in 1.6). Since $V \subseteq X' = X_s \setminus \operatorname{Bs}(\Lambda)$ and X' can be viewed as a subset of $\tilde{X}$, we may write $V = X' \setminus \operatorname{supp}(\tilde{F})$. Let S be the surface obtained from $\tilde{X}$ by shrinking $\tilde{F}$ to a 0-curve (see 1.2, 1.9) and let $m : \tilde{X} \to S$ be the corresponding morphism. Let $W = m(V)$ and let $\mu : W \to V$ be the restriction of m^{-1}. Then (S, μ) is an X-immersion and determines the pair (Λ, F); so (1) is proved.

The fact that equivalent X-immersions determine the same pair (Λ, F) is quite clear. Conversely, suppose that (S, μ) and (S', μ') are X-immersions determining the same pair (Λ, F); we will show that (S, μ) and (S', μ') are equivalent.

For (S, μ), we use the notations W, Z and Σ as in 2.7; for (S', μ'), we use W', Z' and Σ'. Since μ and μ' have the same image $X_s \setminus \operatorname{supp}(F)$, they determine a birational isomorphism $S \to S'$ which restricts to an isomorphism $W \to W'$. So there exists a smooth complete surface Ω and two birational morphisms, $\pi : \Omega \to S$ and $\pi' : \Omega \to$

S', such that if we regard π (resp. π') as a composition of monoidal transformations then each one of these is centered at a point infinitely near $S \setminus W$ (resp. $S' \setminus W'$). We also assume that (Ω, π, π') is minimal, i.e., that the total number of monoidal transformations in π and π' is minimal. We denote this number by $N((S,\mu),(S',\mu'))$. Since we assumed that (S,μ) and (S',μ') determine the same Λ, it follows that $\tilde{\Sigma} = \tilde{\Sigma}'$, where $\tilde{\Sigma}$ (resp. $\tilde{\Sigma}'$) is the strict transform of Σ (resp. Σ') on Ω.

If π is an isomorphism then S' is obtained from S by contracting some irreducible components of $S \setminus W$; since no component of $S \setminus W$ is a (-1)-curve except possibly Σ, and since π' does not shrink Σ (for $\tilde{\Sigma} = \tilde{\Sigma}'$), π' must then be an isomorphism and we are done in this case.

Suppose that π is not an isomorphism; by the above paragraph (with π and π' interchanged), π' is not an isomorphism and we may consider a curve $\tilde{C} \subset \Omega$ which is first to be shrunk by π'. By minimality of (Ω, π, π'), $\tilde{C}$ is the strict transform of some component C of $S \setminus W$ satisfying $C^2 \geq -1$. Since π' does not shrink $\tilde{\Sigma}$, we must have $C = Z$. Thus exactly one of the monoidal transformations making-up π has a center P which is a point of Z. It follows that

$$N(\mathrm{elm}_P(S,\mu),(S',\mu')) < N((S,\mu),(S',\mu'))$$

and we are done by induction. □

Definition 2.9. Suppose that X is a complete normal rational surface and that (S,μ) is an X-immersion. In this paragraph, we define a set $\Pi(S,\mu)$ of birational morphisms and, given $\pi \in \Pi(S,\mu)$, an X-immersion $(S,\mu) * \pi$ determined by (S,μ) and π.

Let $W = \operatorname{dom}\mu$ and let Γ, Z and Σ be the main component, zero-component and section of (S,μ) respectively.

Let $\Pi(S,\mu)$ be the set of birational morphisms $\pi : \tilde{S} \to S$, with $\tilde{S}$ smooth and complete, satisfying:

1. the exceptional locus of π has a unique (-1)-component, which we denote E;
2. $\pi(E)$ is a point of $Z \setminus \Sigma$;
3. $\pi^{-1}(\Gamma)$ is a linear chain and E has two neighbors in it;
4. one of the two branches of $\pi^{-1}(\Gamma)$ at E can be shrunk to a smooth point (this must be the branch containing the strict transforms of Z and Σ; moreover, the first curve to shrink is either Z or Σ).

Given a point P of $Z \setminus \Sigma$, we also define

$$\Pi_P(S,\mu) = \{\pi \in \Pi(S,\mu) \mid \pi \text{ is centered at } P \text{ (i.e., } \pi(E) = P)\}.$$

Given $\pi \in \Pi(S,\mu)$, let $\gamma : \tilde{S} \to S'_+$ be the birational morphism (with S'_+ smooth) whose exceptional locus is the branch of $\pi^{-1}(\Gamma)$ at E containing the strict transforms of Z and Σ. Note that γ is uniquely determined by π and that its exceptional locus has exactly one (-1)-component. Moreover, $\gamma(E)$ is a curve whose self-intersection number is nonnegative; $\gamma(Z)$ is a point of $\gamma(E)$ and $\gamma(E)$ is the only irreducible component of $\gamma(\pi^{-1}(\Gamma))$ containing that point; $\gamma(\pi^{-1}(\Gamma))$ is a linear chain with dual graph $\underset{\gamma(E)}{\overset{x}{\bullet}} - \overset{\omega_1}{\bullet} - \cdots - \overset{\omega_q}{\bullet}$ where $\omega_i \le -2$, $x \ge 0$ and $q > 0$.

Consider the birational morphism $\sigma : S' \to S'_+$ defined as follows:

(a) If $x = 0$, let $S' = S'_+$ and let σ be the identity map.

(b) If $x > 0$, let $P'_+ \in S'_+$ be the unique point of $\gamma(E)$ which also belongs to another irreducible component of $\gamma(\pi^{-1}(\Gamma))$; define σ by blowing-up x times at P'_+, in such a way that the dual graph of $\sigma^{-1}(\gamma(\pi^{-1}(\Gamma)))$ in S' is:

$$\overset{0}{\bullet} - \overset{-1}{\bullet} - \overset{-2}{\bullet} - \cdots - \overset{-2}{\bullet} - \overset{\omega_1 - 1}{\bullet} - \overset{\omega_2}{\bullet} - \cdots - \overset{\omega_q}{\bullet}$$

where the 0-curve is the strict transform of $\gamma(E)$.

Then let $W' = \sigma^{-1}(\gamma(\pi^{-1}(W)))$ and let μ' be the composite

$$W' \xrightarrow{\sigma} \gamma(\pi^{-1}(W)) \xrightarrow{\gamma^{-1}} \pi^{-1}(W) \xrightarrow{\pi} W \xrightarrow{\mu} \mu(W).$$

Then (S', μ') is an X-immersion, determined by (S, μ) and π. We write $(S', \mu') = (S, \mu) * \pi$ and, informally, think of (S', μ') as the result of π "acting" on (S, μ). Note that μ and μ' have the same image.

Definition 2.10. Suppose that X satisfies (†).

(1) Let $\mathcal{C}$ be an equivalence class of X-immersions. Then $\mathcal{C}$ determines a pair (Λ, F) which, in turn, determines $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$ and $\tilde{F} \in \tilde{\Lambda}_*$. As shown in the first paragraph of the proof of Proposition 2.8, contracting $\tilde{F}$ to a 0-curve gives rise to an X-immersion (S, μ) which determines (Λ, F). We call (S, μ) the *distinguished element of* $\mathcal{C}$.

(2) Suppose that Λ is an affine ruling of X such that $\beta(\Lambda) > 0$. Then (by 2.5) Λ_* has exactly one element, say F, and we may consider the distinguished element (S, μ) of the equivalence class of X-immersions which determine (Λ, F). We call (S, μ) *the standard X-immersion associated to* Λ (it is an X-immersion in standard form). Note that (S, μ) comes equipped with a birational morphism $m : \tilde{X} \to S$ (the contraction of $\tilde{F}$ to a 0-curve).

Corollary 2.11 (Reduction Theorem)**.** *Suppose that X satisfies (†) and that Λ is an affine ruling of X such that $\beta(\Lambda) > 0$. Consider the unique element F of Λ_*, the standard X-immersion (S, μ) associated to Λ and the center $P \in S$ of the birational morphism $m : \tilde{X} \to S$ (the contraction of $\tilde{F}$ to a 0-curve). Then, for some $\pi \in \Pi_P(S, \mu)$, the pair (Λ', F') determined by the X-immersion $(S, \mu) * \pi$ satisfies*

$$\beta(\Lambda') = \beta(\Lambda) - 1 \quad \textit{and} \quad \operatorname{supp}(F') = \operatorname{supp}(F).$$

Remark. In the conclusion of Corollary 2.11, we can replace "for some $\pi \in \Pi_P(S, \mu)$" by "for every $\pi \in \Pi_P(S, \mu)$". This is because of part (4) of Lemma 4.4, which also implies that (Λ', F') is uniquely determined by Λ, i.e., is independent of the choice of $\pi \in \Pi_P(S, \mu)$.

Proof of 2.11. Let C be the branching component of D which is closest to H (notations D, H, etc as in the proof of 2.1); then it is easy to see that C satisfies the condition (10) of the proof. As in the proof of 2.1, factor m as $\tilde{X} \to \tilde{S} \xrightarrow{\pi} S$ and consider $S \xleftarrow{\pi} \tilde{S} \xrightarrow{\gamma} S'_+ \xleftarrow{\sigma} S'$. Then it is quite clear that $\pi \in \Pi_P(S,\mu)$ and that the X-immersion (S',μ') constructed in the proof is exactly $(S,\mu)*\pi$. Then (S',μ') determines a pair (Λ',F') and the proof of 2.1 shows that $\beta(\Lambda') < \beta(\Lambda)$. Actually, we have $\beta(\Lambda') = \beta(\Lambda) - 1$ because of how we chose C. $\square$

Lemma 2.12. *Let X be a complete normal rational surface. Suppose that I is an X-immersion, that $\pi \in \Pi_P(I)$ and let $J = I * \pi$ (where P is a point of the 0-component but not of the section of I). Let I^- (resp. J^-) denote the X-immersion obtained from I (resp. J) by performing one subdivisional elementary transformation.*

(1) *$I^- * \pi' = J$ for some $\pi' \in \Pi_{P^-}(I^-)$, where P^- is the point, on the 0-component of I^-, which is the image of the strict transform of the 0-component of I.*
(2) *$I * \pi' = J^-$, for some $\pi' \in \Pi_P(I)$.*
(3) *If $I' \le I$ and $J' \le J$ then there exists $\pi' \in \Pi(I')$ satisfying $I' * \pi' = J'$.*
(4) *There exist $I' \le I$, $J' \le J$, $\pi' \in \Pi(I')$ and $\pi'' \in \Pi(J')$ satisfying*
$$I' * \pi' = J' \quad \text{and} \quad J' * \pi'' = I'.$$

Proof. Write $I = (S,\mu)$ and let Z and Σ be the 0-component and section of I; write $J = (S',\mu') = I * \pi$ and consider $S \xleftarrow{\pi} \tilde{S} \xrightarrow{\gamma} S'_+ \xleftarrow{\sigma} S'$, as in Definition 2.9.

To prove (1), consider the point $\{Q\} = Z \cap \Sigma$, write $(T,\nu) = I^- = \mathrm{elm}_Q(S,\mu)$ and consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \tilde{Y} & \longrightarrow & \tilde{S} & \xrightarrow{\gamma} & S'_+ & \xleftarrow{\sigma} & S' \\
 & & \big\downarrow{\scriptstyle \pi_Y} & & \big\downarrow{\scriptstyle \pi} & & & & \\
T & \xleftarrow{\beta} & Y & \xrightarrow{\alpha} & S & & & &
\end{array}$$

where α is the blowing-up at Q and β contracts the strict transform of Z. Then $\pi' = \beta \circ \pi_Y \in \Pi_{P^-}(I^-)$ and $I^- * \pi' = J$ (where P^- is the center of β).

To prove (2), let Z' and Σ' be the 0-component and section of J, consider the point $\{Q\} = Z' \cap \Sigma'$, write $\mathrm{elm}_Q(J) = J^- = (T,\nu)$, let $\alpha : Y \to S'$ be the blowing-up of S' at Q and $\beta : Y \to T$ the contraction of the strict transform of Z'. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & \tilde{U} & \xrightarrow{\tilde{u}} & U & \xrightarrow{u} & \tilde{S} & \xrightarrow{\pi} & S \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \gamma} & & \\
T & \xleftarrow{\beta} & Y & \xrightarrow{\alpha} & S' & \xrightarrow{\sigma} & S'_+ & &
\end{array}$$

Then $\pi' = \pi \circ u \circ \tilde{u} \in \Pi_P(I)$ and $I * \pi' = J^-$.

Assertion (3) follows immediately from (1) and (2). To prove (4), consider the sections Σ and Σ' of I and J respectively. In view of (3), we may assume that $\Sigma^2 < -1$

and $(\Sigma')^2 < -1$. Then, in the diagram $S \xleftarrow{\pi} \tilde{S} \xrightarrow{\gamma} S'_+ \xleftarrow{\sigma} S'$, the map σ is the identity map, $\gamma \in \Pi(J)$ and $J * \gamma = I$. □

Proposition 2.13. *Let X be a complete normal rational surface and suppose that (S, μ) and (T, ν) are X-immersions. Then the condition $\operatorname{im} \mu = \operatorname{im} \nu$ is equivalent to the existence of a sequence $\{(S_j^*, \mu_j^*)\}_{j=0}^n$ of X-immersions satisfying:*

1. *$(S_0^*, \mu_0^*) \le (S, \mu)$ and $(S_n^*, \mu_n^*) \le (T, \nu)$;*
2. *for all $j = 1, \dots, n$, we have*
$$(S_{j-1}^*, \mu_{j-1}^*) * \pi \text{ is equivalent to } (S_j^*, \mu_j^*), \text{ for some } \pi \in \Pi(S_{j-1}^*, \mu_{j-1}^*).$$

For the proof, we will need the following notations. Given X-immersions $I_j = (S_j, \mu_j)$ $(j = 1, 2)$ such that $\operatorname{im} \mu_1 = \operatorname{im} \mu_2$, let $\mathcal{D}(I_1, I_2)$ denote the set of triples (Ω, π_1, π_2) satisfying:

1. Ω is a smooth complete surface and $\pi_1 : \Omega \to S_1$ and $\pi_2 : \Omega \to S_2$ are birational morphisms;
2. π_j is centered at points of $S_j \setminus \operatorname{dom} \mu_j$ $(j = 1, 2)$ and $\pi_1^{-1}(S_1 \setminus \operatorname{dom} \mu_1) = \pi_2^{-1}(S_2 \setminus \operatorname{dom} \mu_2)$;
3. the birational transformations $\pi_2 \pi_1^{-1}$ and $\mu_2^{-1} \mu_1$, from S_1 to S_2, are equal.

Note that $\mathcal{D}(I_1, I_2)$ is nonempty (because $\operatorname{im} \mu_1 = \operatorname{im} \mu_2$) and that, given any $(\Omega, \pi_1, \pi_2) \in \mathcal{D}(I_1, I_2)$, if one of π_1, π_2 is an isomorphism then both π_1, π_2 are.

Given a birational morphism $f : U \to V$ of smooth complete surfaces, let $N(f) \ge 0$ be the number of monoidal transformations in f.

Given $D = (\Omega, \pi_1, \pi_2) \in \mathcal{D}(I_1, I_2)$, let $N(D) = N(\pi_1) + N(\pi_2)$. Also, let $N(I_1, I_2) = \min_{D \in \mathcal{D}(I_1, I_2)} N(D)$.

Proof of Proposition 2.13. Clearly, the existence of the sequence implies $\operatorname{im} \mu = \operatorname{im} \nu$.

For the converse, let $I = (S, \mu)$ and $J = (T, \nu)$ be X-immersions such that $\operatorname{im} \mu = \operatorname{im} \nu$ and consider the set

$$\mathcal{I}_{(I,J)} = \left\{ (I_1, I_2) \mid I_1 \le (S, \mu),\ I_2 \le (T, \nu),\ \Sigma_1^2 < -1,\ \Sigma_2^2 < -1 \right\},$$

where I_j is an X-immersion and Σ_j is its section. We proceed by induction on the natural number $d(I, J)$ defined by

$$d(I, J) = \min \left\{ N(I_1, I_2) \mid (I_1, I_2) \in \mathcal{I}_{(I,J)} \right\}.$$

If $d(I, J) = 0$ then I and J are equivalent; then it is easy to see that there exists an X-immersion (S_0^*, μ_0^*) satisfying both $(S_0^*, \mu_0^*) \le (S, \mu)$ and $(S_0^*, \mu_0^*) \le (T, \nu)$, so we are done in this case. From now-on, we assume that $d(I, J) > 0$.

Choose $(I_1, I_2) \in \mathcal{I}_{(I,J)}$ such that $N(I_1, I_2) = d(I, J)$; write $I_j = (S_j, \mu_j)$, $D_j = S_j \setminus \operatorname{dom} \mu_j$ and let Z_j and Σ_j be the 0-component and section of I_j. Choose $(\Omega, \pi_1, \pi_2) \in \mathcal{D}(I_1, I_2)$ such that $N(\Omega, \pi_1, \pi_2) = N(I_1, I_2)$.

$$S_1 \xleftarrow{\pi_1} \Omega \xrightarrow{\pi_2} S_2$$

We claim that

(i) Neither of π_1, π_2 is an isomorphism.
(ii) For each $j = 1, 2$, the exceptional locus of π_j has a unique (-1)-component, say $E_j \subset \Omega$, and π_j is centered at a point of Z_j; also, $\pi_2(E_1) = Z_2$ and $\pi_1(E_2) = Z_1$.

Moreover, we claim that (I_1, I_2) and (Ω, π_1, π_2) can be chosen in such a way that the following conditions hold:

(iii) π_j is centered at a point of $Z_j \setminus \Sigma_j$ (for $j = 1, 2$);
(iv) E_j has two neighbors in ${\pi_j}^{-1}(\Gamma_j)$ (for each $j = 1, 2$), where $\Gamma_j \subseteq D_j$ is the main component of (S_j, μ_j).

If (i) is false then, as pointed out just before the proof, both π_1, π_2 are isomorphisms; this contradicts $d(I, J) > 0$, so (i) holds.

By (i), the exceptional locus of π_1 has at least one (-1)-component; let $E_1 \subset \Omega$ be such a component. Since $N(\Omega, \pi_1, \pi_2) = N(I_1, I_2)$, π_2 does not contract E_1. So, $\pi_2(E_1)$ is a non-branching component of D_2 satisfying $\pi_2(E_1)^2 \geq -1$ and consequently $\pi_2(E_1) = Z_2$. In particular, E_1 is unique.

If the center of π_1 is not on Z_1 then the strict transform $\tilde{Z}_1 \subset \Omega$ of Z_1 satisfies $\tilde{Z}_1^2 = 0$. Thus $\pi_2(\tilde{Z}_1)$ is a component of D_2 with nonnegative self-intersection number and, consequently, $\pi_2(\tilde{Z}_1) = Z_2$. This is impossible, because $\pi_2(E_1) = Z_2$ and $E_1 \neq \tilde{Z}_1$. Thus the center of π_1 is on Z_1. Then (ii) follows by symmetry in π_1 and π_2.

For each $j = 1, 2$, let $P_j \in Z_j$ be the center of π_j; define an X-immersion $I'_j = (S'_j, \mu'_j) \leq I_j$ and a morphism $\pi'_j : \Omega \to S'_j$ as follows:

- If $P_j \in Z_j \setminus \Sigma_j$, let $I'_j = I_j$ and $\pi'_j = \pi_j$;
- If $P_j \in Z_j \cap \Sigma_j$, let $I'_j = \mathrm{elm}_{P_j}(I_j)$ and consider

$$\begin{tikzcd} & \Omega \arrow[d, "\pi_j^-"] & \\ S_j & S_j^- \arrow[l, "\alpha_j"'] \arrow[r, "\beta_j"] & S'_j \end{tikzcd}$$

where α_j is the blowing-up of S_j at P_j, β_j is the contraction of the strict transform of Z_j relative to α_j and π_j^- is defined by $\pi_j = \alpha_j \circ \pi_j^-$. Then set $\pi'_j = \beta_j \circ \pi_j^-$.

Then $(I'_1, I'_2) \in \mathcal{I}_{(I,J)}$, $(\Omega, \pi'_1, \pi'_2) \in \mathcal{D}(I'_1, I'_2)$ and $N(\Omega, \pi'_1, \pi'_2) = N(\Omega, \pi_1, \pi_2) = d(I, J)$. Moreover, the center of π'_j is a point of $Z'_j \setminus \Sigma'_j$ (for each $j = 1, 2$), where Z'_j and Σ'_j are the 0-component and section of I'_j respectively. In other words, we may simply assume that (I_1, I_2) and (Ω, π_1, π_2) have been chosen in such a way that (iii) holds. Finally, (iv) follows immediately from (i–iii).

We proved that there exists $(I_1, I_2) \in \mathcal{I}_{(I,J)}$ and $(\Omega, \pi_1, \pi_2) \in \mathcal{D}(I_1, I_2)$ satisfying $N(\Omega, \pi_1, \pi_2) = N(I_1, I_2) = d(I, J)$ and conditions (i–iv). We will now show that $d(I_1 * \pi, I_2) < d(I, J)$ for some $\pi \in \Pi(I_1)$, which will complete the proof.

If $\pi_1^{-1}(\Gamma_1)$ is a linear chain then $\pi_1 \in \Pi(I_1)$ and $I_1 * \pi_1 = I_2$, so we are done in this case.

Assume that $\pi_1^{-1}(\Gamma_1)$ is not a linear chain and consider the branching component C of $\pi_1^{-1}(\Gamma_1)$ which is closest to the strict transform $\tilde{Z}_1 \subset \Omega$ of Z_1. Note that C is contained in the exceptional locus of π_1, for otherwise we would have $C = \tilde{Z}_1$, but $\tilde{Z}_1$ is not branching in $\pi_1^{-1}(\Gamma_1)$ (because Z_1 has one neighbor in Γ_1 and the center of π_1 is one point). Also, $\pi_1^{-1}(\Gamma_1)$ has exactly three branches at C, say $\mathcal{B}$, $\mathcal{B}^u$ and $\mathcal{B}^\ell$, where $\mathcal{B}^u$ contains $\tilde{Z}_1$ and $\mathcal{B}$ contains E_1. Note that $\tilde{Z}_1$ and E_1 are the only (-1)-components of $\pi_1^{-1}(\Gamma_1)$ and that all other components have self-intersection strictly less than -1.

Since $\pi_2(\pi_1^{-1}(\Gamma_1))$ is the linear chain Γ_2, we know that $\mathcal{B}^u$ can be shrunk to a point, i.e., we may factor π_2 as $\Omega \xrightarrow{\alpha} U \xrightarrow{\beta} S_2$, where α is the contraction of $\mathcal{B}^u$. We may also factor π_1 as $\Omega \xrightarrow{\alpha_1} \tilde{S}_1 \xrightarrow{\pi} S_1$, in such a way that $\alpha_1(C)$ is a (-1)-curve on $\tilde{S}_1$; then α_1 is the contraction of $\mathcal{B}$ to a point and $\alpha_1(C)$ is the only (-1)-component of the exceptional locus of π. This gives the first of the following commutative diagrams of smooth complete surfaces and birational morphisms:

$$
\begin{tikzcd}
\Omega \arrow[r, "(\mathcal{B}^u)", "\alpha"'] \arrow[d, "(\mathcal{B})"', "\alpha_1"] & U \arrow[r, "\beta"'] \arrow[d, "(\mathcal{B})"'] & S_2 \\
\tilde{S}_1 \arrow[r, "(\mathcal{B}^u)", "\gamma"'] \arrow[d, "\pi"] & S'_+ & \\
S_1 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega' \arrow[r, "\alpha'"'] \arrow[d, "(\mathcal{B})"', "\pi'_1"] & U \arrow[r, "\beta"'] \arrow[d, "(\mathcal{B})"'] & S_2 \\
S' \arrow[r, "\sigma"'] & S'_+ &
\end{tikzcd}
\tag{19}
$$

where the labels $(\mathcal{B})$ and $(\mathcal{B}^u)$ indicate which set is contracted by each morphism.

Note that $\pi \in \Pi(I_1)$, so we may consider the X-immersion $I'_1 = (S', \mu') = I_1 * \pi$. Recall, from Definition 2.9, that the construction of $I_1 * \pi$ involves a birational morphism $\sigma : S' \to S'_+$ which is the composition of x monoidal transformations, where $x \geq 0$ is the self-intersection number of the curve $(\gamma \circ \alpha_1)(C) \subset S'_+$. Let $\alpha' : \Omega' \to U$ consist of the "same" x monoidal transformations as σ, but performed at the level of U. This gives the second diagram in (19).

Let $\pi'_2 = \beta \circ \alpha' : \Omega' \to S_2$, then $(\Omega', \pi'_1, \pi'_2)$ belongs to $\mathcal{D}(I'_1, I_2)$ but not necessarely to $\mathcal{I}_{(I'_1, I_2)}$. Note that the section Σ'_1 of I'_1 satisfies $(\Sigma'_1)^2 \leq -1$ and let I''_1 be the X-immersion obtained from I'_1 by performing one subdivisional elementary transformation. Then $(I''_1, I_2) \in \mathcal{I}_{(I'_1, I_2)}$, so

$$d(I'_1, I_2) \leq N(I''_1, I_2) \leq N(I'_1, I_2) + 2 \leq N(\Omega', \pi'_1, \pi'_2) + 2. \tag{20}$$

We have $d(I, J) = N(\Omega, \pi_1, \pi_2) = |\mathcal{B}| + N(\pi) + |\mathcal{B}^u| + N(\beta)$ and $N(\Omega', \pi'_1, \pi'_2) = N(\pi'_1) + N(\alpha') + N(\beta) = |\mathcal{B}| + x + N(\beta)$, where $|\mathcal{B}|$ and $|\mathcal{B}^u|$ denote the numbers of irreducible components of $\mathcal{B}$ and $\mathcal{B}^u$. So

$$d(I, J) - N(\Omega', \pi'_1, \pi'_2) = N(\pi) + |\mathcal{B}^u| - x.$$

Note that the self-intersection numbers of $\alpha_1(C) \subset \tilde{S}_1$ and $\gamma(\alpha_1(C)) \subset S'_+$ are -1 and x respectively, so γ increases that number by $x+1$. Since $N(\gamma) = |\mathcal{B}^u|$, we must have $x + 1 \leq |\mathcal{B}^u|$, so

$$d(I, J) - N(\Omega', \pi'_1, \pi'_2) > N(\pi)$$

and, by (20),

$$d(I'_1, I_2) < d(I, J) - N(\pi) + 2.$$

It is easy to see that $N(\pi) \geq 3$, so $d(I'_1, I_2) < d(I, J)$ and we are done. □

Conclusion

2.14. Given a surface X satisfying (†), consider the directed graph $\mathbb{L}(X)$ whose vertices are the affine rulings of X and where, given vertices Λ and Λ', we draw an arrow $\Lambda \to \Lambda'$ if the following condition holds: *There exists an X-immersion I and an element π of $\Pi(I)$ such that (i) I determines (Λ, F) for some $F \in \Lambda_*$ and (ii) $I * \pi$ determines (Λ', F') for some $F' \in \Lambda'_*$.*

Part (4) of Lemma 2.12 implies that if there is an arrow $\Lambda \to \Lambda'$ then there is also an arrow $\Lambda \leftarrow \Lambda'$. Corollary 2.11 implies that each connected component of $\mathbb{L}(X)$ contains a basic affine ruling. Thus, if we want to describe all affine rulings of X, we have to solve the following two problems:

(1) Make a list of all basic rulings of X.
(2) Describe the set $\Pi(S, \mu)$, for each X-immersion (S, μ).[4]

Each one of these problems is nontrivial. The first one is highly dependent on the surface X; [6] solves it for the weighted projective planes (so in particular for $\mathbb{P}^2$). The second problem turns out to be independent of the surface and is completely solved in sections 3 and 4 (see in particular Corollary 4.4).

Remarks. Let X be a surface satisfying (†).

(1) One can show[5] that an affine ruling Λ is an isolated vertex of $\mathbb{L}(X)$ if and only if $\Lambda_* = \emptyset$. Thus, if we make the additional assumption that X satisfies (‡), then no vertex of $\mathbb{L}(X)$ is isolated (see 2.5).
(2) Let us temporarily agree that, given affine rulings Λ and Λ' of X, the phrase "Λ and Λ' have a common member" means that there exists a curve $C \subset X$ and positive integers n and n' satisfying $nC \in \Lambda_*$ and $n'C' \in \Lambda'_*$. Then Proposition 2.13 implies: *Two affine rulings Λ and Λ' of X are in the same connected component of $\mathbb{L}(X)$ if and only if there exists a sequence $\{\Lambda_i\}_{i=0}^n$ of affine rulings of X such that, for each $i < n$, Λ_i and Λ_{i+1} have a common member.*

[4]The point would be in particular to describe explicitly how to *increase* $\beta(\Lambda)$. Section 5 includes a complete answer to this question, as the value of β is easily determined by inspecting the data contained in the "discrete part".

[5]By part (1) of Proposition 2.8 and part (1) of Corollary 4.4.

3. Contraction of weighted trees

We assume familiarity with weighted graphs, their blowing-up and blowing-down. We stress that, in weighted graphs, we do not allow multiple edges between a given pair of vertices. The empty weighted graph is denoted $\varnothing$. A weighted tree without branch points is called a *linear weighted tree* or a *linear chain*.

3.1. Given weighted graphs $\mathcal{G}$ and $\mathcal{G}'$, the symbol $\mathcal{G} \leftarrow \mathcal{G}'$ indicates that $\mathcal{G}'$ is obtained from $\mathcal{G}$ by blowing-up once. In that case, if V (resp. V') denotes the set of vertices of $\mathcal{G}$ (resp. $\mathcal{G}'$) then V can be viewed as a subset of V' and $V' \setminus V$ contains a single vertex, say e. We call e the vertex *created* by $\mathcal{G} \leftarrow \mathcal{G}'$; we also say that $\mathcal{G}$ is the blowing-down of $\mathcal{G}'$ at e. If e has one neighbor v in $\mathcal{G}'$, then v can be viewed as a vertex of $\mathcal{G}$ and $\mathcal{G} \leftarrow \mathcal{G}'$ is called the blowing-up of $\mathcal{G}$ *at the vertex* v. If e has two neighbors u and v in $\mathcal{G}'$, then $\{u, v\}$ is an edge of $\mathcal{G}$ and $\mathcal{G} \leftarrow \mathcal{G}'$ is called the blowing-up of $\mathcal{G}$ *at the edge* $\{u, v\}$.[6] Also, if $\mathcal{G}$ is any weighted graph and $\mathcal{G}'$ is the weighted graph obtained from $\mathcal{G}$ by adding an isolated vertex of weight -1, then we regard $\mathcal{G}'$ as a blowing-up of $\mathcal{G}$.

3.2. Two weighted graphs are *equivalent* if one can be obtained from the other by means of a finite sequence of blowings-up and blowings-down. We will use the symbol "$\sim$" for equivalence of weighted graphs (and "$\approx$" for equivalence of weighted pairs, Definition 3.9).

Blowing-up according to a tableau

3.3. Let $\mathcal{G}_0$ be a weighted graph, e_0 a vertex of $\mathcal{G}_0$ and $c \geq p > 0$ integers. By *blowing-up* $\mathcal{G}_0$ *at* e_0 *according to* $\binom{p}{c}$, we mean producing the sequence $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ defined as follows.

(1) Let $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ be the blowing-up at e_0 and let e_1 be the vertex of $\mathcal{G}_1$ so created. Define $\left(\begin{smallmatrix} u_1 & x_1 \\ v_1 & y_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e_1 & p \\ e_0 & c-p \end{smallmatrix}\right)$.

(2) If $i \geq 1$ is such that $\mathcal{G}_i$, e_i and $\left(\begin{smallmatrix} u_i & x_i \\ v_i & y_i \end{smallmatrix}\right)$ have been defined, then:
   - (a) If $y_i = 0$ then we set $n = i$ and stop.
   - (b) If $y_i \neq 0$ then let $\mathcal{G}_{i+1}$ be the blowing-up of $\mathcal{G}_i$ at the edge $\{u_i, v_i\}$, let e_{i+1} be the vertex of $\mathcal{G}_{i+1}$ so created and define

$$\begin{pmatrix} u_{i+1} & x_{i+1} \\ v_{i+1} & y_{i+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} e_{i+1} & x_i \\ v_i & y_i - x_i \end{pmatrix} & \text{if } x_i \leq y_i, \\ \begin{pmatrix} u_i & x_i - y_i \\ e_{i+1} & y_i \end{pmatrix} & \text{if } x_i > y_i. \end{cases}$$

Remark. In 3.3 we have $n \geq 1$, with equality if and only if $p = c$. Of the n blowings-up in $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$, only $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ is a blowing-up at a vertex.

[6] In [7] and [11], a blowing-up at a vertex (resp. at an edge) is called "sprouting" (resp. "subdivisional").

Definition 3.4. Let $\mathcal{G}_0$ be a weighted graph, e_0 a vertex of $\mathcal{G}_0$ and

$$T = \begin{pmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{pmatrix}$$

a matrix such that $p_i \le c_i$ are positive integers for all i.

We define *the sequence* $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ *obtained by blowing-up* $\mathcal{G}_0$ *at* e_0 *according to* T by induction on k:

- If $k = 0$ (i.e., T is the empty matrix), then $n = 0$ (no blowing-up is performed).
- If $k = 1$, then $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ is defined in 3.3.
- If $k > 1$, then $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ is
$$\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_{m-1} \leftarrow \mathcal{G}_m \leftarrow \mathcal{G}_{m+1} \leftarrow \cdots \leftarrow \mathcal{G}_n,$$
where $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_m$ is the sequence obtained by blowing-up $\mathcal{G}_0$ at e_0 according to $\binom{p_1}{c_1}$ and $\mathcal{G}_m \leftarrow \cdots \leftarrow \mathcal{G}_n$ is obtained by blowing-up $\mathcal{G}_m$ at e_m according to $\left(\begin{smallmatrix} p_2 & \cdots & p_k \\ c_2 & \cdots & c_k \end{smallmatrix}\right)$ (where e_m is the vertex of $\mathcal{G}_m$ created by $\mathcal{G}_{m-1} \leftarrow \mathcal{G}_m$).

Definition 3.5. A *tableau* is a matrix $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)$ whose entries are integers satisfying $c_i \ge p_i \ge 1$ and $\gcd(p_i, c_i) = 1$ for all $i = 1, \ldots, k$. We allow $k = 0$, in which case we say that T is the *empty tableau* and write $T = \mathbf{1}$. The set of all tableaux is denoted $\mathcal{T}$. Given $T \in \mathcal{T}$, let $h(T)$ denote the number of columns of T which are different from $\binom{1}{1}$.

3.6. Let $T' = \begin{pmatrix} p'_1 & \cdots & p'_k \\ c'_1 & \cdots & c'_k \end{pmatrix}$ and $T'' = \begin{pmatrix} p''_1 & \cdots & p''_k \\ c''_1 & \cdots & c''_k \end{pmatrix}$ be two $2 \times k$ matrices as in 3.4. We say that T' and T'' are *equivalent* if there exists a k-tuple $(r_1, \ldots, r_k)$ of positive rational numbers satisfying $\binom{p'_i}{c'_i} = r_i \binom{p''_i}{c''_i}$ for all $i = 1, \ldots, k$. If this is the case then, given a weighted graph $\mathcal{G}_0$ and a vertex e_0 of $\mathcal{G}_0$, blowing-up $\mathcal{G}_0$ at e_0 according to T' or T'' gives the same sequence $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$.

Clearly, each matrix T' as above is equivalent to a unique tableau $T \in \mathcal{T}$ (see 3.5). Also, every Hamburger-Noether tableau $\mathrm{HN} = \begin{pmatrix} p_1 & \cdots & p_{k-1} & p_k \\ c_1 & \cdots & c_{k-1} & c_k \\ \alpha_1 & \cdots & \alpha_{k-1} & \alpha_k \end{pmatrix}$ (as in the appendix of [11]) determines a unique tableau $\overline{\mathrm{HN}} = \begin{pmatrix} \overline{p}_1 & \cdots & \overline{p}_{k-1} & \overline{p}_k \\ \overline{c}_1 & \cdots & \overline{c}_{k-1} & \overline{c}_k \end{pmatrix} \in \mathcal{T}$ where $(\overline{p}_i, \overline{c}_i) = (p_i / \gcd(p_i, c_i), c_i / \gcd(p_i, c_i))$.

3.7. Consider an arbitrary sequence $S: \ \mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ of blowings-up of weighted graphs and, for $i = 1, \ldots n$, let e_i be the vertex of $\mathcal{G}_i$ created by $\mathcal{G}_{i-1} \leftarrow \mathcal{G}_i$. Suppose that S satisfies the two conditions:

(1) if $n \ge 1$ then $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ is the blowing-up *at a vertex* e_0; and
(2) if $n \ge 2$ then, for each $i = 1, \ldots, n-1$, $\mathcal{G}_i \leftarrow \mathcal{G}_{i+1}$ is the blowing-up at the vertex e_i, or at an edge incident to e_i.

Then there exists a unique tableau $T \in \mathcal{T}$ such that S is the blowing-up of $\mathcal{G}_0$ at e_0 according to T. Moreover, those two conditions are necessary for the existence of T.

WEIGHTED PAIRS

Definition 3.8. If $\mathcal{G}$ is a nonempty weighted graph and v a vertex of $\mathcal{G}$ then we say that $(\mathcal{G}, v)$ is a *weighted pair*.

Definition 3.9. Let $(\mathcal{G}, v)$ and $(\mathcal{G}', v')$ be weighted pairs.

Let us say, provisionally, that $(\mathcal{G}', v')$ is an *elementary contraction* of $(\mathcal{G}, v)$ if $\mathcal{G}'$ is the blowing-down of $\mathcal{G}$ at some vertex $e \neq v$ and if the canonical inclusion $V' \hookrightarrow V$ maps v' to v (where V and V' are the sets of vertices of $\mathcal{G}$ and $\mathcal{G}'$ respectively).

We say that $(\mathcal{G}, v)$ is *equivalent* to $(\mathcal{G}', v')$, written $(\mathcal{G}, v) \approx (\mathcal{G}', v')$, if there exists a sequence $(\mathcal{G}_0, v_0), \dots, (\mathcal{G}_n, v_n)$ of weighted pairs satisfying $(\mathcal{G}_0, v_0) = (\mathcal{G}, v)$, $(\mathcal{G}_n, v_n) = (\mathcal{G}', v')$ and such that, for each $i = 1, \dots, n$, one of the following holds:

(1) $(\mathcal{G}_i, v_i)$ is an elementary contraction of $(\mathcal{G}_{i-1}, v_{i-1})$; or
(2) $(\mathcal{G}_{i-1}, v_{i-1})$ is an elementary contraction of $(\mathcal{G}_i, v_i)$.

In the special case where condition (1) holds for all $i = 1, \dots, n$, we say that $(\mathcal{G}, v)$ *contracts* to $(\mathcal{G}', v')$ and we write $(\mathcal{G}, v) \geq (\mathcal{G}', v')$.

When $(\mathcal{G}, v) \approx (\mathcal{G}', v')$, we sometimes identify v with v'.

Definition 3.10. A weighted pair $(\mathcal{G}, v)$ is called a *linear pair* if $\mathcal{G}$ is a linear weighted tree and v has at most one neighbor in $\mathcal{G}$.

Definition 3.11. A weighted pair $(\mathcal{L}, w)$ *satisfies the condition (0)* if $\mathcal{L}$ is a tree of the form

$$\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\overset{\omega_1}{\bullet}\!\!-\cdots\quad-\!\!\overset{\omega_m}{\bullet}\qquad (m \geq 0,\ \omega_i \in \mathbb{Z},\ \omega_i \leq -2)$$

and if w is the vertex of weight 0. (*Remark: Because w is uniquely determined by $\mathcal{L}$, we will often use the symbol $\mathcal{L}$ to represent the pair $(\mathcal{L}, w)$. For instance, we will write $(\mathcal{G}, v) \approx \mathcal{L}$, or we will say that the "weighted pair $(\mathcal{G}, v)$ is equivalent to a tree $\mathcal{L}$ satisfying the condition (0)", when we mean that $(\mathcal{G}, v) \approx (\mathcal{L}, w)$*).

If $\mathcal{L}$ satisfies the condition (0), with notation as above, we define the *transpose* of $\mathcal{L}$ by

$$\mathcal{L}^t\ :\qquad \overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\overset{\omega_m}{\bullet}\!\!-\cdots\quad-\!\!\overset{\omega_1}{\bullet}$$

We also define $\mathcal{L}^{t^i}$ $(i \geq 0)$ the obvious way: $\mathcal{L}^{t^0} = \mathcal{L}$ and $\mathcal{L}^{t^{i+1}} = (\mathcal{L}^{t^i})^t$.

In the special case where either $m = 0$ or $\omega_i = -2$ for all i, we say that $\mathcal{L}$ is *degenerate*.

We now state one of the main results of this paper. In condition (2) of the theorem, $M(\mathcal{L}) \cdot \binom{1}{\nu}$ is the product of the 2×2 matrix $M(\mathcal{L})$ (defined in 3.21, below) with the column $\binom{1}{\nu}$. For the proof, see Theorem 3.32.

Theorem 3.12. *Let $(\mathcal{G}_0, e_0)$ be a weighted pair and $\binom{p}{c} \in \mathcal{T}$, $\binom{p}{c} \neq \binom{1}{1}$. Consider the blowing-up $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ of $\mathcal{G}_0$ at e_0 according to $\binom{p}{c}$ and let e_n be the vertex of $\mathcal{G}_n$ created by $\mathcal{G}_{n-1} \leftarrow \mathcal{G}_n$. Then the following are equivalent:*

(1) *$(\mathcal{G}_n, e_n)$ is equivalent to a linear pair;*

(2) $(\mathcal{G}_0, e_0)$ *is equivalent to a tree* $\mathcal{L}$ *satisfying the condition (0) and* $\binom{p}{c} = M(\mathcal{L})\cdot\binom{1}{\nu}$ *for some integer* $\nu \geq 0$.

Moreover, suppose that these conditions are satisfied, let $\mathcal{G}_n \leftarrow \cdots \leftarrow \mathcal{G}_{n+\nu}$ *be the blowing-up of* $\mathcal{G}_n$ *at* e_n *according to the* $2\times\nu$ *tableau* $\left(\begin{smallmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1\end{smallmatrix}\right)$ *and let* $e_{n+\nu}$ *be the vertex created by* $\mathcal{G}_{n+\nu-1} \leftarrow \mathcal{G}_{n+\nu}$. *Then* $(\mathcal{G}_{n+\nu}, e_{n+\nu})$ *is equivalent to* $\mathcal{L}^t$.

Preliminaries to the proof of Theorem 3.12

Notation 3.13 (Blowing-up as an action)**.** Define a binary operation on the set $\mathcal{T}$ of tableaux (see 3.5) by $\left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k\end{smallmatrix}\right)\left(\begin{smallmatrix} p_{k+1} & \cdots & p_\ell \\ c_{k+1} & \cdots & c_\ell\end{smallmatrix}\right) = \left(\begin{smallmatrix} p_1 & \cdots & p_k & p_{k+1} & \cdots & p_\ell \\ c_1 & \cdots & c_k & c_{k+1} & \cdots & c_\ell\end{smallmatrix}\right)$. Then $\mathcal{T}$ is actually the free monoid on the set of columns $\binom{p}{c}$ where $p \leq c$ are relatively prime positive integers.

Let $(\mathcal{G}_0, e_0)$ be a weighted pair and $T \in \mathcal{T}$ a tableau, consider the blowing-up $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ of $\mathcal{G}_0$ at e_0 according to T and let e_n be the vertex of $\mathcal{G}_n$ created by $\mathcal{G}_{n-1} \leftarrow \mathcal{G}_n$. Then we will write $(\mathcal{G}_0, e_0)T = (\mathcal{G}_n, e_n)$. Hence, blowing-up is a right action of $\mathcal{T}$ on the set of weighted pairs.

3.14. Let $(\mathcal{G}, v)$ and $(\mathcal{G}', v')$ be weighted pairs and $T \in \mathcal{T}$ a tableau. If $(\mathcal{G}, v) \approx (\mathcal{G}', v')$, then $(\mathcal{G}, v)T \approx (\mathcal{G}', v')T$. Hence, blowing-up is also a right action of $\mathcal{T}$ on the set of equivalence classes of weighted pairs.

3.15. Let $\mathcal{G}$ be a weighted graph, $v_1, \dots, v_n$ its vertices and ω_i the weight of v_i. Recall that one defines the determinant of $\mathcal{G}$ by $\det(\mathcal{G}) = \det(-A)$, where A denotes the "intersection matrix" of $\mathcal{G}$, i.e., the $n \times n$ matrix with entries $A_{ii} = \omega_i$ and, if $i \neq j$, $A_{ij} = 1$ (resp. 0) if v_i, v_j are neighbors (resp. are not neighbors). Then $\det(\mathcal{G})$ is independent of the ordering of the vertices and, if $\mathcal{G}$ and $\mathcal{G}'$ are equivalent weighted graphs, $\det(\mathcal{G}) = \det(\mathcal{G}')$.

3.16 ([11], A.14)**.** Let $\mathcal{G}$ be a weighted tree, v a vertex of weight $\Omega(v)$ in $\mathcal{G}$, $\mathcal{G}_1, \dots, \mathcal{G}_n$ the branches of $\mathcal{G}$ at v and v_i the vertex of $\mathcal{G}_i$ which is a neighbor of v in $\mathcal{G}$. If $d_i = \det \mathcal{G}_i$ and $d_i' = \det(\mathcal{G}_i - \{v_i\})$, then

$$\det \mathcal{G} = -\Omega(v)\cdot d_1 \cdots d_n - \sum_{i=1}^{n} d_1 \cdots d_{i-1} d_i' \, d_{i+1} \cdots d_n \,.$$

Notation 3.17. Let $\mathcal{G}$ be a linear weighted tree $\underset{v_1}{\bullet}\!\!-\!\!-\!\!-\cdots -\!\!-\!\!\underset{v_n}{\bullet}$ and $v = v_1$. Then the following abbreviation is very convenient:

$$\det_i(\mathcal{G}, v) = \begin{cases} \det \mathcal{G}, & \text{if } i = 0, \\ \det(\mathcal{G} - \{v_1, \dots, v_i\}), & \text{if } 0 < i < n, \\ 1, & \text{if } i = n, \\ 0, & \text{if } i > n. \end{cases}$$

3.18. Let the notation be as in 3.17 and let $\Omega(v_j)$ be the weight of v_j. Then, by 3.16,

$$\det_i(\mathcal{G}, v) = -\Omega(v_{i+1}) \det_{i+1}(\mathcal{G}, v) - \det_{i+2}(\mathcal{G}, v) \qquad (0 \leq i < n).$$

In particular, if $\Omega(v_1) = 0$ then $\det_2(\mathcal{G}, v) = -\det \mathcal{G}$.

3.19. Recall that an *admissible chain* is a linear tree in which every weight is at most -2. Using 3.18, it is easy to see that every admissible chain has a strictly positive determinant; note, also, that $\varnothing$ is the only admissible chain with determinant 1. We also recall the following fact, which follows easily from 3.16 and 3.18:

Let $\mathcal{G}$ be a linear weighted tree and e a vertex of $\mathcal{G}$. Suppose that all weights in $\mathcal{G}$ are strictly negative, and that e is the only vertex of weight -1. Then:

- *If e has two neighbors and both of them have weight -2, then* $\det(\mathcal{G}) \le 0$.
- *If* $\det(\mathcal{G}) > 0$ *then $\mathcal{G}$ contracts to an admissible chain.*
- *If* $\det(\mathcal{G}) = 1$ *then $\mathcal{G}$ contracts to $\varnothing$.*

Notation 3.20 ([11], A.16)**.** Given relatively prime positive integers a and b, define $\binom{a}{b}^* = \binom{x}{y}$, where x and y are the unique nonnegative integers which satisfy

$$\begin{vmatrix} x & a \\ y & b \end{vmatrix} = -1 \qquad \text{and} \qquad x < a \ \text{ or } \ y < b.$$

Definition 3.21. Given a weighted tree $\mathcal{L}$ satisfying the condition (0), we shall now define a 2×2 matrix $M(\mathcal{L})$, and a subset $\mathcal{T}(\mathcal{L})$ of $\mathcal{T}$. Let v denote the vertex of weight 0 in $\mathcal{L}$ and consider the relatively prime integers $r_0 > r_1 \ge 0$ given by $r_0 = \det_2(\mathcal{L}, v)$ and $r_1 = \det_3(\mathcal{L}, v)$ (see 3.17 and 3.20 for notations). Then define

$$M(\mathcal{L}) = \begin{pmatrix} x & r_0 - r_1 \\ y & r_0 \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_0 - r_1 \\ r_0 \end{pmatrix}^* .$$

Note that $\mathcal{L}$ is completely determined by the second column of $M(\mathcal{L})$.

If $\mathcal{L}$ is nondegenerate (resp. degenerate) then, for each integer $\nu \ge 0$ (resp. $\nu > 0$), let T_ν temporarily denote the $2 \times (\nu + 1)$ matrix $\left(\begin{smallmatrix} p & 1 & \cdots & 1 \\ c & 1 & \cdots & 1 \end{smallmatrix}\right)$, where $\binom{p}{c} = M(\mathcal{L}) \cdot \binom{1}{\nu}$. Then $T_\nu \in \mathcal{T}$ and the first column of T_ν is not $\binom{1}{1}$. Define

$$\mathcal{T}(\mathcal{L}) = \{ T_\nu \mid \nu \ge 0 \ \ (\text{resp. } \nu > 0) \} .$$

Given $k \in \mathbb{N}$, we also define $\mathcal{T}_k(\mathcal{L}) = \left\{ T \in \mathcal{T} \mid T\binom{1}{1}^k \in \mathcal{T}(\mathcal{L}) \right\}$ (so $\mathcal{T}_0(\mathcal{L}) = \mathcal{T}(\mathcal{L})$). Here, $T\binom{1}{1}^k$ is a product in the monoid $\mathcal{T}$.

In the following statement, we abbreviate $\det(\overset{\omega_1}{\bullet}\!\!-\!\!\!-\!\!\cdot \ \cdots \ \cdot\!\!-\!\!\!-\!\!\overset{\omega_m}{\bullet})$ by $\det(\omega_1, \dots, \omega_m)$.

Lemma 3.22. *Let $\omega_1, \dots, \omega_m \le -2$ be integers (where $m \ge 1$) and define*

$$\begin{aligned} b &= \det(\omega_1, \dots, \omega_m), \\ a &= \det(\omega_2, \dots, \omega_m), \ \ a' = \det(\omega_1, \dots, \omega_{m-1}) \quad (a = 1 = a' \textit{ if } m = 1) \\ a'' &= \det(\omega_2, \dots, \omega_{m-1}) \quad (a'' = 0 \textit{ if } m = 1). \end{aligned}$$

Then:

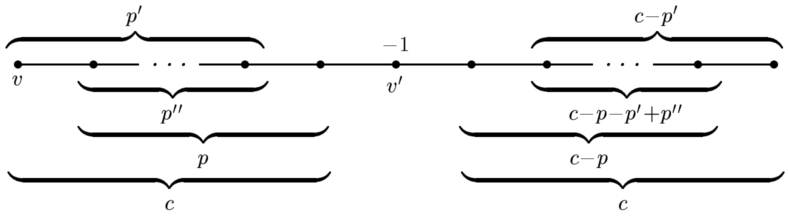
(1) $\binom{a}{b}^* = \binom{a''}{a'}$, $\binom{b-a}{b}^* = \binom{b-a-a'+a''}{b-a'}$ *and* $\binom{b-a'}{b}^* = \binom{b-a-a'+a''}{b-a}$;

(2) $\det(\omega_1, \dots, \omega_{m-1}) = b - y$ *and* $\det(\omega_2, \dots, \omega_{m-1}) = a + x - y$, *where* $\binom{x}{y} = \binom{b-a}{b}^*$.

Proof. Lemma 3.6 of [7] gives $aa' - ba'' = 1$, $0 \le a'' < \min(a, a')$ and $\max(a, a') < b$; this gives $\binom{a}{b}^* = \binom{a''}{a'}$ and it also follows that $\begin{vmatrix} b-a-a'+a'' & b-a \\ b-a' & b \end{vmatrix} = -1$. Since b, $b-a$ and $b-a'$ are positive integers, $(b-a-a'+a'')b = (b-a)(b-a')-1 \ge 0$, so $b-a-a'+a'' \ge 0$ and we obtain the second equality of assertion (1). The third equality follows from the second by symmetry, i.e., by interchanging a and a'. Assertion (2) follows from (1). □

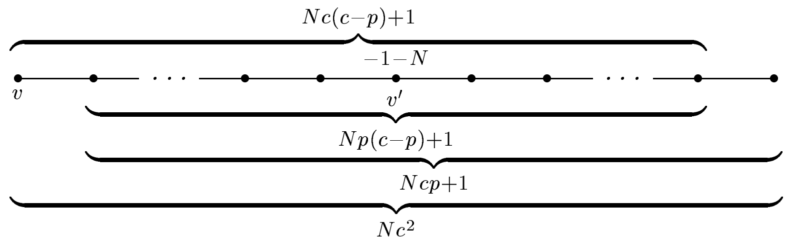
Lemma 3.23. *Let $c > p > 0$ be relatively prime integers, let $\mathcal{G}$ be the weighted graph which consists of a single vertex v of weight zero, and let $(\mathcal{G}', v') = (\mathcal{G}, v)\binom{p}{c}$. Then $\mathcal{G}'$ has two branches at v', with determinants of subtrees as follows:*

$(\mathcal{G}', v')$:



where we define $\binom{p''}{p'} = \binom{p}{c}^*$.

Moreover, if we let $(\mathcal{G}'', v'') = (\mathcal{G}', v')\binom{1}{N}$ (with $N \ge 1$) then the connected component of $\mathcal{G}'' \setminus \{v''\}$ containing v and v' is as follows:



Proof. This follows from 3.22 and from A.18.2 and A.18.3 of [11]. □

Lemma 3.24. *If $\mathcal{L}$ is a tree satisfying the condition (0) then $M(\mathcal{L}^t) = M(\mathcal{L})^t$.*

Proof. Use the notation of 3.11 for $\mathcal{L}$. If $m \ge 1$, the result follows from Lemma 3.22; if $m = 0$, it is trivial. □

We recall two properties of weigthed graphs[7] and state them in the language of weighted pairs. First, if a weighted graph is equivalent to a linear weighted graph, then it contracts to a linear weighted graph. For weighted pairs, one has:

3.25. If a weighted pair is equivalent to a linear pair, then it contracts to a linear pair.

For the second property, consider a sequence $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ of blowings-up of weighted graphs satisfying the two conditions of 3.7 and such that $(\mathcal{G}_n, e_n)$ contracts to a linear pair; then $(\mathcal{G}_j, e_j)$ contracts to a linear pair, for every $j < n$ satisfying:

[7]The first of these two facts is proved in [1], I.4.13. We don't know a reference for the second one.

$\mathcal{G}_j \leftarrow \mathcal{G}_{j+1}$ is a blowing-up *at a vertex.* This can be conveniently expressed as part (1) of:

3.26. Let $(\mathcal{G}, v)$ be a weighted pair.

(1) If there exists $T \in \mathcal{T}$ such that $(\mathcal{G}, v)T$ contracts to a linear pair, then $(\mathcal{G}, v)$ contracts to a linear pair.
(2) $(\mathcal{G}, v)$ contracts to a linear pair if and only if $(\mathcal{G}, v)\binom{1}{1}$ contracts to a linear pair.

Definition 3.27. A weighted pair $(\mathcal{L}, w)$ *satisfies the condition* $(+)$ if $\mathcal{L}$ is a tree of the form

$$\overset{\alpha}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\omega_1}{\bullet}\!\!-\!\cdots\;\;-\!\!\overset{\omega_m}{\bullet} \qquad (\alpha > 0,\ m \geq 0,\ \omega_i \in \mathbb{Z},\ \omega_i \leq -2)$$

and if w is the vertex of positive weight.

3.28. If $(\mathcal{G}, v) \approx (\mathcal{L}, w)$ are weighted pairs and $(\mathcal{L}, w)$ satisfies the condition $(+)$, then $(\mathcal{G}, v) \geq (\mathcal{L}, w)$.

The (straightforward) proof of 3.28 is left to the reader. Statement 3.29 follows immediately from 3.28:

3.29. Let $\mathcal{C}$ be an equivalence class of weighted pairs. Then:

(1) The class $\mathcal{C}$ contains a pair satisfying the condition (0) if and only if it contains one satisfying the condition $(+)$.
(2) The class $\mathcal{C}$ contains at most one pair satisfying the condition (0) and at most one pair satisfying the condition $(+)$.

Definition 3.30. A weighted pair $(\mathcal{G}, v)$ is *contractible* if it is equivalent to some pair $(\mathcal{L}, w)$ which satisfies the condition (0). Then $\mathcal{L}$ is unique (by 3.29) and we say that *$(\mathcal{G}, v)$ is of type $\mathcal{L}$.*

Lemma 3.31. *If $(\mathcal{G}, e)$ is any weighted pair then at most one integer $r \geq 0$ is such that $(\mathcal{G}, e)\binom{1}{1}^r$ is contractible.*

Proof. It suffices to show that, if $r > 0$ and $(\mathcal{L}, w)$ satisfies $(+)$, then $(\mathcal{L}', w') = (\mathcal{L}, w)\binom{1}{1}^r$ does not contract to a pair which satisfies the condition $(+)$. But this is trivial. $\square$

Main result

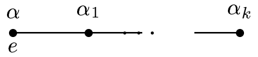
Except for notation, the following is exactly the same as Theorem 3.12.

Theorem 3.32. *Let $(\mathcal{G}, e)$ be a weighted pair and $\binom{p}{c} \in \mathcal{T}$, $\binom{p}{c} \neq \binom{1}{1}$. Then the following are equivalent:*

(1) *$(\mathcal{G}, e)\binom{p}{c}$ contracts to some linear pair;*
(2) *$(\mathcal{G}, e)$ is contractible, and $\binom{p}{c} = M(\mathcal{L}) \cdot \binom{1}{\nu}$ for some integer $\nu \geq 0$, where $\mathcal{L}$ is the type of $(\mathcal{G}, e)$.*

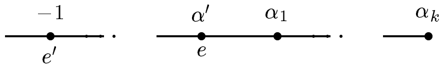
Moreover, if these conditions are satisfied then $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^\nu$ is equivalent to $\mathcal{L}^t$.

Proof. If condition (1) holds then, by 3.26, $(\mathcal{G}, e)$ contracts to a linear pair $(\mathcal{M}, e)$ which has no vertex of weight -1, except possibly e:

$(\mathcal{M}, e)$: 

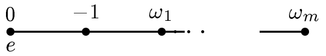
$(k \geq 0,\ \alpha_i \neq -1)$.

We claim that $(\mathcal{M}, e)$ satisfies the condition (+). Indeed, let $(\mathcal{M}', e') = (\mathcal{M}, e)\binom{p}{c}$:

$(\mathcal{M}', e')$: 

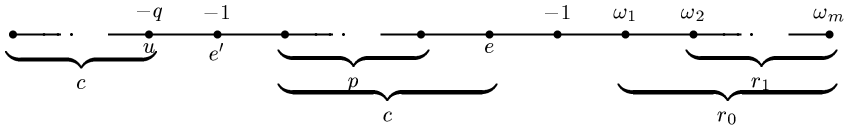
Because $p \neq c$, we know that $\mathcal{M}'$ has two branches $\mathcal{C}$ and $\mathcal{C}'$ at e'; let $\mathcal{C}$ be the one which contains e. Since $(\mathcal{M}', e')$ contracts to a linear pair (by 3.14 and 3.25), and since every vertex of $\mathcal{C}'$ has weight at most -2, $\mathcal{C}$ must be equivalent to the empty graph. This implies that all α_i are negative, so $\alpha_i \leq -2$ for all i. Another consequence is that $\alpha' = -1$, because all vertices of $\mathcal{C}$ other than e have weight at most -2. We also have $\alpha' \leq \alpha - 2$, because $\binom{p}{c}$ produces at least two blowings-up, the first blowing-up is at the vertex e and the second one is at an edge incident to e. We conclude that $\alpha > 0$, so $\mathcal{M}$ satisfies the condition (+).

In view of 3.29, condition (1) implies that $(\mathcal{G}, e)$ is equivalent to a pair $(\mathcal{L}, e)$ satisfying the condition (0); thus, in order to prove that conditions (1) and (2) are equivalent, we may assume that $(\mathcal{G}, e) \approx (\mathcal{L}, e)$:

$(\mathcal{G}, e) \approx (\mathcal{L}, e) =$ 

$(m \geq 0$ and $\omega_i \leq -2)$.

Consider the integers $r_0 = \det_2(\mathcal{L}, e)$ and $r_1 = \det_3(\mathcal{L}, e)$ used in the definition of $M(\mathcal{L})$. Write $(\mathcal{L}', e') = (\mathcal{L}, e)\binom{p}{c}$, then:

(21) $(\mathcal{L}', e')$: 

where the numbers under the braces represent the determinants of the indicated subtrees of $\mathcal{L}'$ (in particular, the p and c in the left part of the picture are obtained from 3.23). Note that the extra assumptions made for drawing this picture (e.g., $m > 1$) have no effect on the following argument.

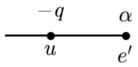
Let $\mathcal{B}$ and $\mathcal{B}'$ be the two branches of $\mathcal{L}'$ at e', where $\mathcal{B}$ is the one containing e; then, by 3.16, $\det \mathcal{B} = cr_0 - pr_0 - cr_1$. Now condition (1) of the Theorem is equivalent to $\det \mathcal{B} = 1$, hence to

$$\begin{vmatrix} p & r_0 - r_1 \\ c & r_0 \end{vmatrix} = -1. \tag{22}$$

This holds if and only if $\binom{p}{c} - \nu\binom{r_0 - r_1}{r_0} = \binom{r_0 - r_1}{r_0}^*$ for some $\nu \geq 0$, and this is equivalent to condition (2) of the Theorem. Hence, conditions (1) and (2) of the Theorem are equivalent.

Assume that conditions (1) and (2) hold; continuing with the same notation, there remains to prove that $(\mathcal{L}', e')\binom{1}{1}^\nu$ is equivalent to $\mathcal{L}^t$.

The pair $(\mathcal{L}', e')$ contracts to a linear pair $(\mathcal{L}'', e')$:

(23) $(\mathcal{L}'', e') : \cdots$  $(\alpha \geq 0,\ q \geq 2)$

where $\mathcal{L}'' - \{e'\}$ is identical to the branch $\mathcal{B}'$ of $\mathcal{L}'$ at e'. Since $\mathcal{B}'$ is nonempty and every weight in it is at most -2, we have

$$c = \det_1(\mathcal{L}'', e') > \det_2(\mathcal{L}'', e') > \det_3(\mathcal{L}'', e') \geq 0, \tag{24}$$

where the equality comes from the fact that $\det \mathcal{B}' = c$ (see the picture at line (21)).

By 3.18, $\det \mathcal{L} = -\det_2(\mathcal{L}, e) = -r_0$, so

$$\det \mathcal{H} = -r_0, \qquad \text{for each weighted graph } \mathcal{H} \text{ equivalent to } \mathcal{L}. \tag{25}$$

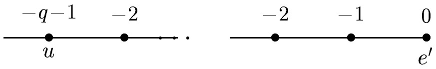
So we have $-r_0 = \det \mathcal{L}'' = -\alpha \det_1(\mathcal{L}'', e') - \det_2(\mathcal{L}'', e')$, i.e.,

$$r_0 = \alpha c + \det_2(\mathcal{L}'', e'). \tag{26}$$

We have to separate two cases.

Case $\alpha > 0$. Since $\det_2(\mathcal{L}'', e') > 0$, we have $c < r_0$ by equation (26). From this and equation (22), we deduce that $\binom{p}{c} = \binom{r_0 - r_1}{r_0}^*$ and hence that $\nu = 0$. So we have to show that $(\mathcal{L}'', e')$ is equivalent to $\mathcal{L}^t$.

Observe that $(\mathcal{L}'', e') \approx (\mathcal{L}^{(3)}, e')$, where

$(\mathcal{L}^{(3)}, e') : \cdots$ 

satisfies the condition (0) ($\mathcal{L}^{(3)}$ is obtained from $\mathcal{L}''$ by blowing-up α times). We have $\det_2(\mathcal{L}^{(3)}, e') = -\det \mathcal{L}^{(3)}$ by 3.18, and since $\det \mathcal{L}^{(3)} = -r_0$ by equation (25),

$$\det_2(\mathcal{L}^{(3)}, e') = r_0. \tag{27}$$

We have $\det_1(\mathcal{L}^{(3)}, e') = 1 \cdot \det_2(\mathcal{L}^{(3)}, e') - \det_3(\mathcal{L}^{(3)}, e') = r_0 - \det_3(\mathcal{L}^{(3)}, e')$. Since the weighted trees $\mathcal{L}^{(3)} - \{e'\}$ and $\mathcal{L}'' - \{e'\}$ are equivalent, we also have $\det_1(\mathcal{L}^{(3)}, e') = \det_1(\mathcal{L}'', e') = c$. Thus

$$\det_3(\mathcal{L}^{(3)}, e') = r_0 - c. \tag{28}$$

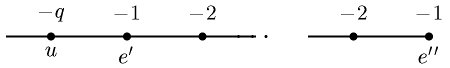
From equations (27) and (28), we obtain that the second column of $M(\mathcal{L}^{(3)})$ is $\binom{c}{r_0}$, which is identical to the second column of $M(\mathcal{L})^t = M(\mathcal{L}^t)$ (by 3.24). Hence, $\mathcal{L}^{(3)} = \mathcal{L}^t$, i.e., $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^\nu$ is equivalent to $\mathcal{L}^t$ in this case.

Case $\alpha = 0$. This time we have $\det_2(\mathcal{L}'', e') = r_0$ by (26), and $c = \det_1(\mathcal{L}'', e') = q\det_2(\mathcal{L}'', e') - \det_3(\mathcal{L}'', e')$; so, if we write $\rho = \det_3(\mathcal{L}'', e')$,

$$c = qr_0 - \rho \qquad (q \geq 2,\ \ 0 \leq \rho < r_0).$$

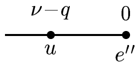
In particular we have $c > r_0$, so $\nu > 0$. Since $\alpha = 0$ and $\nu > 0$, the pair $(\mathcal{L}^{(3)}, e'') = (\mathcal{L}'', e')\binom{1}{1}^{\nu}$ looks like this:

(29) $(\mathcal{L}^{(3)}, e'') : \cdots$



and $(\mathcal{L}^{(3)}, e'') \geq (\mathcal{L}^{(4)}, e'')$, where

$(\mathcal{L}^{(4)}, e'') : \cdots$



On the other hand, if we write $M(\mathcal{L}) = \left(\begin{smallmatrix} x & r_0 - r_1 \\ y & r_0 \end{smallmatrix}\right)$ then by definition of ν we have $c = \nu r_0 + y = (\nu+1)r_0 - (r_0 - y)$ with $\nu + 1 \geq 2$ and $0 \leq r_0 - y < r_0$. So $q = \nu + 1$ and $\rho = r_0 - y$. In particular, $(\mathcal{L}^{(4)}, e'')$ satisfies the condition (0).

Since $\mathcal{L}^{(4)} - \{e'', u\}$ is identical to $\mathcal{L}'' - \{e', u\}$, we have

$$\det_i(\mathcal{L}^{(4)}, e'') = \det_i(\mathcal{L}'', e') \qquad (\text{all } i \geq 2)$$

so, in particular,

$$\det_2(\mathcal{L}^{(4)}, e'') = r_0 \quad \text{and} \quad \det_3(\mathcal{L}^{(4)}, e'') = \rho = r_0 - y.$$

So the second column of $M(\mathcal{L}^{(4)})$ is $\binom{y}{r_0}$, which is identical to the second column of $M(\mathcal{L})^t = M(\mathcal{L}^t)$. Hence, $\mathcal{L}^{(4)} = \mathcal{L}^t$, i.e., $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{\nu}$ is equivalent to $\mathcal{L}^t$. $\square$

We now give some corollaries to Theorem 3.32. See Definition 3.21 for $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}_k(\mathcal{L})$.

Corollary 3.33. *Let $(\mathcal{G}, e)$ be a weighted pair and $T = \binom{p}{c}\binom{1}{1}^r \in \mathcal{T}$, where $r \geq 0$ and $\binom{p}{c} \neq \binom{1}{1}$. Then the following are equivalent:*

1. *$(\mathcal{G}, e)T$ is contractible;*
2. *$(\mathcal{G}, e)$ is contractible of type $\mathcal{L}$ and $T \in \mathcal{T}(\mathcal{L})$.*

Moreover, if these conditions hold then $(\mathcal{G}, e)T$ is equivalent to $\mathcal{L}^t$.

Remark. By definition of $\mathcal{T}(\mathcal{L})$ (3.21), $r = 0$ can occur if and only if $\mathcal{L}$ is nondegenerate.

Proof. Suppose that condition (1) holds. Then, in particular, $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^r$ contracts to a linear pair, so $(\mathcal{G}, e)\binom{p}{c}$ contracts to a linear pair by 3.26. By Theorem 3.32, we obtain that $(\mathcal{G}, e)$ is equivalent to a tree $\mathcal{L}$ which satisfies the condition (0), that $\binom{p}{c} = M(\mathcal{L}) \cdot \binom{1}{\nu}$ for some $\nu \geq 0$ and that $(\mathcal{G}, e)\binom{p}{c}\binom{1}{1}^{\nu}$ is equivalent to $\mathcal{L}^t$. By Lemma 3.31 we get $r = \nu$; hence, $T \in \mathcal{T}(\mathcal{L})$ and $(\mathcal{G}, e)T$ is equivalent to $\mathcal{L}^t$.

The proof that (2) implies (1) is left to the reader. $\square$

Notation 3.34. $\mathcal{T}^{\#} = \mathcal{T} \setminus \binom{1}{1}\mathcal{T}$

Hence, $\mathcal{T}^{\#}$ contains the empty tableau, and all nonempty tableaux whose first column is not $\binom{1}{1}$. This is a submonoid of $\mathcal{T}$ with the property that each $T \in \mathcal{T}^{\#}$ has a unique factorization into irreducibles: $T = T_r \cdots T_1$, $T_i \in \mathcal{T}^{\#}$, $h(T_i) = 1$. Note also that $\mathcal{T}_k(\mathcal{L}) \subset \mathcal{T}^{\#}$, for any $k \in \mathbb{N}$ and $\mathcal{L}$ satisfying the condition (0).

Iterating Corollary 3.33 gives:

Corollary 3.35. *Let A be a weighted pair and $T \in \mathcal{T}^{\#}$. If $T = T_r \cdots T_1$ is the irreducible factorization of T in $\mathcal{T}^{\#}$, then the following are equivalent:*

(1) *AT is contractible of type $\mathcal{L}$,*
(2) *A is contractible of type $\mathcal{L}^{t^r}$, and $T_i \in \mathcal{T}(\mathcal{L}^{t^i})$ for all $i = 1, \dots, r$.*

Definition 3.36. A weighted pair $P = (\mathcal{G}, v)$ is *pseudo-linear* if v has exactly one neighbor v' in $\mathcal{G}$ and the connected component Γ of $\mathcal{G}$ containing v has the form:

$$\Gamma : \quad \underset{v}{\overset{0}{\bullet}} \!\!-\!\!\!-\!\!\!-\!\! \underset{v'}{\overset{x}{\bullet}} \!\!-\!\!\!-\!\!\!-\!\! \overset{\omega_1}{\bullet} \cdots \!\!-\!\!\!-\!\! \overset{\omega_n}{\bullet} \qquad (n \geq 0,\ x, \omega_i \in \mathbb{Z},\ x \leq -1,\ \omega_i \leq -2).$$

We also say that P is *pseudo-linear of type* $(-1-x, \mathcal{L})$, where $\mathcal{L}$ is the weighted pair (satisfying the condition (0)) obtained from the above picture by replacing the "x" by a "-1". If P is pseudo-linear, with Γ as in the above picture, let P^t be the weighted pair obtained from P by changing the weights in Γ, so as to obtain

$$\underset{v}{\overset{0}{\bullet}} \!\!-\!\!\!-\!\!\!-\!\! \underset{v'}{\overset{x}{\bullet}} \!\!-\!\!\!-\!\!\!-\!\! \overset{\omega_n}{\bullet} \cdots \!\!-\!\!\!-\!\! \overset{\omega_1}{\bullet},$$

and by leaving the other connected components unchanged. Note that if P is pseudo-linear of type $(k, \mathcal{L})$, then P^t is pseudo-linear of type $(k, \mathcal{L}^t)$.

If P is pseudo-linear of type $(k, \mathcal{L})$ then any weighted pair equivalent to P is said to be *pseudo-contractible*, or *pseudo-contractible of type* $(k, \mathcal{L})$ (note that the type is well-defined). If a weighted pair P is pseudo-contractible of type $(k, \mathcal{L})$, then $k \in \mathbb{N}$; if $k > 0$, then $P\binom{1}{1}$ is pseudo-contractible of type $(k-1, \mathcal{L})$.

As an immediate consequence of Corollary 3.35, we have:

Corollary 3.37. *Let P be a weighted pair and $T \in \mathcal{T}^{\#} \setminus \{\mathbf{1}\}$. If $T = T_r \cdots T_1$ is the irreducible factorization of T in $\mathcal{T}^{\#}$, then the following are equivalent:*

(1) *PT is pseudo-contractible of type $(k, \mathcal{L})$,*
(2) *P is pseudo-contractible of type $(0, \mathcal{L}^{t^r})$ and*

$$T_i \in \begin{cases} \mathcal{T}_k(\mathcal{L}^t), & \text{if } i = 1, \\ \mathcal{T}(\mathcal{L}^{t^i}), & \text{for all } i = 2, \dots, r. \end{cases}$$

4. Description of the set $\Pi(S, \mu)$

4.1. Let $f : X \to Y$ be a birational morphism of smooth complete surfaces and D a nonzero divisor of Y with strong normal crossings. We say that $\overline{\mathrm{HN}}(f, D)$ *is defined* if the following condition holds:

> If $\operatorname{center}(f) \cap \operatorname{supp}(D)$ is nonempty then it is a single point P, P belongs to exactly one component Z of D and $f^{-1}(P)$ contains exactly one (-1)-curve.

If this condition holds, then we define $\overline{\mathrm{HN}}(f, D) \in \mathcal{T}$ as follows.

- If $\operatorname{center}(f) \cap \operatorname{supp}(D) = \emptyset$, define $\overline{\mathrm{HN}}(f, D) = \mathbf{1}$ (the empty tableau).

- If $\text{center}(f) \cap \text{supp}(D) = \{P\}$, let $E \subset X$ denote the unique (-1)-curve in $f^{-1}(P)$ and choose local coordinates (ξ, η) of Y at P such that ξ is a local equation of Z. Consider the finite Hamburger-Noether tableau

$$\text{HN} = \text{HN}(E; \xi, \eta) = \text{HN}(f; \xi, \eta) = \begin{pmatrix} p_1 & \cdots & p_{k-1} & p_k \\ c_1 & \cdots & c_{k-1} & c_k \\ \alpha_1 & \cdots & \alpha_{k-1} & \alpha_k \end{pmatrix}$$

as defined in the appendix of [11]. Then HN uniquely determines a tableau $\overline{\text{HN}} \in \mathcal{T}$ (3.6) and $\overline{\text{HN}}$ is independent of the choice of (ξ, η). We define $\overline{\text{HN}}(f, D) = \overline{\text{HN}}$. Note that $\overline{\text{HN}}(f, D) = \overline{\text{HN}}(f, Z)$.

We state two important properties of $\overline{\text{HN}}(f, D)$. Recall that $\mathcal{G}(D, Y)$ denotes the dual graph of D in Y.

(1) Consider the weighted pairs $R = (\mathcal{G}(D, Y), Z)$ and $R' = (\mathcal{G}(f^{-1}(D), X), E)$, where we regard $f^{-1}(D)$ as a reduced effective divisor (with strong normal crossings) of X, and where

$$E = \begin{cases} f^{-1}(Z), & \text{if } \text{center}(f) \cap \text{supp}(D) = \emptyset, \\ \text{the } (-1)\text{-curve in } f^{-1}(P), & \text{if } \text{center}(f) \cap \text{supp}(D) = \{P\}. \end{cases}$$

Then $R' = R\,\overline{\text{HN}}(f, D)$.

(2) (a) Suppose that f factors as $X \xrightarrow{\alpha} S \xrightarrow{\beta} Y$ and that $\text{center}(\alpha) \cap \beta^{-1}(D)$, if nonempty, belongs to a unique component of $\beta^{-1}(D)$. Then

$$\overline{\text{HN}}(\beta \circ \alpha, D) = \overline{\text{HN}}(\beta, D)\; \overline{\text{HN}}(\alpha, \beta^{-1}(D)).$$

(b) Conversely, given any factorization $\overline{\text{HN}}(f, D) = BA$ with $A, B \in \mathcal{T}$, there is an essentially unique way to factor f as $X \xrightarrow{\alpha} S \xrightarrow{\beta} Y$ such that $\text{center}(\alpha) \cap \beta^{-1}(D)$, if nonempty, belongs to a unique component of $\beta^{-1}(D)$, $\overline{\text{HN}}(\beta, D) = B$ and $\overline{\text{HN}}(\alpha, \beta^{-1}(D)) = A$.

Definition 4.2. Suppose that X is a complete normal rational surface and that $I = (S, \mu)$ is an X-immersion. Let Γ, Z and Σ be the main component, 0-component and section of I, respectively, and let D be the divisor of S, with strong normal crossings, with support $S \setminus \text{dom}\,\mu$. We define two weighted pairs determined by I:

$$\mathcal{P}(I) = (\mathcal{G}(D, S), Z) \text{ and } \mathcal{L}(I) = (\mathcal{G}(\Gamma, S), Z).$$

Note that $\mathcal{L}(I)$ is the connected component of $\mathcal{P}(I)$ containing the distinguished vertex; also, if $\Sigma^2 < 0$ then $\mathcal{P}(I)$ and $\mathcal{L}(I)$ are pseudo-linear of type $(-1-\Sigma^2, \mathcal{L})$ (3.36), where $\mathcal{L}$ is the weighted pair obtained from $\mathcal{L}(I)$ by replacing the weight of Σ by "-1".

4.3. Suppose that X is a complete normal rational surface and that $I = (S, \mu)$ is an X-immersion. Let Γ, Z and Σ be the main component, 0-component and section of I, respectively.

Given any morphism $\pi : \tilde{S} \to S$ satisfying conditions (1) and (2) of 2.9, the tableau $\overline{\mathrm{HN}}(\pi, \Gamma)$ (see 4.1) contains enough information to decide whether π also satisfies conditions (3) and (4) of 2.9. Indeed,

$$\mathcal{L}(I)\, \overline{\mathrm{HN}}(\pi, \Gamma) = (\mathcal{G}(\pi^{-1}(\Gamma), \tilde{S}), E),$$

and we immediately see that conditions (3) and (4) are equivalent to

(3′) $\overline{\mathrm{HN}}(\pi, \Gamma) = \binom{1}{1}^r \binom{p}{c}$, for some $r \geq 0$ and $\binom{p}{c} \neq \binom{1}{1}$;

(4′) $\mathcal{L}(I)\, \binom{1}{1}^r \binom{p}{c}$ contracts to some linear pair.

Hence, Theorem 3.32 allows us to give a complete description of $\Pi_P(S, \mu)$ (see 2.9). In particular, if condition (4′) holds then $\mathcal{L}(I)\, \binom{1}{1}^r$ is contractible. Note that this implies $r = -1 - \Sigma^2$; it also follows that there exists an X-immersion (S_0, μ_0) in standard form, obtained from (S, μ) via a sequence of r elementary transformations of sprouting type, and there exists $\pi_0 \in \Pi(S_0, \mu_0)$, such that $(S, \mu) * \pi = (S_0, \mu_0) * \pi_0$.

Corollary 4.4. *Suppose that X is a complete normal rational surface and that (S, μ) is an X-immersion; let Γ, Z and Σ be the main component, 0-component and section of (S, μ) respectively. Let $P \in Z \setminus \Sigma$.*

(1) *$\Pi_P(S, \mu) \neq \emptyset$ if and only if $\Sigma^2 < 0$.*

(2) *If $\pi \in \Pi(S, \mu)$ then there exists an X-immersion $(S_0, \mu_0) \geq (S, \mu)$ in standard form satisfying $(S, \mu) * \pi = (S_0, \mu_0) * \pi_0$ for some $\pi_0 \in \Pi(S_0, \mu_0)$.*

(3) *Suppose that $I = (S, \mu)$ is in standard form. If $\mathcal{L}(I)$ is non-degenerate (resp. degenerate) then*

$$\Pi_P(S, \mu) = \{\pi_\nu \mid \nu \in \mathbb{N} \text{ (resp. } \nu \in \mathbb{N} \setminus \{0\})\},$$

where $\pi_\nu : \tilde{S}_\nu \to S$ is the unique birational morphism which is centered at P, whose exceptional locus has a unique (-1)-component, and which satisfies

$$\overline{\mathrm{HN}}(\pi_\nu, \Gamma) = M(\mathcal{L}(I)) \cdot \begin{pmatrix} 1 \\ \nu \end{pmatrix}.$$

Moreover, (i) the section Σ_ν of the X-immersion $(S, \mu) * \pi_\nu$ satisfies $\Sigma_\nu^2 = -1 - \nu$; and (ii) if I' is an X-immersion equivalent to $(S, \mu) * \pi_\nu$ and in standard form then $\mathcal{L}(I')$ is the transpose of $\mathcal{L}(I)$.

(4) *Suppose that (S, μ) is in standard form. Given any $\pi, \pi' \in \Pi_P(S, \mu)$, the X-immersions $(S, \mu) * \pi$ and $(S, \mu) * \pi'$ are equivalent.*

Proof. Assertion (3) is a direct consequence of Theorem 3.32. Assertion (2) was pointed out in 4.3 and the “only if” part of (1) follows from (2). Observe that (3) implies, in particular, that $\Pi_P(I)$ is nonempty whenever I is in standard form; the “if” part of (1) easily follows from this and part (1) of Lemma 2.12.

In view of (3), it suffices to prove (4) in the special case where $\pi = \pi_\nu$ and $\pi' = \pi_{\nu+1}$. Write $J = I * \pi_\nu$ and consider the X-immersion J^- obtained from J by performing one elementary transformation of subdivisional type. By part (2) of Lemma 2.12, there exists $\pi'' \in \Pi_P(I)$ such that $I * \pi'' = J^-$. By (3), the section of J has self-intersection

$-1-\nu$, so that of J^- has self-intersection $-1-(\nu+1)$. Again by part (3), we have $\pi''=\pi_n$ for some n and the section of $J^-=I*\pi''$ has self-intersection $-1-n$. Hence, $n=\nu+1$. Consequently, $I*\pi_{\nu+1}=I*\pi''=J^-$ is equivalent to $J=I*\pi_\nu$. □

5. Description of affine rulings by discrete data

See 5.3, below, for an introduction to this section.

5.1. Let X be a surface satisfying (†) and Λ an affine ruling of X.

Let $(\tilde{X},\tilde{\Lambda})=(X,\Lambda)^\sim$ be as in Proposition 1.5 and recall that $X'=X_s\setminus \mathrm{Bs}(\Lambda)\subset X$ is embedded in $\tilde{X}$ as the complement of a divisor D with strong normal crossings, and that exactly one component H of D is a section of $\tilde{\Lambda}$ (see 1.8). Let $m=-H^2\geq 1$. In view of 1.2 and convention 1.9, there is a unique birational morphism $\pi:\tilde{X}\to\mathbb{F}_m$ which contracts each reducible member of $\tilde{\Lambda}$ to a 0-curve and whose exceptional locus is disjoint from H.

Assume that Λ_* is nonempty. Each choice of an element $F\in\Lambda_*$ determines a factorization

$$\tilde{X}\xrightarrow{\pi_2}S\xrightarrow{\pi_1}\mathbb{F}_m,$$

of π, where:

- π_2 is the contraction of $\tilde{F}$ to a 0-curve, where $\tilde{F}\in\tilde{\Lambda}_*$ is the image of $F\in\Lambda_*$ under the bijection $\Lambda\to\tilde{\Lambda}$ of 1.6. (Note that π_2 is the identity map when $\tilde{F}$ is irreducible, or equivalently when F is a reduced member of Λ.)
- If some member of $\tilde{\Lambda}\setminus\{\tilde{F}\}$ is reducible then it is unique (by definition of Λ_*) and we denote it by $\tilde{G}$; if there is no such member, let $\tilde{G}$ be any member of $\tilde{\Lambda}\setminus\{\tilde{F}\}$. Let π_1 be the contraction of $\tilde{G}$ (or rather, of $\pi_2(\tilde{G})$) to a 0-curve. (This gives $\pi_1=\mathrm{id}$ when every member of $\tilde{\Lambda}\setminus\{\tilde{F}\}$ is irreducible.)

We will sometimes refer to π_1 and π_2 as *the pair of morphisms determined by* (Λ,F).

Regard $D_2=\pi_2(\mathrm{supp}(\tilde{F}+D))$ as a reduced effective divisor of S (with strong normal crossings) and observe that it has no branching component (because $F\in\Lambda_*$); note that $Z_2=\pi_2(\mathrm{supp}\,\tilde{F})$ and $\Sigma_2=\pi_2(H)$ are respectively a 0-component and a $(-m)$-component of D_2. The curve $\Sigma_1=\pi_1(\Sigma_2)=\pi_1(\pi_2(H))\subset\mathbb{F}_m$ is the negative section of the standard ruling of $\mathbb{F}_m$; also, $Z_1=\pi_1(\pi_2(\mathrm{supp}\,\tilde{G}))$ and $\pi_1(Z_2)$ are distinct members of that ruling and $D_1=Z_1+\Sigma_1+\pi_1(Z_2)$ is a divisor of $\mathbb{F}_m$ with strong normal crossings.

For each $i\in\{1,2\}$, the exceptional locus of π_i contains at most one (-1)-curve and, if $\pi_i\neq\mathrm{id}$, the center P_i of π_i is a single point and belongs to $Z_i\setminus\mathrm{supp}(D_i-Z_i)$. Thus we may consider $T_i=\overline{\mathrm{HN}}(\pi_i,D_i)\in\mathcal{T}$, as defined in 4.1. In this way, (Λ,F) determines a unique triple $(m,T_1,T_2)\in\mathbb{Z}^+\times\mathcal{T}\times\mathcal{T}$, which we call the *discrete part* of (Λ,F) (or of (X,Λ,F)).

Definition 5.2. (1) Given a triple (X,Λ,F), where X is a surface satisfying (†), Λ is an affine ruling of X and $F\in\Lambda_*$, the *discrete part* of (X,Λ,F) is the triple (m, T_1, T_2) defined in 5.1. The notation is $\mathrm{disc}(X, \Lambda, F) = (m, T_1, T_2)$. We sometimes call (m, T_1, T_2) the discrete part *of* (Λ, F).

(2) Given a surface X satisfying (†), $\mathbb{T}(X)$ denotes the set of $\mathrm{disc}(X, \Lambda, F)$ such that Λ is an affine ruling of X and $F \in \Lambda_*$; $\mathbb{T}_0(X) \subseteq \mathbb{T}(X)$ denotes the set of $\mathrm{disc}(X, \Lambda, F)$ such that Λ is a *basic* affine ruling of X and $F \in \Lambda_*$.

5.3. Let X be a surface satisfying (†). Can a description of the set $\mathbb{T}(X)$ be regarded as a solution to Problem 1 for X? There are two difficulties:

(D1) X may admit affine rulings Λ such that $\Lambda_* = \emptyset$, and $\mathbb{T}(X)$ contains no information about such rulings.

Note that if we assume that all basic affine rulings of X are known then, in particular, all Λ satisfying $\Lambda_* = \emptyset$ are known (see 2.5); this is why (D1) did not cause problems in sections 2 and 4. In this section, however, (D1) can only be resolved by assuming that X satisfies (‡), in which case all Λ satisfy $\Lambda_* \neq \emptyset$ (by 2.5 again).

(D2) Given $\tau = (m, T_1, T_2) \in \mathbb{T}(X)$, we need a method for constructing all (Λ, F) (on X) such that $\mathrm{disc}(X, \Lambda, F) = \tau$.

Paragraph 5.29, below, describes a method for constructing all (X', Λ', F') such that $\mathrm{disc}(X', \Lambda', F') = \tau$, and this is good enough for (D2) if one can prove that all such X' are isomorphic to X. Thus Corollary 5.32 implies that, if X satisfies (‡), describing $\mathbb{T}(X)$ does solve Problem 1 for X.

Some of the results of this section (5.17, 5.22, 5.23, 5.39) describe $\mathbb{T}(X)$ in terms of $\mathbb{T}_0(X)$, or in terms of the subset $\min \mathbb{T}(X)$ of $\mathbb{T}_0(X)$. So, given X satisfying (‡),

this section reduces Problem 1 to the problem of describing $\mathbb{T}_0(X)$ *or* $\min \mathbb{T}(X)$.

Definition 5.4. (1) Let $n \geq 1$. By a *weighted n-tuple*, we mean an ordered n-tuple $S = (\mathcal{G}, v_1, \dots, v_{n-1})$ where $\mathcal{G}$ is a weighted graph and $v_1, \dots, v_{n-1}$ are distinct vertices of $\mathcal{G}$ (when $n = 1$, S is a weighted graph; when $n = 2$, it is a weighted pair 3.8).

(2) Let S be a weighted n-tuple, with $n \geq 2$. Given $T \in \mathcal{T}$, we define a weighted n-tuple ST and a weighted $(n-1)$-tuple $S \ominus T$ as follows. Write $S = (\mathcal{G}, v_1, \dots, v_{n-1})$ and let $(\mathcal{G}', e)$ denote the weighted pair $(\mathcal{G}, v_1)T$. Note that $v_2, \dots, v_{n-1}$ can be regarded as vertices of $\mathcal{G}' \setminus \{e\}$. Then we define

$$ST = (\mathcal{G}', e, v_2, \dots, v_{n-1}) \quad \text{and} \quad S \ominus T = (\mathcal{G}' \setminus \{e\}, v_2, \dots, v_{n-1}).$$

Remarks. Let $S = (\mathcal{G}, v_1, \dots, v_{n-1})$ be a weighted n-tuple.

(1) When $n = 2$, the definition of ST given in 5.4 agrees with the one given in section 3.

(2) The above definition gives $S\mathbf{1} = S$ and $S \ominus \mathbf{1} = (\mathcal{G} \setminus \{v_1\}, v_2, \dots, v_{n-1})$ (where $\mathbf{1}$ is the empty tableau). So, given $T, T' \in \mathcal{T}$, $S \ominus T = ST \ominus \mathbf{1}$ and $S \ominus (TT') = (ST) \ominus T'$.

(3) Let P and P' be weighted pairs and $T \in \mathcal{T}$. If $P \approx P'$ then, by 3.14, $P \ominus T \sim P' \ominus T$ (where "$\approx$" (resp. "$\sim$") means equivalence of weighted pairs (resp. weighted graphs)).

Notation 5.5. Given $x \in \mathbb{Z}$, let $\mathcal{G}_{(x)}$ denote the weighted triple $(\mathcal{G}, v_1, v_2)$, where $\mathcal{G}$ is the weighted graph $\overset{0}{\underset{v_1}{\bullet}} \text{——} \overset{x}{\bullet} \text{——} \overset{0}{\underset{v_2}{\bullet}}$.

5.6. *Consider the weighted pair S consisting of a single vertex of weight zero. For any $T \in \mathcal{T}$, the condition*

$S \ominus T$ *has no branch point and every weight in it is strictly less than* -1

holds if and only if one of the following holds:

(1) $T = \mathbf{1}$;
(2) $T = \binom{p}{c}$, *where* $\binom{p}{c} \neq \binom{1}{1}$;
(3) $T = \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$, *where* $\binom{p}{c} \neq \binom{1}{1}$ *and* $N \geq 1$.

5.7. *Let x be a negative integer and $T_1, T_2 \in \mathcal{T}$.*

(1) *The condition*

$$\mathcal{G}_{(x)} \ominus T_1 \text{ is pseudo-linear}$$

holds if and only if T_1 satisfies one of conditions (1–3) of 5.6. *Moreover, if $\mathcal{G}_{(x)} \ominus T_1$ is pseudo-linear then it has at most two connected components and the one which does not contain the distinguished vertex is an admissible chain.*

(2) *The condition*

$(\mathcal{G}_{(x)} \ominus T_1) \ominus T_2$ *has no branch point and every weight in it, except possibly that of the middle vertex of $\mathcal{G}_{(x)}$, is strictly less than -1,*

holds if and only if each of T_1, T_2 satisfies one of conditions (1–3) of 5.6.

Proof. To prove (1), write $\mathcal{G}_{(x)} = (\mathcal{G}, v_1, v_2)$ and consider the weighted pair $S = (\{v_1\}, v_1)$ (a single vertex of weight 0). We may regard $S \ominus T$ as the graph obtained from the weighted pair $P = \mathcal{G}_{(x)} \ominus T$ by deleting the distinguished vertex (i.e., v_2), its unique neighbor and all edges incident to these two vertices. Note that P has at most two connected components, say $\mathcal{L}$ and A, where $\mathcal{L}$ contains the distinguished vertex and A is a (possibly empty) admissible chain. If P is pseudo-linear, $S \ominus T$ has no branch point (otherwise $\mathcal{L}$ would have one) and every weight in $S \ominus T$ is strictly less than -1; thus (by 5.6) T satisfies one of conditions (1–3) of 5.6. The converse is equally trivial, as is assertion (2). $\square$

Notation 5.8. Given $T \in \mathcal{T}$ satisfying one of the conditions (1–3) of 5.6, we define $\check{T} \in \mathcal{T}$ as follows:

$$\check{T} = \begin{cases} \mathbf{1}, & \text{if } T \text{ satisfies 5.6.1;} \\ \binom{p'}{c}, & \text{if } T \text{ satisfies 5.6.2, where } p' \text{ is given by } \binom{p''}{p'} = \binom{p}{c}^* \text{ (see 3.20);} \\ \left(\begin{smallmatrix} c-p & 1 \\ c & N \end{smallmatrix}\right), & \text{if } T \text{ satisfies 5.6.3.} \end{cases}$$

Note that if T satisfies condition 5.6.i (where $i \in \{1,2,3\}$) then so does $\check{T}$. If s is a positive integer, write $T^{(\check{}s)} = \left(T^{(\check{}(s-1))}\right)\check{}$, where $T^{(\check{}0)} = T$. Note that $T^{(\check{}2)} = T$.

Lemma 5.9. *Let (m, T_1, T_2) be the discrete part of (X, Λ, F), where X is a surface satisfying $(\dagger)$, Λ is an affine ruling of X and $F \in \Lambda_*$.*

(1) *The weighted pair $P = \mathcal{G}_{(-m)} \ominus T_1$ is isomorphic to $\mathcal{P}(I)$ (see 4.2), where I is the distinguished element of the equivalence class of X-immersions determining (Λ, F). In particular, P is pseudo-linear of type $(m-1, \mathcal{L})$ for some $\mathcal{L}$; moreover, P has at most one connected component A other than the one containing the distinguished vertex, and A is an admissible chain.*

(2) *There is an isomorphism of weighted graphs $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2 \to \mathcal{G}(\Lambda)$ which maps the middle vertex of $\mathcal{G}_{(-m)}$ to the vertex H of $\mathcal{G}(\Lambda)$ (see 1.13 for the definition of $\mathcal{G}(\Lambda)$; H denotes the unique component of $\tilde{X} \setminus X'$ which is a section of $\tilde{\Lambda}$).*

Proof. Let the notation (S, D_2, etc.) be as in 5.1. By definition of I (2.10), we have $I = (S, \mu)$ for some μ and, moreover, $S \setminus \operatorname{dom} \mu = \operatorname{supp}(D_2)$. So we have $\mathcal{P}(I) = (\mathcal{G}(D_2, S), Z_2)$. For each $i = 1, 2$, let

$$E_i = \begin{cases} \text{the unique } (-1)\text{-curve in } \pi_i^{-1}(P_i), & \text{if } \pi_i \neq \mathrm{id}, \\ Z_i, & \text{if } \pi_i = \mathrm{id}. \end{cases}$$

Consider the weighted triple $(\mathcal{G}(D_1, \mathbb{F}_m), Z_1, \pi_1(Z_2)) = \mathcal{G}_{(-m)}$. Since $\pi_1^{-1}(\operatorname{supp}(D_1)) = \operatorname{supp}(D_2) \cup E_1$ and E_1 is not a component of D_2, we have $\mathcal{G}_{(-m)} \ominus T_1 = \mathcal{P}(I)$ and (1) holds. Since $\pi_2^{-1}(D_2) = \operatorname{supp}(D) \cup E_2$ and E_2 is not a component of D, $\mathcal{P}(I) \ominus T_2 = \mathcal{G}(D, \tilde{X}) = \mathcal{G}(\Lambda)$. □

Notation 5.10. (1) Let $\mathbb{T}$ be the set of triples $(m, T_1, T_2) \in \mathbb{Z}^+ \times \mathcal{T} \times \mathcal{T}$ such that $T_2 \in \mathcal{T}^{\#}$ (3.34) and T_1 satisfies one of the conditions (1–3) of 5.6.

(2) Let $\mathbb{T}(\dagger)$ be the set of $(m, T_1, T_2) \in \mathbb{T}$ such that the intersection matrix (3.15) of the weighted graph $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ is negative definite.

The following says, in particular, that $\mathbb{T}(X) \subseteq \mathbb{T}(\dagger)$ for each X satisfying $(\dagger)$.

Lemma 5.11. *Let (m, T_1, T_2) be the discrete part of (X, Λ, F), where X is a surface satisfying $(\dagger)$, Λ is an affine ruling of X and $F \in \Lambda_*$. Then $(m, T_1, T_2) \in \mathbb{T}(\dagger)$ and the following are equivalent:*

(1) *X satisfies $(\ddagger)$ and Λ is basic;*
(2) *T_2 satisfies one of the conditions (1–3) of 5.6.*

Proof. By 5.7 and part (1) of 5.9, T_1 satisfies one of the conditions (1–3) of 5.6.

By part (2) of Lemma 5.9, every vertex of $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$, except possibly the middle vertex of $\mathcal{G}_{(-m)}$, has weight strictly less than -1. Write $\mathcal{G}_{(-m)} = (\mathcal{G}, v_1, v_2)$ and note that the distinguished vertex v_2 of the weighted pair $\mathcal{G}_{(-m)} \ominus T_1$ has weight

0. If $T_2 \notin \mathcal{T}^{\#}$ then $T_2 \neq \mathbf{1}$ and the first column of T_2 is $\binom{1}{1}$, so the weight of v_2 in $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ is -1, contradicting the above observation. Hence, $T_2 \in \mathcal{T}^{\#}$.

Let $\hat{X} \to X$ be the minimal resolution of singularities of X and let $\hat{E} \subset \hat{X}$ be the exceptional locus; since X is normal, the divisor $\hat{E}$ has a negative definite intersection matrix; since $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2 \cong \mathcal{G}(\Lambda)$ by 5.9, and $\mathcal{G}(\Lambda)$ contracts to $\mathcal{G}(\hat{E}, \hat{X})$, we get $(m, T_1, T_2) \in \mathbb{T}(\dagger)$.

By 5.7, $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ (hence $\mathcal{G}(\Lambda)$) has no branch point if and only if T_2 satisfies one of the conditions (1–3) of 5.6. Hence, (1) and (2) are equivalent. $\square$

Definition 5.12. Given $(n, T_1, T_2), (m, T_1', T_2') \in \mathbb{T}$, write $(n, T_1, T_2) \equiv (m, T_1', T_2')$ to indicate that

$$(\mathcal{G}_{(-n)} \ominus T_1)T_2 \approx (\mathcal{G}_{(-m)} \ominus T_1')T_2'$$

(equivalence of weighted pairs). Note that "$\equiv$" is an equivalence relation on the set $\mathbb{T}$.

Theorem 5.13. *Let $\tau, \tau' \in \mathbb{T}$ be such that $\tau \equiv \tau'$. Suppose that $\tau = \operatorname{disc}(X, \Lambda, F)$, where X is a surface satisfying ($\dagger$), Λ is an affine ruling of X and $F \in \Lambda_*$. Then there exist an affine ruling Λ' of X and $F' \in \Lambda'_*$ such that $\tau' = \operatorname{disc}(X, \Lambda', F')$ and $\operatorname{supp}(F') = \operatorname{supp}(F)$.*

In view of 2.14, the above result relates the viewpoint of this section with the operation "$*$" of sections 2 and 4. See also Proposition 5.23.

The proof requires 5.14 and 5.15:

Lemma 5.14. *If P is a pseudo-linear weighted pair then:*

(1) *At most one pair $(x, T) \in \mathbb{Z} \times \mathcal{T}$ satisfies $\mathcal{G}_{(x)} \ominus T = P$.*
(2) *Suppose that $\mathcal{G}_{(x)} \ominus T = P$. Then T satisfies one of conditions (1–3) of 5.6 and $\mathcal{G}_{(x)} \ominus \check{T} = (\mathcal{G}_{(x)} \ominus T)^t = P^t$.*

Proof. Write $P = (\mathcal{G}, v)$. We may assume that $P = \mathcal{G}_{(x)} \ominus T$ for some (x, T). Then v has a unique neighbor v' in $\mathcal{G}$, and the weight of v' is x; hence, x is uniquely determined. By 5.7, T satisfies one of conditions (1–3) of 5.6 (which proves part of assertion (2)). Note also that $\mathcal{G}$ has either one or two connected components; we say that $\mathcal{G}$ has two connected components, $\mathcal{L}$ and A, where $\mathcal{L}$ contains v and A is a (possibly empty) admissible chain. Moreover, $\mathcal{L}$ is as follows:

$$\mathcal{L}: \quad \underset{v}{\overset{0}{\bullet}} \!-\!-\!-\! \underset{v'}{\overset{x}{\bullet}} \!-\!-\!-\! \underset{y_1}{\overset{\omega_1}{\bullet}} \!-\!- \cdots \;\; -\!-\! \underset{y_n}{\overset{\omega_n}{\bullet}} \qquad (n \geq 0,\ \omega_i \in \mathbb{Z},\ \omega_i \leq -2).$$

We now show that T is unique. If $n = 0$ (resp. $n = 1$) then T must be $\mathbf{1}$ (resp. $\binom{1}{-\omega_1}$), so we may assume that $n \geq 2$. We consider two cases.

If A is nonempty and contains a weight other than -2, then $T = \binom{p}{c}$ for some p, c satisfying $1 < p < c$. Then Lemma 3.23 implies that $c = \det(\omega_1, \ldots, \omega_n)$ and $p = \det(\omega_2, \ldots, \omega_n)$, so T is unique (notation as in Lemma 3.22).

Before treating the second case, let us observe that at most one $i \in \{1, \ldots, n\}$ can satisfy $\det(\omega_1, \ldots, \omega_{i-1}) = \det(\omega_{i+1}, \ldots, \omega_n)$, because the left-hand-side is a strictly increasing function of i, while the right-hand-side is strictly decreasing.

If A is a chain of $N-1$ vertices of weight -2 (where $N \geq 1$), then $T = \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$, for some $\binom{p}{c} \neq \binom{1}{1}$. Consider the vertex e (of weight -1) which is deleted from $\mathcal{G}_{(x)}T$ in order to define $\mathcal{G}_{(x)} \ominus T$; then e has a unique neighbor among $\{y_1, \dots, y_n\}$, say y_j. By Lemma 3.23 applied to the first column $\binom{p}{c}$ of T, we have $\det(\omega_1, \dots, \omega_{j-1}) = c = \det(\omega_{j+1}, \dots, \omega_n)$ and $\det(\omega_2, \dots, \omega_{j-1}) = p$. So j must be the unique i of the preceding paragraph; since j is uniquely determined, so are $c = \det(\omega_1, \dots, \omega_{j-1})$ and $p = \det(\omega_2, \dots, \omega_{j-1})$. This proves assertion (1).

Assertion (2) is obtained from the following observation, which is a consequence of Lemma 3.23: Let $(\mathcal{G}, v)$ be the weighted pair consisting of a single vertex of weight 0, let $\binom{p}{c} \in \mathcal{T}$, $\binom{p}{c} \neq \binom{1}{1}$, and consider the weighted pair $(\mathcal{G}', v') = (\mathcal{G}, v)\binom{p}{c}$. Use the following notation for the weights in $(\mathcal{G}', v')$:

$$(\mathcal{G}', v') = (\mathcal{G}, v)\binom{p}{c} \ : \qquad \underset{v}{\overset{\omega_1}{\bullet}} \!-\!\!-\; \cdots \;-\!\!-\!\!-\!\!-\!\!\overset{\omega_n}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{v'}{\overset{-1}{\bullet}}\!\!-\!\!-\!\!-\!\!\overset{a_1}{\bullet}\!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!\overset{a_m}{\bullet}\,.$$

If we define p' by $\binom{p''}{p'} = \binom{p}{c}^*$, then:

$$(\mathcal{G}, v)\binom{p'}{c} \ : \qquad \underset{v}{\overset{\omega_n}{\bullet}} \!-\!\!-\; \cdots \;-\!\!-\!\!-\!\!-\!\!\overset{\omega_1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{a_m}{\bullet}\!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!\overset{a_1}{\bullet}\,.$$

On the other hand,

$$(\mathcal{G}, v)\binom{c-p}{c} \ : \qquad \underset{v}{\overset{a_m}{\bullet}} \!-\!\!-\; \cdots \;-\!\!-\!\!-\!\!-\!\!\overset{a_1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{\omega_n}{\bullet}\!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!\overset{\omega_1}{\bullet}\,.$$

□

5.15. Let S be a smooth complete surface, D a divisor of S with strong normal crossings and such that each component of D is rational, and $\mathcal{G} = \mathcal{G}(D, S)$, the dual graph of D in S. Let also $\mathcal{G}'$ be a weighted graph.

(1) Suppose that $\mathcal{G}$ can be contracted to $\mathcal{G}'$. Let $v_1, \dots, v_n$ be the vertices of $\mathcal{G}$ which disappear in that process and let $D_1, \dots, D_n$ be the corresponding components of D. Then there is an essentially unique birational morphism $\pi : S \to S'$ whose exceptional locus is $D_1 \cup \cdots \cup D_n$ (where S' is a smooth complete surface). Then the divisor $D' = \pi(D)$ of S' with strong normal crossings has dual graph $\mathcal{G}'$.

(2) Suppose that $\mathcal{G}'$ can be contracted to $\mathcal{G}$. Then there exists a (not necessarely unique) birational morphism $\pi : S' \to S$ (where S' is a smooth complete surface) such that the divisor $D' = \pi^{-1}(D)$ of S', with strong normal crossings, satisfies $\mathcal{G}(D', S') = \mathcal{G}'$. The exceptional locus of π consists of the components of D' corresponding to the vertices of $\mathcal{G}'$ which disappear in the contraction to $\mathcal{G}$.

In case (1) (resp. (2)), we call π simply "the (resp. a) birational morphism corresponding to $\mathcal{G} \geq \mathcal{G}'$ (resp. $\mathcal{G}' \geq \mathcal{G}$)"; it is tacitely assumed that the above conditions are satisfied. Similar remarks hold if both $\mathcal{G}$ and $\mathcal{G}'$ are weighted pairs; in this case, we have the additional information that π does not shrink the curve which corresponds to the distinguished vertex.

Proof of Theorem 5.13. Write $\tau = (n, T_1, T_2)$ and $\tau' = (m, T_1', T_2')$.

Consider $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$ and recall that $X \setminus (\operatorname{Sing} X \cup \operatorname{Bs} \Lambda)$ is embedded in $\tilde{X}$ as the complement of a divisor D with strong normal crossings. Also, consider the curve $C = C_{\tilde{F}}$ in $\tilde{X}$ (notation as in 1.8), where $\tilde{F} \in \tilde{\Lambda}$ corresponds to $F \in \Lambda$ via the bijection $\Lambda \to \tilde{\Lambda}$ (1.6). Then we have $(\mathcal{G}_{(-n)} \ominus T_1)T_2 = (\mathcal{G}(D + C, \tilde{X}), C)$.

Since $\tau \equiv \tau'$, we have $(\mathcal{G}(D + C, \tilde{X}), C) \approx (\mathcal{G}_{(-m)} \ominus T_1')T_2'$; this can be written as $(\mathcal{G}(D + C, \tilde{X}), C) \leq \mathcal{P} \geq (\mathcal{G}_{(-m)} \ominus T_1')T_2'$, where $\mathcal{P}$ is a weighted pair and the inequalities indicate contractions of weighted pairs. In view of 5.15 we may consider a diagram $\tilde{X} \xleftarrow{\omega} \Omega \xrightarrow{\omega'} \tilde{Y}$, where Ω and $\tilde{Y}$ are smooth complete surfaces and ω and ω' are birational morphisms corresponding to $(\mathcal{G}(D+C, \tilde{X}), C) \leq \mathcal{P}$ and $\mathcal{P} \geq (\mathcal{G}_{(-m)} \ominus T_1')T_2'$ respectively. Define $D' = \omega'(\omega^{-1}D)$ and $C' = \omega'(\tilde{C})$, where $\tilde{C} \subset \Omega$ is the strict transform of C. Then $D' + C'$ is a divisor of $\tilde{Y}$ with strong normal crossings and $(\mathcal{G}_{(-m)} \ominus T_1')T_2' = (\mathcal{G}(D' + C', \tilde{Y}), C')$. Moreover, $\tilde{X} \leftarrow \Omega \to \tilde{Y}$ gives an isomorphism $\tilde{Y} \setminus \operatorname{supp} D' \to \tilde{X} \setminus \operatorname{supp} D$ which maps C' onto C.

Since $(\mathcal{G}(D'+C', \tilde{Y}), C') = (\mathcal{G}_{(-m)} \ominus T_1')T_2'$, the weighted graph $\mathcal{G}(D'+C', \tilde{Y})$ contracts to the underlying weighted graph of $\mathcal{G}_{(-m)} \ominus T_1'$; by 5.15 again, this contraction gives a birational morphism $\pi_2' : \tilde{Y} \to S'$, where S' is smooth. Consider the divisor $M' = \pi_2'(D' + C')$ of S' (with strong normal crossings); then $\mathcal{G}(M', S')$ is the underlying weighted graph of $\mathcal{G}_{(-m)} \ominus T_1'$, i.e., $(\mathcal{G}(M', S'), Z') = \mathcal{G}_{(-m)} \ominus T_1'$ for some component Z' of M'. By 5.7, $\mathcal{G}_{(-m)} \ominus T_1'$ is pseudo-linear, has at most two connected componenents, and the connected component which does not contain the distinguished vertex is an admissible chain. Thus we obtain an X-immersion (S', μ'), where $\mu' : S' \setminus \operatorname{supp}(M') \to X_s \setminus \operatorname{supp}(F)$ is the isomorphism determined by π_2', ω', ω and $\tilde{X} \setminus \operatorname{supp}(D + C) \cong X_s \setminus \operatorname{supp}(F)$. The X-immersion (S', μ') determines an affine ruling Λ' of X and an element F' of Λ_*' satisfying $\operatorname{supp}(F') = \operatorname{supp}(F)$ (because the image of μ' is $X_s \setminus \operatorname{supp}(F)$). Also, Z' is the 0-component of (S', μ') and let Σ' be the section of (S', μ'). Since the unique neighbor of the distinguished vertex of $\mathcal{G}_{(-m)} \ominus T_1'$ has weight $-m$, we have $(\Sigma')^2 = -m$. Note that $(X, \Lambda')^\sim = (\tilde{Y}, |Z'|^\sim)$, where $|Z'|^\sim$ denotes the strict transform of $|Z'|$. Also, $\overline{\mathrm{HN}}(\pi_2', M')$ is defined and is equal to T_2'.

Let $\pi_1' : S' \to \mathbb{F}_m$ be the unique birational morphism which contracts each reducible member of $|Z'|$ to a 0-curve and whose exceptional locus is disjoint from Σ' (1.2).

We claim that $N(\pi_1') = m' - 2$, where m' is the number of irreducible components of M' and $N(\pi_1')$ is (as usual) the number of irreducible components in the exceptional locus of π_1'. To see this, let π be the composition $\Omega \xrightarrow{\omega} \tilde{X} \xrightarrow{\pi_2} S \xrightarrow{\pi_1} \mathbb{F}_n$, where π_1 and π_2 are the two morphisms determined by (X, Λ, F) as in 5.1. Then $N(\pi) = |\mathcal{P}| - 2$ and consequently $N(\pi') = |\mathcal{P}| - 2$, where π' is the composition $\Omega \xrightarrow{\omega'} \tilde{Y} \xrightarrow{\pi_2'} S' \xrightarrow{\pi_1'} \mathbb{F}_m$. Since ω' and π_2' correspond to contractions of graphs, it follows that $N(\pi_1') = |\mathcal{G}_{(-m)} \ominus T_1'| - 2$, from which the claim follows.

Note that $\pi_1'(\Sigma')$ is the negative section of the standard ruling Λ_m of $\mathbb{F}_m$ and that $\pi_1'(Z')$ is a member of Λ_m. We claim that, for some member L of Λ_m other than $\pi_1'(Z')$,

(30) $\pi_1'(M') \subseteq L + \pi_1'(\Sigma') + \pi_1'(Z')$, center$(\pi_1') \subset L \setminus \pi_1'(\Sigma')$ and $\overline{\mathrm{HN}}(\pi_1', L) = T_1'$.

The verification of this splits into two cases.

If $m' = 2$ then $|\mathcal{G}_{(-m)} \ominus T_1'| = 2$, so T_1' is the empty tableau. On the other hand, $N(\pi_1') = m' - 2 = 0$ implies that π_1' is an isomorphism. If we let L be any member of Λ_m other than $\pi_1'(Z')$, then (30) holds.

If $m' > 2$ then $\mathcal{G}_{(-m)} \ominus T_1'$ has more than 2 vertices, so the middle vertex of $\mathcal{G}_{(-m)}$ has exactly two neighbors, i.e., Σ' has two neighbors Z' and Z'' in M'. Let $M_1', \dots, M_{m'-3}'$ be the components of M' other than Z', Σ' and Z''; since each M_i' is contained in a member of the ruling $|Z'|$ (because $M_i' \cap Z' = \emptyset$) and is disjoint from Σ', each M_i' is shrunk by π_1'. Since $N(\pi_1') = m'-2$, the exceptional locus of π_1' is $E \cup M_1' \cup \dots \cup M_{m'-3}'$, for some curve E not contained in $M_1' \cup \dots \cup M_{m'-3}'$. Since $(M_i')^2 < -1$ for all i, E is the unique (-1)-component of the exceptional locus of π_1'. Let $L = \pi_1'(Z'')$ and note that L is a member of Λ_m other than $\pi_1'(Z')$ and satisfying:

$$\pi_1'(M') = L + \pi_1'(\Sigma') + \pi_1'(Z') \quad \text{and} \quad \text{center}(\pi_1') \subset L \setminus \pi_1'(\Sigma').$$

Using $\mathcal{G}_{(-m)} = (\mathcal{G}(\pi_1'(M'), \mathbb{F}_m), L, \pi_1'(Z'))$, we obtain

$$\begin{aligned} \mathcal{G}_{(-m)} \ominus \overline{\mathrm{HN}}(\pi_1', L) &= (\mathcal{G}(\pi_1'(M'), \mathbb{F}_m), L, \pi_1'(Z')) \ominus \overline{\mathrm{HN}}(\pi_1', L) \\ &= (\mathcal{G}(M', S'), Z') = \mathcal{G}_{(-m)} \ominus T_1', \end{aligned}$$

so $\overline{\mathrm{HN}}(\pi_1', L) = T_1'$ by 5.14 and (30) holds in this case too.

We conclude that π_1' and π_2' are the two morphisms determined by (X, Λ', F') (5.1) and that disc$(X, \Lambda', F') = (m, T_1', T_2')$. □

The following order relation is useful for describing $\mathbb{T}(X)$ explicitely:

Definition 5.16. We define a transitive relation $>$ on the set $\mathbb{T}$ by declaring that $(n, T_1, T_2) > (m, T_1', T_2')$ if $n = 1$ and the following holds:

> Let $\mathcal{L}$ be the weighted pair such that $\mathcal{G}_{(-m)} \ominus T_1'$ is pseudo-linear of type $(m-1, \mathcal{L})$. Then there exist an integer $s \geq 1$ and tableaux $X_1, \dots, X_s$ such that $T_1 = (T_1')^{(\backsim s)}$, $T_2 = X_s \cdots X_1 T_2'$ and $X_i \in \mathcal{T}_{k_i}(\mathcal{L}^{t^i})$, where $k_1 = m - 1$ and $k_i = 0$ for all $i > 1$.

We define the symbols $<$, $\geq$ and $\leq$ the usual way. (See Definition 3.21 for $\mathcal{T}_k(\mathcal{L})$.)

Remark. There cannot be an infinite descending sequence $\tau_1 > \tau_2 > \cdots$ in $\mathbb{T}$. Indeed, if $(n, T_1, T_2) > (m, T_1', T_2')$ then the number of columns of T_2' is strictly less than that of T_2.

Note that, if $\tau' \in \mathbb{T}$ is given, we may explicitely describe all $\tau \in \mathbb{T}$ satisfying $\tau > \tau'$ (this is done in 5.39, below). Thus the following (see also Corollary 5.22) describes $\mathbb{T}(X)$ in terms of $\mathbb{T}_0(X)$:

Corollary 5.17. *Let X be a surface satisfying (†).*

(1) *If $\tau, \tau' \in \mathbb{T}$ are such that $\tau > \tau'$, then $\tau \in \mathbb{T}(X) \iff \tau' \in \mathbb{T}(X)$.*
(2) *Given any $\tau \in \mathbb{T}(X) \setminus \mathbb{T}_0(X)$, there exists $\tau' \in \mathbb{T}_0(X)$ such that $\tau > \tau'$.*

Although 5.17 is essentially a corollary of Theorem 5.13, its proof requires some preparation.

Lemma 5.18. *Let $(n, T_1, T_2), (m, T_1', T_2') \in \mathbb{T}$.*

(1) *If $(n, T_1, T_2) > (m, T_1', T_2')$ then $(n, T_1, T_2) \equiv (m, T_1', T_2')$.*
(2) *If $(n, T_1, T_2) \equiv (m, T_1', T_2')$ then $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$ and $(\mathcal{G}_{(-m)} \ominus T_1') \ominus T_2'$ are equivalent weighted graphs and consequently*
$$(n, T_1, T_2) \in \mathbb{T}(\dagger) \iff (m, T_1', T_2') \in \mathbb{T}(\dagger).$$

Proof. Suppose that $(n, T_1, T_2) > (m, T_1', T_2')$. Recall that $n = 1$ and let the notations $(\mathcal{L},\ s,\ X_1, \dots, X_s,\ k_i)$ be as in Definition 5.16. Since $\mathcal{G}_{(-m)} \ominus T_1'$ is pseudo-linear of type $(m-1, \mathcal{L})$, it follows that $\mathcal{G}_{(-1)} \ominus T_1'$ is pseudo-linear of type $(0, \mathcal{L})$, so $\mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)} = (\mathcal{G}_{(-1)} \ominus T_1')^{t^s}$ is pseudo-linear of type $(0, \mathcal{L}^{t^s})$. By Corollary 3.37, $(\mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)})X_s \cdots X_1$ is pseudo-contractible of type $(m-1, \mathcal{L})$.

Let $P \asymp P'$ mean, temporarily, that the weighted pairs P and P' are the same outside of the connected component containing the distinguished vertex. Then
$$\mathcal{G}_{(-m)} \ominus T_1' \asymp \mathcal{G}_{(-1)} \ominus T_1' \asymp \mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)} \asymp (\mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)})X_s \cdots X_1,$$
where the second "$\asymp$" follows from part (2) of 5.14 and the other two are obvious. Thus the weighted pairs $(\mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)})X_s \cdots X_1$ and $\mathcal{G}_{(-m)} \ominus T_1'$ are pseudo-contractible of the same type, and identical outside of the connected component containing the distinguished vertex; it follows that
$$(\mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)})X_s \cdots X_1 \approx \mathcal{G}_{(-m)} \ominus T_1' \tag{31}$$
and consequently
$$(\mathcal{G}_{(-1)} \ominus T_1)T_2 = (\mathcal{G}_{(-1)} \ominus (T_1')^{(\backsim s)})X_s \cdots X_1 T_2' \approx (\mathcal{G}_{(-m)} \ominus T_1')T_2',$$
which proves assertion (1). If $(n, T_1, T_2) \equiv (m, T_1', T_2')$ then $(\mathcal{G}_{(-n)} \ominus T_1)T_2 \approx (\mathcal{G}_{(-m)} \ominus T_1')T_2'$, so
$$(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2 = (\mathcal{G}_{(-n)} \ominus T_1)T_2 \ominus \mathbf{1} \sim (\mathcal{G}_{(-m)} \ominus T_1')T_2' \ominus \mathbf{1} = (\mathcal{G}_{(-m)} \ominus T_1') \ominus T_2'$$
and (2) holds. □

Lemma 5.19. *Let $\tau = (n, T_1, T_2) \in \mathbb{T}(\dagger)$. If the weighted graph $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$ can be contracted to a weighted graph whose number of branch points is strictly less than that of $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$, then $\tau > \tau'$ for some $\tau' \in \mathbb{T}(\dagger)$.*

Proof. Let $\mathcal{L}$ be the weighted pair such that $P = \mathcal{G}_{(-n)} \ominus T_1$ is pseudo-linear of type $(n-1, \mathcal{L}^t)$. Note that $T_2 \in \mathcal{T}^\#$ but that, since $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$ has a branch point, T_2 satisfies none of the conditions of 5.6 (this follows from 5.7). Thus, if we write $T_2 = CT$ with $C, T \in \mathcal{T}$ and C a single column, we have $C \neq \binom{1}{1}$, $T \neq \mathbf{1}$ and if T is a single column then it is not of the form $\binom{1}{k}$. Consider the weighted pair $PC = (\mathcal{H}, e)$ and regard e as a vertex of $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2 = (\mathcal{H}, e) \ominus T$. Then $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$ has three branches at e, say $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$, where $\mathcal{B}$ contains the vertices of P, $\mathcal{B} \cup \mathcal{B}'$ contains no branch point of $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$ and every weight in $\mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}''$ is strictly less than -1, except possibly the middle vertex of $\mathcal{G}_{(-n)}$ (which belongs to $\mathcal{B}$ and has weight $-n$). Since $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2$ contracts to a graph with less branch points, $n = 1$ (so P is of type $(0, \mathcal{L}^t)$) and $\mathcal{B}$ shrinks. In other words, the connected component $\mathcal{L}^t C$ of PC (regard $\mathcal{L}^t C$ as a weighted pair) contracts to a linear pair. By 3.32, $\mathcal{L}^t C \binom{1}{1}^\nu$ is contractible of type $\mathcal{L}$ and $C\binom{1}{1}^\nu \in \mathcal{T}(\mathcal{L}^t)$, for some $\nu \in \mathbb{N}$; so PC is pseudo-contractible of type $(\nu, \mathcal{L})$.

We may write $T_2 = X_1 T_2'$ with $X_1 = C\binom{1}{1}^\ell \in \mathcal{T}$ (some $\ell \in \mathbb{N}$) and $T_2' \in \mathcal{T}^\#$. If $\ell > \nu$ then PX_1 contracts to a weighted pair (W, w) which contains a vertex $v \neq w$ of nonnegative weight; then $(\mathcal{G}_{(-1)} \ominus T_1) \ominus T_2 = PX_1 \ominus T_2'$ contracts to a weighted graph containing a nonnegative weight, contradicting the fact that its intersection matrix is negative definite. So $\ell \leq \nu$ and consequently $X_1 \in \mathcal{T}_{\nu-\ell}(\mathcal{L}^t)$, which we rewrite as $X_1 \in \mathcal{T}_{m-1}(\mathcal{L}^t)$, where $m \geq 1$. It is then clear that the triple $\tau' = (m, \check{T}_1, T_2')$ belongs to $\mathbb{T}$ and satisfies $\tau > \tau'$. By 5.18, $\tau' \in \mathbb{T}(\dagger)$. $\square$

Proof of Corollary 5.17. Since $\tau > \tau'$ implies $\tau \equiv \tau'$ by Lemma 5.18, assertion (1) of 5.17 follows from 5.13. Also, (2) follows from (1): If $\tau \in \mathbb{T}(X) \setminus \mathbb{T}_0(X)$ then Lemma 5.19 implies that τ is not minimal in $(\mathbb{T}(\dagger), <)$; since there is no infinite descending sequence in $(\mathbb{T}(\dagger), <)$, we may therefore choose a minimal τ' in $(\mathbb{T}(\dagger), <)$ such that $\tau > \tau'$; then (1) implies $\tau' \in \mathbb{T}(X)$, so $\tau' \in \mathbb{T}_0(X)$ by Lemma 5.19. $\square$

Notation 5.20. (1) Given $\tau \in \mathbb{T}$, define

$$[\tau, \infty) = \{\tau' \in \mathbb{T} \mid \tau' \geq \tau\} \quad \text{and} \quad (-\infty, \tau] = \{\tau' \in \mathbb{T} \mid \tau' \leq \tau\}.$$

(2) If X satisfies $(\dagger)$, $\min(\mathbb{T}(X)) = \{\tau \in \mathbb{T}(X) \mid \tau \text{ is a minimal element of } (\mathbb{T}(\dagger), <)\}$.

Remark. By Lemma 5.19, $\min(\mathbb{T}(X)) \subseteq \mathbb{T}_0(X)$.

Lemma 5.21. *Given $\tau \in \mathbb{T}$, the set $(-\infty, \tau]$ is totally ordered and finite. Consequently, if τ is not minimal in $\mathbb{T}$ then there exists exactly one $\tau^- \in \mathbb{T}$ such that*

$$\tau > \tau^- \quad \text{and no } \tau^* \in \mathbb{T} \text{ satisfies } \tau > \tau^* > \tau^-.$$

We call τ^- the immediate predecessor *of τ.*

Proof. We show that if $\tau, \tau', \tau'' \in \mathbb{T}$ satisfy $\tau > \tau'$ and $\tau > \tau''$, then $\tau' \geq \tau''$ or $\tau' \leq \tau''$. Write $\tau = (1, T_1, T_2)$, $\tau' = (m', T_1', T_2')$ and $\tau'' = (m'', T_1'', T_2'')$ and let q' (resp. q'') be the number of columns of T_2' (resp. T_2''). We may assume that $q' \leq q''$.

Since $\tau > \tau'$, we have $T_2 = X_s \cdots X_1 T_2'$ (notation as in Definition 5.16) and similarly $\tau > \tau''$ gives $T_2 = Y_r \cdots Y_1 T_2''$. Thus T_2' (resp. T_2'') consists of the rightmost q' (resp. q'') columns of T_2; since $q' \leq q''$, it follows that $T_2'' = W T_2'$ for some $W \in \mathcal{T}$ (and $T_2'' \in \mathcal{T}^\#$ implies $W \in \mathcal{T}^\#$). So $X_s \cdots X_1 T_2' = T_2 = Y_r \cdots Y_1 W T_2'$ and consequently $X_s \cdots X_1 = Y_r \cdots Y_1 W$. Since the X_i are irreducible elements of the monoid $\mathcal{T}^\#$, it follows that $W = X_j \cdots X_1$ (some $j \geq 0$) by unique factorization in $\mathcal{T}^{\#}$ (see 3.34). Thus $T_2'' = X_j \cdots X_1 T_2'$ and it follows that $\tau'' > \tau'$ or $\tau'' = \tau'$. This shows that $(-\infty, \tau]$ is totally ordered; the other assertions are trivial. $\square$

Corollary 5.22. *If X satisfies (†) then $\{[\tau, \infty) \mid \tau \in \min(\mathbb{T}(X))\}$ is a partition of $\mathbb{T}(X)$.*

Remark. $[\tau, \infty)$ is described explicitely in 5.39, below.

Proof. By Corollary 5.17, the union of the sets $[\tau, \infty)$ is $\mathbb{T}(X)$. If $\tau', \tau'' \in \min(\mathbb{T}(X))$ are such that $[\tau', \infty) \cap [\tau'', \infty) \neq \emptyset$, then Lemma 5.21 implies $\tau' = \tau''$. $\square$

In relation with the reduction process of Corollary 2.11, we give:

Proposition 5.23. *Let X be a surface satisfying (†), Λ a non-basic affine ruling of X and F the unique element of Λ_*. Consider the pair (Λ^-, F^-) obtained from (Λ, F) by means of the reduction process of Corollary 2.11, i.e., if $I = (S, \mu)$ is the distinguished X-immersion determining (Λ, F), $P \in S$ the center of the morphism $\pi_2 : \tilde{X} \to S$ which contracts $\tilde{F}$ to a 0-curve and π any[8] element of $\Pi_P(I)$, then (Λ^-, F^-) is the pair determined by the X-immersion $I * \pi$. Let τ and τ^- denote the discrete parts of (Λ, F) and (Λ^-, F^-) respectively. Then τ^- is the immediate predecessor of τ (see 5.21).*

Proof. Write $\tau = (n, T_1, T_2)$. Since Λ is non-basic, $(\mathcal{G}_{(-n)} \ominus T_1) \ominus T_2 \cong \mathcal{G}(\Lambda)$ can be contracted to a weighted graph with less branch points; then the proof of Lemma 5.19 produces a $\tau' = (m, T_1', T_2')$ such that $\tau > \tau'$, $T_2 = X_1 T_2'$ and $X_1 = C\binom{1}{1}^{\ell}$ (note that τ' is the immediate predecessor of τ). Since $\tau > \tau'$ implies $\tau \equiv \tau'$, the proof of Theorem 5.13 produces a pair (Λ', F') whose discrete part is τ'. The factorization $T_2 = C\binom{1}{1}^{\ell} T_2'$ determines a factorization of π_2 as

$$\tilde{X} \xrightarrow[T_2']{} R'' \xrightarrow[\binom{1}{1}^{\ell}]{} \tilde{S} \xrightarrow[C]{\pi} S \tag{32}$$

and $C \in \mathcal{T}_\nu(\mathcal{L}^t)$ implies that $\pi \in \Pi_P(S, \mu)$. Then we see that the X-immersion (S', μ') (in the proof of Theorem 5.13) is equivalent to $I * \pi$. Since (S', μ') determines (Λ', F') and $I * \pi$ determines (Λ^-, F^-), this means that $(\Lambda', F') = (\Lambda^-, F^-)$. So $\tau' = \tau^-$ and we are done. $\square$

Surfaces satisfying (‡)

Notation 5.24. (1) Consider triples (X, Λ, F) where X satisfies (†), Λ is an affine ruling of X and $F \in \Lambda_*$. Two such triples are *equivalent*, $(X, \Lambda, F) \sim (X', \Lambda', F')$, when there exists an isomorphism $X \to X'$ which transforms Λ into Λ' and F into F'. If this is the case then $\mathrm{disc}(X, \Lambda, F) = \mathrm{disc}(X', \Lambda', F')$, so we may speak of the discrete part of the equivalence class $[X, \Lambda, F]$ of (X, Λ, F). So

[8] For the fact that (Λ^-, F^-) is independent of the choice of $\pi \in \Pi_P(I)$, see the last assertion of Corollary 4.4.

we obtain a map disc : $\mathbb{S}(\dagger) \to \mathbb{T}(\dagger)$, where $\mathbb{S}(\dagger)$ denotes the set of equivalence classes $[X, \Lambda, F]$.

(2) We will also consider the restriction disc : $\mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$ of the above map $\mathbb{S}(\dagger) \to \mathbb{T}(\dagger)$, where $\mathbb{S}_0(\ddagger) = \{[X, \Lambda, F] \in \mathbb{S}(\dagger) \mid X$ satisfies $(\ddagger)$ and Λ is basic$\}$ and where $\mathbb{T}_0(\ddagger)$ is the set of $(m, T_1, T_2) \in \mathbb{T}$ such that (i) each of T_1, T_2 satisfies one of conditions (1–3) of 5.6; and (ii) the weighted graph $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ has a negative definite intersection matrix. (See Lemma 5.11 for the fact that disc maps $\mathbb{S}_0(\ddagger)$ in $\mathbb{T}_0(\ddagger)$; see also 5.41.)

(3) Let $\overline{\mathbb{S}}(\ddagger)$ be the set of isomorphism classes of surfaces satisfying $(\ddagger)$. The isomorphism class of X is denoted $[X]$. Then $[X, \Lambda, F] \mapsto [X]$ defines a map $\mathbb{S}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$.

In particular, we will show:

Proposition 5.25. *$\mathbb{S}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$ and $\mathbb{S}(\dagger) \to \mathbb{T}(\dagger)$ are surjective and $\mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$ is bijective.*

Proof that $\mathbb{S}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$ is surjective. If X is any surface satisfying $(\ddagger)$, then X admits a basic affine ruling Λ by Theorem 2.1 and $\Lambda_* \neq \emptyset$ by 2.5; thus $[X]$ is in the image of $\mathbb{S}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$. $\square$

The proof of the other assertions requires some preparation.

Definition 5.26. Let m be a positive integer, Λ_m the standard ruling of $\mathbb{F}_m$ and $\Sigma_m \subset \mathbb{F}_m$ the negative section of Λ_m ($\Sigma_m^2 = -m$). Let $T_1, T_2 \in \mathcal{T}$.

(1) By a *blowing-up of $\mathbb{F}_m$ according to* (T_1, T_2), we mean a triple (π, P_1, P_2) where $\pi : Y \to \mathbb{F}_m$ is a birational morphism (with Y smooth and complete), P_1, P_2 are points of $\mathbb{F}_m \setminus \Sigma_m$ belonging to distinct members of Λ_m ($P_i \in Z_i \in \Lambda_m$, $Z_1 \neq Z_2$), center$(\pi) \subseteq \{P_1, P_2\}$ and, for each $i = 1, 2$, $\pi^{-1}(P_i)$ contains at most one (-1)-curve and $\overline{\mathrm{HN}}(\pi, Z_i) = T_i$.

(2) Let $\beta = (\pi, P_1, P_2)$ and $\beta' = (\pi', P_1', P_2')$ be two blowings-up of $\mathbb{F}_m$ according to (T_1, T_2). We say that *β is equivalent to β'* if there exists a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\cong} & Y' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{F}_m & \xrightarrow[\varphi]{\cong} & \mathbb{F}_m \end{array}$$

where the horizontal arrows are isomorphisms and, for each $i = 1, 2$, $\varphi(P_i) = P_i'$.

Lemma 5.27. *Let $(m, T_1, T_2) \in \mathbb{Z}^+ \times \mathcal{T} \times \mathcal{T}$ be such that:*

(i) *Each T_i satisfies one of conditions (1–3) of 5.6; and*

(ii) *if both T_i are nonempty then $mc_1c_2 - c_1p_2 - c_2p_1 \neq 0$, where $\binom{p_i}{c_i}$ is the first column of T_i.*

Then any two blowings-up of $\mathbb{F}_m$ according to (T_1, T_2) are equivalent.

Proof. Let $\beta = (\pi, P_1, P_2)$ and $\beta' = (\pi', P_1', P_2')$ be two blowings-up of $\mathbb{F}_m$ according to (T_1, T_2). Since there exists an automorphism[9] of $\mathbb{F}_m$ which maps P_1 and P_2 to P_1' and P_2' respectively, we may assume that $(P_1, P_2) = (P_1', P_2')$. Let Z_i be the member of Λ_m containing P_i ($Z_1 \neq Z_2$) and choose a section S of Λ_m such that $S \cap \Sigma_m = \emptyset$ and $P_1, P_2 \in S$. We can write $\mathbb{F}_m \setminus \Sigma_m = \operatorname{Spec} \mathbf{k}[x_1, y_1] \cup \operatorname{Spec} \mathbf{k}[x_2, y_2]$, where x_i, y_i are local equations at P_i for Z_i and S respectively, $x_2 = x_1^{-1}$ and $y_2 = y_1 x_1^{-m}$. Then the Hamburger-Noether tableaux $\mathrm{HN}_i = \mathrm{HN}(\pi; x_i, y_i)$ and $\mathrm{HN}_i' = \mathrm{HN}(\pi'; x_i, y_i)$ satisfy $\overline{\mathrm{HN}}_i = T_i = \overline{\mathrm{HN}}_i'$ $(i = 1, 2)$.

Note that, for each $(\sigma, \tau) \in (\mathbf{k}^*)^2$, $x_1 \mapsto \sigma x_1$, $y_1 \mapsto \tau y_1$ induces an automorphism $\varphi_{\sigma,\tau}$ of $\mathbb{F}_m$ which leaves (P_1, P_2, S) unchanged.

Let $I = \{i \in \{1, 2\} \mid T_i \text{ has two columns}\}$ and, for each $i \in I$, define $\alpha_i, \alpha_i' \in \mathbf{k}^*$ by saying that $\left(\begin{smallmatrix} * \\ * \\ \alpha_i \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} * \\ * \\ \alpha_i' \end{smallmatrix}\right)$) is the first column of HN_i (resp. HN_i'). If the two sequences $\{\alpha_i\}_{i \in I}$ and $\{\alpha_i'\}_{i \in I}$ are equal, then the assertion is trivial. So it suffices to show that the sequence $\{\alpha_i\}_{i \in I}$ can be transformed into the constant sequence with value 1 by composing π with automorphisms $\varphi_{\sigma,\tau}$.

Let us study the following situation. Let P_i, Z_i, S and (x_i, y_i) be as above. Let $\alpha_1, \alpha_2 \in \mathbf{k}^*$ and $p_1, c_1, p_2, c_2 \in \mathbb{N}$ be such that $0 < p_i \leq c_i$ are relatively prime $(i = 1, 2)$ and $mc_1c_2 - c_1p_2 - c_2p_1 \neq 0$; consider a birational morphism $f : Y \to \mathbb{F}_m$ (Y smooth and complete) satisfying $\operatorname{center}(f) = \{P_1, P_2\}$ and, for each $i = 1, 2$, $f^{-1}(P_i)$ contains a unique (-1)-curve E_i and $\mathrm{HN}(f; x_i, y_i) = \left(\begin{smallmatrix} p_i \\ c_i \\ \alpha_i \end{smallmatrix}\right)$. For each $i = 1, 2$, the HN-algorithm of [11] produces a parameter u_i for $E_i \cong \mathbb{P}^1$ and the condition $u_i = \alpha_i$ determines a point on E_i. Moreover, $u_i = y_i^{c_i}/x_i^{p_i}$ or $u_i = x_i^{p_i}/y_i^{c_i}$. We have $\varphi_{\sigma,\tau}(u_i) = \sigma^{\nu_i}\tau^{\mu_i}u_i$, with $(\nu_1, \mu_1) = \pm(-p_1, c_1)$ and $(\nu_2, \mu_2) = \pm(p_2 - mc_2, c_2)$. Since $\left|\begin{smallmatrix} \nu_1 & \mu_1 \\ \nu_2 & \mu_2 \end{smallmatrix}\right| = \pm(mc_1c_2 - c_1p_2 - c_2p_1) \neq 0$, we may choose (σ, τ) such that

$$\mathrm{HN}(\varphi_{\sigma,\tau} \circ f; x_1, y_1) = \left(\begin{smallmatrix} p_1 \\ c_1 \\ 1 \end{smallmatrix}\right) \quad \text{and} \quad \mathrm{HN}(\varphi_{\sigma,\tau} \circ f; x_2, y_2) = \left(\begin{smallmatrix} p_2 \\ c_2 \\ 1 \end{smallmatrix}\right).$$

□

Lemma 5.28. *Let $(m, T_1, T_2) \in \mathbb{T}_0(\ddagger)$.*

(1) *If both T_i are nonempty then $mc_1c_2 - c_1p_2 - c_2p_1 > 0$, where $\binom{p_i}{c_i}$ is the first column of T_i.*

(2) *The blowing-up of $\mathbb{F}_m$ according to (T_1, T_2) is unique, up to equivalence.*

Proof. If both T_i are nonempty then let Γ be the connected component of

$$\left(\mathcal{G}_{(-m)} \ominus \begin{pmatrix} p_1 \\ c_1 \end{pmatrix}\right) \ominus \begin{pmatrix} p_2 \\ c_2 \end{pmatrix}$$

containing the vertices of $\mathcal{G}_{(-m)}$. Since Γ is a subgraph of $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$, it must have a negative definite intersection matrix. In particular, $\det(\Gamma) > 0$. By 3.16 and

[9]The automorphism preserves fibres and Σ_m, since $m > 0$.

Lemma 3.23, $\det(\Gamma) = mc_1c_2 - c_1p_2 - c_2p_1$. This proves (1), and (2) follows from (1) and Lemma 5.27. $\square$

5.29 (Proof of Proposition 5.25, continued)**.** Given $\tau = (m, T_1, T_2) \in \mathbb{T}(\dagger)$, we describe a method for constructing all (X, Λ, F) such that $\mathrm{disc}(X, \Lambda, F) = \tau$ (where X satisfies $(\dagger)$, Λ is an affine ruling of X and $F \in \Lambda_*$). This will show, in particular, that $\mathrm{disc} : \mathbb{S}(\dagger) \to \mathbb{T}(\dagger)$ is surjective.

Choose a blowing-up

$$\left(\tilde{X} \xrightarrow{\pi} \mathbb{F}_m, P_1, P_2\right)$$

of $\mathbb{F}_m$ according to (T_1, T_2) and let Z_1 and Z_2 be the elements of Λ_m satisfying $P_i \in Z_i$. Recall that $\mathrm{center}(\pi) \subseteq \{P_1, P_2\}$ and $\overline{\mathrm{HN}}(\pi, Z_i) = T_i$. For $i = 1, 2$, define

$$E_i = \begin{cases} \pi^{-1}(Z_i), & \text{if } P_i \notin \mathrm{center}(\pi), \\ \text{the } (-1)\text{-curve in } \pi^{-1}(P_i), & \text{if } P_i \in \mathrm{center}(\pi) \end{cases}$$

and let D be the divisor of $\tilde{X}$ with strong normal crossings defined by $\pi^{-1}(\mathrm{supp}(Z_1 + \Sigma_m + Z_2)) = \mathrm{supp}(E_1 + D + E_2)$ and $E_1, E_2 \not\subseteq \mathrm{supp}(D)$. Then $\mathcal{G}(D, \tilde{X}) = (\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ and consequently D has a negative definite intersection matrix (because $\tau \in \mathbb{T}(\dagger)$). So there exists a complete normal surface X and a birational morphism $\tilde{X} \to X$ with exceptional locus $\mathrm{supp}(D)$. Note that Λ_m determines an affine ruling Λ of X, because $\mathbb{F}_m \leftarrow \tilde{X} \to X$ restrict to an isomorphism between $\mathbb{F}_m \setminus \mathrm{supp}(Z_1 + \Sigma_m + Z_2)$ and an open subset of X. Moreover, if $\tilde{\Lambda}$ is the strict transform of Λ_m with respect to π, then $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$. Equation (4) of 1.7 implies that $\mathrm{Pic}(X_s)$ has rank 1, so X satisfies $(\dagger)$. Note that the image of E_i under $\tilde{X} \to X$ is the support of some $F_i \in \Lambda$; moreover, $F_2 \in \Lambda_*$ and $\mathrm{disc}(X, \Lambda, F_2) = \tau$. It is clear, also, that (X, Λ, F_2) is determined by the choice of the blowing-up (π, P_1, P_2) and that every triple (X, Λ, F) with discrete part τ can be obtained in this way, i.e., by choosing a suitable blowing-up. $\square$

5.30 (End of proof of Proposition 5.25)**.** We show that $\mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$ is bijective. Given $\tau = (m, T_1, T_2) \in \mathbb{T}_0(\ddagger)$, consider a triple (X, Λ, F_2) constructed as in 5.29. By Lemma 5.11, X satisfies $(\ddagger)$ and Λ is basic, so $[X, \Lambda, F_2] \in \mathbb{S}_0(\ddagger)$. Also, uniqueness (5.28) of the blowing-up (π, P_1, P_2) up to equivalence implies uniqueness of (X, Λ, F_2) up to equivalence; in other words, $\tau \mapsto [X, \Lambda, F_2]$ is a well-defined map $\mathbb{T}_0(\ddagger) \to \mathbb{S}_0(\ddagger)$, and this is the inverse of the "discrete part" map $\mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$. $\square$

Corollary 5.31. *There exists a surjective map $f : \mathbb{T}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$ satisfying:*

> *Given $\tau \in \mathbb{T}_0(\ddagger)$ and X satisfying $(\ddagger)$, $f(\tau) = [X]$ if and only if there exists an affine ruling Λ of X and an $F \in \Lambda_*$ such that τ is the discrete part of (Λ, F).*

Remark. One interesting aspect of the surjection $f : \mathbb{T}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$ of Corollary 5.31 is that, given $\tau \in \mathbb{T}_0(\ddagger)$, we may construct, in a very explicit way, a surface X such that $f(\tau) = [X]$ (the construction is carried out in 5.29). Since the elements of $\mathbb{T}_0(\ddagger)$

can be described explicitely (5.41), this gives an interesting description of the class of surfaces satisfying (‡).

Corollary 5.32. *Let X_1 and X_2 be surfaces satisfying* (†) *and such that $\mathbb{T}(X_1) \cap \mathbb{T}(X_2) \neq \emptyset$. Then:*

(1) $\mathbb{T}_0(X_1) \cap \mathbb{T}_0(X_2) \neq \emptyset$.
(2) *If at least one of X_1, X_2 satisfies* (‡)*, then $X_1 \cong X_2$.*

Proof. Assertion (1) follows immediately from Corollary 5.17. To prove (2), assume that X_1 satisfies (‡) and consider $\tau = (m, T_1, T_2) \in \mathbb{T}_0(X_1) \cap \mathbb{T}_0(X_2)$. By Lemma 5.11, each of T_1, T_2 satisfies one of the conditions (1–3) of 5.6; since $\tau \in \mathbb{T}(X_2)$, 5.11 implies that X_2 satisfies (‡). Then the surjection $f : \mathbb{T}_0(\ddagger) \to \overline{\mathbb{S}}(\ddagger)$ of Corollary 5.31 satisfies $f(\tau) = [X_1]$ and $f(\tau) = [X_2]$, so $[X_1] = [X_2]$. □

5.33. Consider the equivalence relation "$\sim$" on $\mathbb{T}$ which is *generated* by declaring that $\tau \sim \tau'$ whenever $\tau < \tau'$. Then $\tau \sim \tau' \implies \tau \equiv \tau'$, but *the converse does not hold.* Indeed, 5.21 implies that $\mathbb{T}/{\sim} = \{[\tau, \infty) \mid \tau \in \min(\mathbb{T})\}$, so each equivalence class with respect to $\sim$ contains exactly one minimal element of $(\mathbb{T}, <)$. However, if $\tau = (1, \left(\begin{smallmatrix} 16 & 1 \\ 39 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 135 & 1 \\ 229 & 4 \end{smallmatrix}\right))$ and $\tau' = (1, \left(\begin{smallmatrix} 23 & 1 \\ 39 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 5 & 4 \end{smallmatrix}\right))$ then τ and τ' are distinct minimal elements of $(\mathbb{T}, <)$ and $\tau \equiv \tau'$.

Regarding the relation $\sim$ of 5.33, we have the following:

Corollary 5.34. *For $i = 1, 2$, let X_i be a surface satisfying* (‡)*, let Λ_i be an affine ruling of X_i and let $F_i \in (\Lambda_i)_*$. If $\operatorname{disc}(X_1, \Lambda_1, F_1) \sim \operatorname{disc}(X_2, \Lambda_2, F_2)$, then there exist (Λ_1', F_1') and (Λ_2', F_2') satisfying:*

(1) *For each i, Λ_i' is a basic affine ruling of X_i, $F_i' \in (\Lambda_i')_*$ and $\operatorname{supp}(F_i') = \operatorname{supp}(F_i)$;*
(2) *there exists an isomorphism $X_1 \to X_2$ which carries Λ_1' to Λ_2' and F_1' to F_2'.*

In particular, there exists an isomorphism $X_1 \to X_2$ which maps $\operatorname{supp}(F_1)$ onto $\operatorname{supp}(F_2)$.

Proof. Let $\tau_i \in \mathbb{T}$ be the discrete part of (Λ_i, F_i). Then Lemma 5.21 implies that there exists $\tau' \in \mathbb{T}$ such that (for all i) $\tau_i \geq \tau'$; clearly, τ' may be chosen so that it is a minimal element of $\mathbb{T}$. By Theorem 5.13, for each i there exists an affine ruling Λ_i' of X_i and $F_i' \in (\Lambda_i')_*$ satisfying $\operatorname{supp}(F_i') = \operatorname{supp}(F_i)$ and such that the discrete part of (X_i, Λ_i', F_i') is τ'. So (5.19) Λ_i' is basic and the two elements $[X_1, \Lambda_1', F_1']$ and $[X_2, \Lambda_2', F_2']$ of $\mathbb{S}_0(\ddagger)$ have the same image (namely, τ') under the bijective map $\mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$. Hence, $[X_1, \Lambda_1', F_1'] = [X_2, \Lambda_2', F_2']$. □

Multiplicities

Definition 5.35. Given a tableau $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right) \in \mathcal{T}$, we define

$$\mu(T) = \begin{cases} 1, & \text{if } T = \mathbf{1}, \\ c_1 \cdots c_k, & \text{else.} \end{cases}$$

Note that $\mu : \mathcal{T} \to \mathbb{N} \setminus \{0\}$ is a homomorphism of multiplicative monoids.

Remark. Given a finite Hamburger-Noether tableau $\mathrm{HN} = \left(\begin{smallmatrix} p_1 & \cdots \\ c_1 & \cdots \\ \alpha_1 & \cdots \end{smallmatrix}\right)$, consider $T = \overline{\mathrm{HN}} \in \mathcal{T}$ defined as in 3.6. Then $\mu(T) = c_1$ (or 1, if HN is empty).

By the above remark and A.10.1 of [11], we have

5.36. Let $f : X \to Y$ be a birational morphism of smooth complete surfaces and D a nonzero divisor of Y with strong normal crossings. Assume that the exceptional locus of f contains at most one (-1)-curve and that the center of f, if nonempty, is a point P belonging to exactly one component Z of D. Let

$$E = \begin{cases} f^{-1}(Z), & \text{if } f \text{ is an isomorphism,} \\ \text{the } (-1)\text{-curve in } f^{-1}(P), & \text{if } f \text{ is not an isomorphism.} \end{cases}$$

Then the multiplicity of E in the total transform of D is equal to $\mu(\overline{\mathrm{HN}}(f, D))$.

The above statement and Proposition 1.8 give:

Corollary 5.37. *Let X be a surface satisfying (†). If (m, T_1, T_2) is the discrete part of (Λ, F), where Λ is an affine ruling of X and $F \in \Lambda_*$, and if $G \in \Lambda \setminus \{F\}$ is such that $\{F, G\}$ contains all multiple members of Λ (such a G exists, by definition of Λ_*), then*

$$F = \mu(T_2)C_2 \quad \text{and} \quad G = \mu(T_1)C_1,$$

where $C_1, C_2 \subset X$ are (irreducible) curves. Moreover, $\mathrm{Pic}(X_s) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(\mu(T_1), \mu(T_2))$.

Remark. If $X = \mathbb{P}^2$, or more generally a weighted projective plane $\mathbb{P}(a, b, c)$ where a, b, c are pairwise relatively prime, then $\mu(T_1) = \deg C_2$ and $\mu(T_2) = \deg C_1$. (In view of the above result, this follows immediately from $\gcd(\deg C_1, \deg C_2) = 1$, for which we refer to [5] or [6].)

See also Corollary 5.40.

Some explicit computations

5.38. Let $m > 0$ be an integer and suppose that $T \in \mathcal{T}$ satisfies one of conditions (1–3) of 5.6.

(1) Recall that $\mathcal{G}_{(-m)} \ominus T$ is pseudo-linear of type $(m-1, \mathcal{L})$, where $\mathcal{L}$ is a weighted pair satisfying the condition (0), uniquely determined by T. Then Lemma 3.23 gives:

$$M(\mathcal{L}) = \begin{cases} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) & \text{if } T \text{ satisfies 5.6.1,} \\ \left(\begin{smallmatrix} c-p-p'+p'' & c-p \\ c-p' & c \end{smallmatrix}\right) & \text{if } T \text{ satisfies 5.6.2,} \\ \left(\begin{smallmatrix} Np(c-p)-1 & Nc^2-Ncp-1 \\ Ncp-1 & Nc^2 \end{smallmatrix}\right) & \text{if } T \text{ satisfies 5.6.3,} \end{cases}$$

where p' and p'' are defined by $\binom{p''}{p'} = \binom{p}{c}^*$. Note that $\mathcal{L}$ is degenerate if and only if $T \in \{\mathbf{1}, \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right)\} \cup \left\{ \binom{n}{n+1} \mid n \geq 1 \right\}$.

(2) The conditions

$$M(\mathcal{L}) = \begin{pmatrix} \dot\gamma(T) & \alpha_1 \\ \alpha_2 & \gamma(T) \end{pmatrix} \quad \text{and} \quad \alpha_{i+2} = \alpha_i \ (i \geq 1)$$

define positive integers $\dot\gamma(T)$ and $\gamma(T)$ and an infinite sequence $\alpha(T) = (\alpha_1, \alpha_2, \dots)$ of positive integers. Note that these are uniquely determined by T and can be computed from (1). They satisfy $M(\mathcal{L}^{t^i}) = \begin{pmatrix} \dot\gamma(T) & \alpha_{i+1} \\ \alpha_i & \gamma(T) \end{pmatrix}$ for all $i \geq 1$.

(3) A sequence $\nu = (\nu_1, \dots, \nu_s)$ of natural numbers is said to be (m,T)-*admissible* if $s \geq 1$ and the following conditions hold:
   (a) If $T \in \{\mathbf{1}, \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right)\} \cup \left\{ \left(\begin{smallmatrix} n \\ n+1 \end{smallmatrix}\right) \mid n \geq 1 \right\}$, $\nu_1 \geq \max(1, m-1)$ and $\nu_i \geq 1$ for all $i > 1$.
   (b) For all other T, $\nu_1 \geq m-1$ and $\nu_i \geq 0$ for all $i > 1$.

(4) Given an (m,T)-admissible sequence $\nu = (\nu_1, \dots, \nu_s)$, consider the sequence of tableaux $(X_1, \dots, X_s) \in \mathcal{T}^s$ given by $X_i = \left(\begin{smallmatrix} p_i \\ c_i \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)^{\nu_i - k_i}$, where $\left(\begin{smallmatrix} p_i \\ c_i \end{smallmatrix}\right)$ is the matrix product $M(\mathcal{L}^{t^i}) \left(\begin{smallmatrix} 1 \\ \nu_i \end{smallmatrix}\right)$, $k_1 = m-1$ and $k_i = 0$ for all $i \geq 1$. Then $X_i \in \mathcal{T}_{k_i}(\mathcal{L}^{t^i})$ for all $i = 1, \dots, s$.

5.39. *Let $\tau = (m, T_1, T_2) \in \mathbb{T}$. For each (m, T_1)-admissible sequence $\nu = (\nu_1, \dots, \nu_s)$, define $\tau_\nu \in \mathbb{T}$ by $\tau_\nu = (1, (T_1)^{(\smile s)}, X_s \cdots X_1 T_2)$, where $(X_1, \dots, X_s)$ is determined by ν and (m, T_1) as in part* 4 *of* 5.38. *Then*

$$[\tau, \infty) = \{\tau\} \cup \{\tau_\nu \mid \nu \textit{ is an } (m, T_1)\textit{-admissible sequence}\}.$$

Corollary 5.40. *Let X be a surface satisfying (†) and suppose that $(m, T_1', T_2') \in \mathbb{T}(X)$. Let $\gamma = \gamma(T_1')$ and $\alpha(T_1') = (\alpha_1, \alpha_2, \dots)$. Then the set*

$$\{(\mu(T_1), \mu(T_2)) \mid (1, T_1, T_2) \in \mathbb{T}(X) \textit{ and } (1, T_1, T_2) > (m, T_1', T_2')\}$$

is equal to

$$\left\{ \left(\mu(T_1'),\ \mu(T_2') \cdot \prod_{i=1}^{s} (\alpha_i + \nu_i \gamma) \right) \middle| \ (\nu_1, \dots, \nu_s) \textit{ is } (m, T_1')\textit{-admissible} \right\}.$$

5.41. We describe the elements of $\mathbb{T}_0(‡)$. Consider a triple $\tau = (m, T_1, T_2)$ where m is a positive integer and each T_i is a tableau ($T_i \in \mathcal{T}$) satisfying one of conditions (1–3) of 5.6 (each element of $\mathbb{T}_0(‡)$ is such a triple). Consider the connected component Γ of $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ containing the vertices of $\mathcal{G}_{(-m)}$. Then every connected component of $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ is a linear chain and every vertex, except possibly the middle vertex of $\mathcal{G}_{(-m)}$ (which has weight $-m$), has weight strictly less than -1. So $\tau \in \mathbb{T}_0(‡) \Leftrightarrow \det(\Gamma) > 0$ and in particular:

- If $m > 1$ then $\tau \in \mathbb{T}_0(‡)$;
- if $\mathbf{1} \in \{T_1, T_2\}$ then $\tau \in \mathbb{T}_0(‡)$.

Assume that $m = 1$ and that neither of T_1, T_2 is empty; then T_i is either $\left(\begin{smallmatrix} p_i \\ c_i \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} p_i & 1 \\ c_i & x_i \end{smallmatrix}\right)$ with $x_i \geq 1$. We may then compute $\det(\Gamma)$ in each case and conclude:

(1) If $T_1 = \binom{p_1}{c_1}$ and $T_2 = \binom{p_2}{c_2}$, $\tau \in \mathbb{T}_0(\ddagger) \iff \Delta > 0$;
(2) if $T_i = \binom{p_i}{c_i}$ and $T_j = \left(\begin{smallmatrix} p_j & 1 \\ c_j & x_j \end{smallmatrix}\right)$, $\tau \in \mathbb{T}_0(\ddagger) \iff \Delta c_j x_j - c_i > 0$;
(3) if $T_1 = \left(\begin{smallmatrix} p_1 & 1 \\ c_1 & x_1 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & x_2 \end{smallmatrix}\right)$, $\tau \in \mathbb{T}_0(\ddagger) \iff \Delta c_1 c_2 x_1 x_2 - c_1^2 x_1 - c_2^2 x_2 > 0$,

where $\Delta = mc_1c_2 - c_1p_2 - c_2p_1 = c_1c_2 - c_1p_2 - c_2p_1$.

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