ON LOG $\mathbb{Q}$-HOMOLOGY PLANES AND WEIGHTED PROJECTIVE PLANES

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Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and consider algebraic surfaces over $\mathbf{k}$. Let $\mathcal{M}$ be the class of normal affine surfaces U satisfying $\mathrm{ML}(U) = \mathbf{k}$, where $\mathrm{ML}(U)$ stands for the Makar-Limanov invariant of U, that is, the intersection of the kernels of all locally nilpotent derivations $\mathcal{O}_U(U) \to \mathcal{O}_U(U)$. The aim of this paper is to describe the class

$$\mathcal{M}_0 = \big\{\, U \in \mathcal{M} \mid \mathrm{Pic}(U_s) \text{ is a finite group} \,\big\},$$

where U_s denotes $U \setminus \mathrm{Sing}(U)$. In the special case where $\mathbf{k} = \mathbb{C}$, the class $\mathcal{M}_0$ can also be defined as that of log $\mathbb{Q}$-homology planes with trivial Makar-Limanov invariant. This explains our title but note that we will never assume that $\mathbf{k} = \mathbb{C}$.

1. Statement of some results

Given positive integers a_0, a_1, a_2, the weighted projective plane $\mathbb{P}(a_0, a_1, a_2)$ is defined by $\mathbb{P}(a_0, a_1, a_2) = \mathrm{Proj}\, R$, where $R = \mathbf{k}[X_0, X_1, X_2]$ is the polynomial ring in three variables with the grading given by $\deg(X_i) = a_i$. As is well-known, $\mathbb{P}(a_0, a_1, a_2)$ is isomorphic to $\mathbb{P}(a_0', a_1', a_2')$ for some *pairwise relatively prime* positive integers a_0', a_1', a_2'. Whenever we consider a weighted projective plane $\mathbb{P}(a_0, a_1, a_2)$, we will choose a_0, a_1, a_2 to be pairwise relatively prime.

We use the set $\mathcal{T}_*$, defined in 3.1, to parametrize the class $\mathcal{M}_0$. One of our main results is:

Theorem A. *Given $T \in \mathcal{T}_*$, define $(a_0, a_1, a_2) \in \mathbb{Z}^3$ and $f \in \mathbf{k}[X_0, X_1, X_2]$ as follows:*

(i) *If $T = \binom{p}{c} \in \mathcal{T}_1$, let $(a_0, a_1, a_2) = (c-p, c, 1)$ and $f = X_1$.*

(ii) *If $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$, there exists $a_2 \in \mathbb{Z}$ satisfying:*

$$(c-p)a_2 \equiv 1 \pmod{c}, \quad \gcd(a_2, a) = 1 \quad \textit{and} \quad 0 < a_2 < ac.$$

Choose any such a_2 and define $a_0 = ac - a_2$, $a_1 = a$ and $f = X_0X_2 + X_1^c$.

Then a_0, a_1, a_2 are pairwise relatively prime positive integers and f is homogeneous with respect to the grading of $\mathbf{k}[X_0, X_1, X_2]$ given by $\deg X_i = a_i$ ($i = 0, 1, 2$). Let the surface U_T be the complement of the curve $f = 0$ in $\mathbb{P}(a_0, a_1, a_2)$. Then:

(1) *Up to isomorphism, U_T depends only on T (i.e., U_T is independent of the choice of a_2 made in case* (ii)*, in the above statement).*

(2) *U_T belongs to the class $\mathcal{M}_0$ and each member of $\mathcal{M}_0$ is a U_T for some T.*

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(3) *Given $T, T' \in \mathcal{T}_*$, U_T is isomorphic to $U_{T'}$ if and only if $T' \in \{T, \check{T}\}$ (see 3.3 for $\check{T}$).*

Remarks. (1) Since U_T is the complement of the curve $f = 0$ in $\mathbb{P}(a_0, a_1, a_2)$, it follows that it is a quotient of the hypersurface $f = 1$ of $\mathbb{A}^3$ by a finite cyclic group action.

More precisely, in case (ii) U_T is the quotient of the surface Y in $\mathbb{A}^3$ with equation $X_0X_2 + X_1^c = 1$ by the group Ω_{ac} of ac-roots of 1, acting with weights $-a_2$, $a = a_1$, a_2. Since $\gcd(a_2, ac) = 1$ we can pick j such that $ja_2 \equiv 1 \pmod{ac}$. Then $j \equiv c - p \pmod{c}$ and we can replace the above weights by -1, $a(c-p)$, 1. Precisely the points $\{ (0, x_1, 0) \mid x_1^c = 1 \}$ have non-trivial stabilizer, namely the group Ω_a of a-roots of 1. The invariants for Ω_a are generated by $y_0 = x_0^a$, $y_2 = x_2^a$, x_0x_2, $y_1 = x_1$. Hence Y/Ω_a, which is the universal covering space of U_T in case $\mathbf{k} = \mathbb{C}$, is the Danielewski surface with equation $Y_0Y_2 = (1 - Y_1^c)^a$.

In case (i) of Theorem A, a similar argument shows that U_T is a quotient of $\mathbb{A}^2$ by a cyclic group. So, if $\mathbf{k} = \mathbb{C}$, U_T is contractible and is its own universal covering space.

(2) Some surfaces belonging to $\mathcal{M}_0$ may be embedded in infinitely many (non isomorphic) weighted projective planes; one can see that this is the case for U_T if and only if $T \in \mathcal{T}_1$ (this particular claim is not proved in the present paper but see 7.2–7.5 for related questions). Also note that a_2, in Theorem A(ii), is not always unique.

Let us explain how Theorem A follows from the results of this paper. In 6.1 we define a map $\bar{\mathfrak{f}} : \mathcal{T}_* \to \overline{\mathcal{M}}_0$, where $\overline{\mathcal{M}}_0$ is the set of isomorphism classes $[U]$ of surfaces $U \in \mathcal{M}_0$. The fact that $\bar{\mathfrak{f}}$ is a well-defined map is not trivial: It is based on 5.4, 3.9 and ultimately result 5.25 of [5]. In 7.1, we show that $\bar{\mathfrak{f}}(T)$ is the isomorphism class of the surface U_T defined in Theorem A; this proves assertion (1) of Theorem A, as well as the first half of assertion (2). The second half of (2) is the fact that $\bar{\mathfrak{f}}$ is surjective (a consequence of 6.3) and assertion (3) is a rephrasing of 6.7.

See also result 6.9, which describes the properties of U_T (or $\bar{\mathfrak{f}}(T)$) in terms of T.

There is another way to present the classification of surfaces belonging to $\mathcal{M}_0$. Given a normal surface U, let $\mathcal{G}_{\text{res}}[U]$ denote the resolution graph of U (see 2.1) and let $\mathcal{G}_\infty[U]$ denote the equivalence class of weighted graphs which contains the dual graph of any divisor at infinity of U (see the beginning of section 2 for details). Then 6.9 contains a description of the set

$$(1) \qquad \{ (\mathcal{G}_\infty[U], \mathcal{G}_{\text{res}}[U]) \mid U \in \mathcal{M}_0 \}.$$

From this description, it follows in particular that $\{ \mathcal{G}_{\text{res}}[U] \mid U \in \mathcal{M}_0 \}$ is exactly the set of all admissible chains (an *admissible chain* is a weighted graph which is a linear chain and in which every weight is strictly less than -1; the empty graph is considered to be an admissible chain). It also follows that an equivalence class $\mathcal{C}$ of weighted graphs is an element of $\{ \mathcal{G}_\infty[U] \mid U \in \mathcal{M}_0 \}$ if and only if some element of $\mathcal{C}$ has the form:

$$\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\omega_1}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{\omega_q}{\bullet} \qquad \text{(where } q \geq 0 \text{ and } \forall_i\ \omega_i \leq -2\text{).} \tag{2}$$

To decide whether two graphs of the form (2) are equivalent, see 2.17. We have:

Theorem B (See 6.5)**.** *Let U and U' be surfaces belonging to the class $\mathcal{M}_0$. If $\mathcal{G}_\infty[U] = \mathcal{G}_\infty[U']$ and $\mathcal{G}_{\mathrm{res}}[U] = \mathcal{G}_{\mathrm{res}}[U']$ then $U \cong U'$.*

Clearly, this result and the description (6.9) of the set (1) constitute a classification of $\mathcal{M}_0$. Note that it is easy to go back and forth between $T \in \mathcal{T}_*$ and the graphs $\mathcal{G}_\infty[U_T]$ and $\mathcal{G}_{\mathrm{res}}[U_T]$, see section 8.

We also classify G_a-actions on these surfaces. Given a surface $U = \operatorname{Spec} R$ belonging to $\mathcal{M}$, define:

$$\textsc{klnd}(R) = \{ \ker D \mid D : R \to R \text{ is a } \textit{nonzero} \text{ locally nilpotent derivation} \},$$

which is a collection of subalgebras of R. It is well-known that the problem of describing the G_a-actions on U is equivalent to that of describing the set $\textsc{klnd}(R)$. One can show that $\textsc{klnd}(R)$ has the same cardinality as the ground field $\mathbf{k}$ (whenever $U \in \mathcal{M}$), but it is more interesting to count the orbits, relative to the obvious action of the group $\operatorname{Aut}_{\mathbf{k}}(R)$ on the set $\textsc{klnd}(R)$ (that is, if $\theta \in \operatorname{Aut}_{\mathbf{k}}(R)$ and $A \in \textsc{klnd}(R)$, define $\theta A = \theta(A)$). For instance, it was suggested that the action might be transitive whenever $U \in \mathcal{M}$ is a smooth surface. The following fact shows that that idea was not correct:

Theorem C. *Let $T \in \mathcal{T}_*$ and let $U_T = \operatorname{Spec} R$ be the surface defined in Theorem A. Then the action of $\operatorname{Aut}_{\mathbf{k}}(R)$ on $\textsc{klnd}(R)$ is such that:*

(1) *The number of orbits is either one or two.*
(2) *The action is transitive if and only if $T = \check{T}$.*

This result is a reformulation of 6.8, taking into account the fact (7.1) that $\bar{\mathfrak{f}}(T)$ is the isomorphism class of U_T.

We may give a geometric interpretation of the condition $T = \check{T}$. Let us say that a surface $U \in \mathcal{M}$ is *symmetric at infinity* if the following equivalent conditions are satisfied:

- For some element of $\mathcal{G}_\infty[U]$ of the form (2), $(\omega_1, \dots, \omega_q) = (\omega_q, \dots, \omega_1)$.
- For every element of $\mathcal{G}_\infty[U]$ of the form (2), $(\omega_1, \dots, \omega_q) = (\omega_q, \dots, \omega_1)$.

(Equivalence follows from 2.17.) Now, by 6.9, the condition $T = \check{T}$ is equivalent to U_T being symmetric at infinity; so Theorem C implies:

$$\textit{The action is transitive} \iff U_T \textit{ is symmetric at infinity}$$

where in fact "$\implies$" is valid for all $U \in \mathcal{M}$.

In the case $\mathbf{k} = \mathbb{C}$, some of our results can also be found in [10] (see also [11] for the smooth case). That paper describes the set (1) and shows that each member of $\mathcal{M}_0$ can be realized as a quotient of the type described in the remark following Theorem A.

However, [10] does not give embeddings into weighted projective planes (i.e., Theorem A(2)) and does not address the question of deciding when two members of $\mathcal{M}_0$ are isomorphic (i.e., Theorem A(3) or Theorem B). Also, Theorem C is not contained in [10]. Note that [10] investigates the relation between the condition $\mathrm{ML}(U) = \mathbb{C}$ and finiteness of the fundamental group at infinity of U; we don't consider that question here.

This work is, in essence, an elaborate corollary of our results in [5] and [6] on the classification of affine rulings of weighted projective planes. Theorem A(3) and Theorem C, in particular, are quite delicate results that appear to require the very precise methods developed there.

2. Preliminaries on surfaces

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. All algebraic varieties are assumed to be $\mathbf{k}$-varieties. All divisors are Weil divisors.

Let D be a divisor of a complete normal surface S. By a *component* of D we always mean an irreducible component. We call D an *SNC-divisor* of S if it satisfies:

(a) D is effective and reduced
(b) $\mathrm{supp}(D) \subset S \setminus \mathrm{Sing}(S)$
(c) each component of D is a smooth curve
(d) if C, C' are distinct components of D then $C \cdot C' \leq 1$
(e) if C, C', C'' are distinct components of D then $C \cap C' \cap C'' = \varnothing$.

If D is an SNC-divisor of S then we may consider the dual graph of D in S, which we denote $\mathcal{G}(D, S)$. For the definition of $\mathcal{G}(D, S)$ and the theory of weighted graphs, see for instance [13].

By *"graph"* we mean a finite undirected graph in which no edge connects a vertex to itself and at most one edge joins any given pair of vertices. A vertex adjacent to a vertex v is also called a *neighbor* of v; we say that v is a *branch point* if it has more than two neighbors. A tree without branch points is called a *linear chain*. An *admissible chain* is a (possibly empty) weighted graph which is a linear chain and in which every weight is strictly less than -1. If D is an SNC-divisor of S then a *branching component* of D is an irreducible component which is a branch point in the dual graph $\mathcal{G}(D, S)$.

If U is a normal surface then there exists a *smooth-normal compactification of U,* by which we mean an open immersion $u : U \hookrightarrow S$ such that S is a complete normal surface and $S \setminus U$ is the support of an SNC-divisor of S (so in particular $\mathrm{Sing}(S) = \mathrm{Sing}(U)$). If $u : U \hookrightarrow S$ is a smooth-normal compactification of U then we define $\mathcal{G}(u) = \mathcal{G}(D, S)$, where D is the SNC-divisor of S satisfying $S \setminus U = \mathrm{supp}(D)$. Note that the equivalence class of the weighted graph $\mathcal{G}(u)$ depends only on the isomorphism class $[U]$ of U. This equivalence class of weighted graphs is denoted $\mathcal{G}_\infty[U]$.

A normal surface U is said to be *completable by rational curves* if there exists a smooth-normal compactification $u : U \hookrightarrow S$ of U such that every curve in $S \setminus U$ is a rational curve. If this is the case then every smooth-normal compactification of U has that property.

If U is a normal surface then

$$\left\{ \mathcal{G}(u) \mid u \text{ is a smooth-normal compactification of } U \right\}$$

is a nonempty subset of $\mathcal{G}_\infty[U]$; it is equal to $\mathcal{G}_\infty[U]$ whenever U is completable by rational curves.

If $f : X \to Y$ is a birational morphism of surfaces, we use the notations

$$\begin{aligned} \operatorname{cent}(f) &= \big\{ P \in Y \mid f^{-1}(P) \text{ contains more than one point} \big\} \\ \operatorname{exc}(f) &= f^{-1}\big(\operatorname{cent}(f)\big) \end{aligned}$$

for the *center* and *exceptional locus* of f respectively.

2.1. Recall from [6]-1.19 that the *resolution graph* of a normal surface U is the dual graph of $\mathcal{E}$ in $\hat{U}$, where $\mathcal{E}$ is the exceptional locus of the minimal SNC resolution of singularities $\hat{U} \to U$. We will denote that weighted graph by $\mathcal{G}_{\text{res}}[U]$, which reminds us that the resolution graph depends only on the isomorphism class $[U]$.

G_a-actions and $\mathbb{A}^1$-fibrations

2.2. **Notations.** In addition to the classes $\mathcal{M}$ and $\mathcal{M}_0$ defined in the introduction, we will also consider the larger class $\mathcal{N}$ of normal algebraic surfaces.

Given a class $\mathcal{S}$ of algebraic surfaces (e.g., $\mathcal{M}$, $\mathcal{M}_0$ or $\mathcal{N}$), let

$$\mathcal{S}^+ = \big\{ (U, \rho) \mid U \in \mathcal{S} \text{ and } \rho : U \to \mathbb{A}^1 \text{ is a morphism whose general fiber is } \mathbb{A}^1 \big\}.$$

Note that surjectivity of the map $\rho : U \to \mathbb{A}^1$ is *not* required in the definition of $\mathcal{S}^+$. Define an equivalence relation on the set $\mathcal{S}^+$ by declaring that $(U, \rho) \sim (U', \rho')$ if there exists a commutative diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\rho"'] & U' \arrow[d, "\rho'"] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

where the horizontal arrows are isomorphisms of varieties. Let $[U, \rho]$ denote the equivalence class of the pair (U, ρ) and $[U]$ the isomorphism class of U. Then define:

$$\overline{\mathcal{S}} = \big\{ [U] \mid U \in \mathcal{S} \big\} \quad \text{and} \quad \overline{\mathcal{S}}^+ = \big\{ [U, \rho] \mid (U, \rho) \in \mathcal{S}^+ \big\}.$$

We also consider the set map $\overline{\mathcal{S}}^+ \longrightarrow \overline{\mathcal{S}}$, defined by $[U, \rho] \mapsto [U]$, which we simply call *the projection.* By 2.3, the projection is surjective when $\mathcal{S} = \mathcal{M}$ or $\mathcal{M}_0$.

2.3. Let $U = \operatorname{Spec} R$ be a normal affine surface and recall, from the introduction, that the study of G_a-actions on U reduces to describing the set $\text{KLND}(R)$. The following facts are well-known and easy to prove.

(1) $\text{ML}(U) = \mathbf{k}$ if and only if $\text{KLND}(R)$ contains more than one element.
(2) If $\text{ML}(U) = \mathbf{k}$ then $R^* = \mathbf{k}^*$ and U is rational and completable by rational curves.
(3) If $\rho : U \to \mathbb{A}^1$ is a surjective morphism with general fiber $\mathbb{A}^1$, then the image of the corresponding homomorphism $\rho^* : \mathbf{k}[t] \hookrightarrow R$ is an element of $\text{KLND}(R)$. If U is rational and $R^* = \mathbf{k}^*$ then all elements of $\text{KLND}(R)$ can be obtained in this way.

(4) Two morphisms $U \to \mathbb{A}^1$ as in statement (3) correspond to the same element of $\mathrm{KLND}(R)$ if and only if they differ by an automorphism of $\mathbb{A}^1$.

2.4. This paragraph makes the link between Theorem C and the other results of this paper. Consider a surface $U = \mathrm{Spec}(R)$ belonging to $\mathcal{M}$. Let $E \subset \overline{\mathcal{M}}^+$ be the inverse image of $[U] \in \overline{\mathcal{M}}$ by the projection $\overline{\mathcal{M}}^+ \to \overline{\mathcal{M}}$.

Fix $A \in \mathrm{KLND}(R)$; by 2.3, there exist surjective morphisms $\rho : U \to \mathbb{A}^1$ with general fiber $\mathbb{A}^1$ and satisfying $A = \mathrm{im}(\rho^*)$. Moreover, any two such morphisms differ by an automorphism of $\mathbb{A}^1$ and consequently the element $[U, \rho]$ of E is completely determined by A. This defines a set map

$$\begin{aligned} F : \mathrm{KLND}(R) &\longrightarrow E \\ A &\longmapsto [U, \rho] \end{aligned}$$

Note that F is surjective ($R^* = \mathbf{k}^*$ by 2.3, so every dominant morphism $\rho : U \to \mathbb{A}^1$ is surjective) and that, given $A, A' \in \mathrm{KLND}(R)$,

$$F(A) = F(A') \iff \exists_{\theta \in \mathrm{Aut}_{\mathbf{k}}(R)}\ \theta(A) = A'.$$

Thus, if we consider the natural action of the group $\mathrm{Aut}_{\mathbf{k}}(R)$ on the set $\mathrm{KLND}(R)$, the number of orbits is equal to the cardinality of E.

Completions and minimal completions

2.5. **Definition.** Let (U, ρ) be an element of $\mathcal{N}^+$ (see 2.2). A *completion of* (U, ρ) is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & S \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\bar\rho} \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array} \tag{3}$$

where $u : U \to S$ is a smooth-normal compactification of U, $\mathbb{A}^1 \to \mathbb{P}^1$ is an open immersion and $\bar\rho$ is a morphism.

Let D be the SNC-divisor of S such that $S \setminus U = \mathrm{supp}(D)$. A component C of D is called *ρ-vertical* if $\bar\rho(C)$ is a point; if C is not ρ-vertical, we call it *ρ-horizontal.* The assumption that the general fiber of ρ is $\mathbb{A}^1$ implies that the general fiber of $\bar\rho$ meets D in one point. It easily follows that

D has exactly one ρ-horizontal component

and that, if the unique ρ-horizontal component is denoted H, then

$\bar\rho$ restricts to an isomorphism $H \to \mathbb{P}^1$.

We call D and H the *boundary divisor* and the *horizontal component* of the completion (3), respectively.

It is clear that each element (U, ρ) of $\mathcal{N}^+$ has a completion. Observe that given any completion (3) of (U, ρ), the morphism $\bar\rho : S \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration (i.e., the general fiber of $\bar\rho$ is a $\mathbb{P}^1$). $\mathbb{P}^1$-fibrations have been studied extensively, and the structure of their fibers is understood. Since all ρ-vertical components of the boundary divisor D are (by definition) contained in fibers of $\bar{\rho}$, one can obtain information about the structure of D. In particular, the properties of $\mathbb{P}^1$-fibrations imply the existence of minimal completions:

2.6. **Definition.** Let (U, ρ) be an element of $\mathcal{N}^+$. By a *minimal completion* of (U, ρ), we mean a completion (3) of (U, ρ) satisfying the additional condition:

Every ρ-vertical component C of the boundary divisor D satisfies $C^2 \neq -1$.

The following is well-known:

2.7. **Lemma.** *Given any $(U, \rho) \in \mathcal{N}^+$ and $x \in \mathbb{Z}$, there exists a minimal completion of (U, ρ) whose horizontal component H satisfies $H^2 = x$.*

We shall now see that each element $[U, \rho]$ of $\overline{\mathcal{N}}^+$ (see 2.2) determines a specific element $\mathcal{G}_\infty[U, \rho]$ of $\mathcal{G}_\infty[U]$.

2.8. **Definition.** Let (U, ρ) be an element of $\mathcal{N}^+$. By 2.7, we may choose a minimal completion (3) of (U, ρ) whose horizontal component H satisfies $H^2 = -1$. Let D be the boundary divisor of such a completion. We claim:

The weighted graph $\mathcal{G}(D, S)$ is independent of the choice of the completion. In fact, it depends only on $[U, \rho]$.

Assuming that this is true, define $\mathcal{G}_\infty[U, \rho] = \mathcal{G}(D, S)$ and note that $\mathcal{G}_\infty[U, \rho] \in \mathcal{G}_\infty[U]$.

The meaning of the above claim is that

$$[U, \rho] \longmapsto \mathcal{G}_\infty[U, \rho]$$

is a well-defined map, going from $\overline{\mathcal{N}}^+$ to the set of all weighted graphs. This claim is justified by part (2) of 2.15, below.

Note that $\mathcal{G}_\infty[U, \rho]$ is *not* independent of ρ.[1] For instance, there exists $U \in \mathcal{M}_0$ and morphisms $\rho, \rho' : U \to \mathbb{A}^1$ (whose general fibers are $\mathbb{A}^1$) such that $\mathcal{G}_\infty[U, \rho]$ and $\mathcal{G}_\infty[U, \rho']$ are (79) and (81) respectively. The proof of Theorem C is based on this type of phenomenon.

To prove that Definition 2.8 is sound, as well as several other results of this paper, the following combinatorial object is very convenient:

2.9. **Definition.** A *weighted pair* is an ordered pair $\mathcal{P} = (\mathcal{G}, v)$ where $\mathcal{G}$ is a nonempty weighted graph and v is a vertex of $\mathcal{G}$. We call v and $\mathcal{G}$ the *distinguished vertex* and the *underlying weighted graph* of $\mathcal{P}$. One defines an equivalence relation $\approx$ on the set of weighted pairs by declaring that $(\mathcal{G}, v) \approx (\mathcal{G}', v')$ means that $\mathcal{G}$ can be transformed into $\mathcal{G}'$ by a finite sequence of blowings-up and blowings-down of weighted graphs, *in such a way that the distinguished vertex is preserved.* Refer to [6]-2.5 for the precise definition of $\approx$. Note in particular that the condition $(\mathcal{G}, v) \approx (\mathcal{G}', v')$ implies that $\mathcal{G}$ and $\mathcal{G}'$ are equivalent weighted graphs.

[1] So result 1.9 of [2] (as far as we understand it) is false.

2.10. **Definition.** Let (U, ρ) be an element of $\mathcal{N}^+$. Given a completion (3) of (U, ρ), with boundary divisor D and horizontal component H, define the weighted pair

$$\mathcal{P}(U, \rho, u) = \big(\mathcal{G}(D, S), H\big).$$

We will see in 2.15 that, up to equivalence $\approx$ of weighted pairs, $\mathcal{P}(U, \rho, u)$ is independent of the choice of a completion.

2.11. **Definition.** We say that a weighted pair $\mathcal{P} = (\mathcal{G}, v)$ is *of type* $(\star)$ if it satisfies the conditions:

1. (1) Every connected component of $\mathcal{G}$ is a tree.
2. (2) If w is the weight of a vertex of $\mathcal{G} \setminus \{v\}$ then $w = 0$ or $w \leq -2$.
3. (3) Some vertex of $\mathcal{G} \setminus \{v\}$ has weight zero. Moreover, any such vertex has exactly one neighbor in $\mathcal{G}$ and that neighbor is v.

We say that $\mathcal{P}$ is *of type* $(\star\star)$ if it satisfies (1–3) and:

4. (4) $\mathcal{G}$ is connected and exactly one vertex of $\mathcal{G} \setminus \{v\}$ has weight zero.

From the properties of $\mathbb{P}^1$-fibrations, we now derive:

2.12. **Lemma.** *If* (3) *is a minimal completion of* $(U, \rho) \in \mathcal{N}^+$ *then* $\mathcal{P}(U, \rho, u)$ *is a weighted pair of type* $(\star)$. *If we also assume that* U *is affine and has trivial units (i.e.,* $\mathcal{O}_U(U)^* = \mathbf{k}^*$*), then* $\mathcal{P}(U, \rho, u)$ *is of type* $(\star\star)$.

Proof. Let the boundary divisor and horizontal component of the minimal completion (3) be denoted by D and H respectively. Let $\mathcal{G} = \mathcal{G}(D, S)$, then $\mathcal{P}(U, \rho, u) = (\mathcal{G}, H)$.

It is known that every fiber of $\bar{\rho}$ is a tree; since distinct fibers are disjoint, and since D has only one horizontal component, it follows that 2.11(1) holds.

Let C be a vertex of $\mathcal{G} \setminus \{H\}$. Then C is a component of a fiber of $\bar{\rho}$. From known properties of the fibers, it follows that $C^2 \leq 0$. Since C is a ρ-vertical component of D, and since (3) is a minimal completion, $C^2 \neq -1$ and 2.11(2) holds.

We may pick a point P of $\mathbb{P}^1$ which is not in the image of the composite $U \xrightarrow{\rho} \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Consider the fiber $L = \bar{\rho}^{-1}(P)$. Then L is entirely contained in supp(D) and, consequently, no component of L has self-intersection equal to -1. From known properties of the fibers of $\bar{\rho}$, it follows that L is an irreducible curve and that $L^2 = 0$. Hence L is a vertex of $\mathcal{G} \setminus \{H\}$ and has weight zero.

Let L be any vertex of weight zero in $\mathcal{G} \setminus \{H\}$. Then L is an irreducible component of some fiber $\bar{\rho}^{-1}(P)$, and $L^2 = 0$. From known properties of the fibers of $\bar{\rho}$, it follows that $L = \bar{\rho}^{-1}(P)$. Thus $L \cap H \neq \varnothing$ and (since distinct fibers are disjoint) H is the only neighbor of L. This proves 2.11(3) and hence that $\mathcal{P}(U, \rho, u)$ is of type $(\star)$.

Note that the above argument shows that the number of vertices of weight zero in $\mathcal{G} \setminus \{H\}$ is equal to the cardinality of $\mathbb{P}^1$ minus image of $U \xrightarrow{\rho} \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$.

As is well-known, if U is affine then $\mathcal{G}$ is connected; if also U has trivial units then every dominant morphism $U \to \mathbb{A}^1$ is surjective, so $\rho : U \to \mathbb{A}^1$ is surjective. By the above paragraph, there is only one vertex of weight zero in $\mathcal{G} \setminus \{H\}$, so $\mathcal{P}(U, \rho, u)$ is of type $(\star\star)$. $\square$

2.13. **Definition.** A weighted pair $\mathcal{P} = (\mathcal{G}, v)$ is *minimal* if every vertex in $\mathcal{G} \setminus \{v\}$ of weight -1 has more than two neighbors in $\mathcal{G}$.

Note that every weighted pair of type $(\star)$ is minimal. We leave it to the reader to verify:

2.14. **Lemma.** *Consider equivalent weighted pairs $\mathcal{P} \approx \mathcal{P}'$, both of which are minimal. If one of them is of type $(\star)$ then $\mathcal{P}$ and $\mathcal{P}'$ are identical, except possibly for the weight of the distinguished vertex. In particular, both of them are of type $(\star)$.*

2.15. **Lemma.** *Let (U, ρ) and (U', ρ') be equivalent elements of $\mathcal{N}^+$ and let*

$$\begin{array}{ccc} U & \xrightarrow{u} & S \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\bar\rho} \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array} \quad \textit{and} \quad \begin{array}{ccc} U' & \xrightarrow{u'} & S' \\ {\scriptstyle\rho'}\downarrow & & \downarrow{\scriptstyle\bar\rho'} \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

be completions of (U, ρ) and (U', ρ') respectively.

(1) $\mathcal{P}(U, \rho, u) \approx \mathcal{P}(U', \rho', u')$

(2) *If both completions are minimal then the weighted pairs $\mathcal{P}(U, \rho, u)$ and $\mathcal{P}(U', \rho', u')$ are identical, except possibly for the weight of the distinguished vertex.*

Proof. Let D be the SNC-divisor of S such that $U = S \setminus \operatorname{supp}(D)$ and let H be the unique ρ-horizontal component of D; let D' and H' be the corresponding objects in the second diagram. There exists a birational isomorphism $\beta : S \dashrightarrow S'$ which restricts to an isomorphism $U \to U'$ and which gives a commutative diagram:

$$\begin{array}{ccc} S & \overset{\beta}{\dashrightarrow} & S' \\ {\scriptstyle\bar\rho}\downarrow & & \downarrow{\scriptstyle\bar\rho'} \\ \mathbb{P}^1 & \xrightarrow{\cong} & \mathbb{P}^1 \end{array} \tag{4}$$

We may consider a complete normal surface W and birational morphisms $S \xleftarrow{w} W \xrightarrow{w'} S'$ such that $w' \circ w^{-1} = \beta$. Let $\tilde H \subset W$ be the proper transform of $H \subset S$ with respect to w. Since every fiber of $\bar\rho$ meets H, it follows that every fiber of $\bar\rho \circ w$ meets $\tilde H$; so, by commutativity of (4), every fiber of $\bar\rho' \circ w'$ meets $\tilde H$; thus w' cannot contract $\tilde H$ to a point, i.e., $w'\big(\tilde H\big) \subset S'$ is one of the components of D'. Since H' is the only component of D' which is ρ'-horizontal, we must have

$$w'\big(\tilde H\big) = H'.$$

Hence, by blowing-up and blowing-down, it is possible to transform $\mathcal{G}(D, S)$ into $\mathcal{G}(D', S')$ *without ever contracting* H. So, by definition [6]-2.5 of $\approx$, we have

$$\mathcal{P}(U, \rho, u) = \big(\mathcal{G}(D, S), H\big) \approx \big(\mathcal{G}(D', S'), H'\big) = \mathcal{P}(U', \rho', u'),$$

which proves assertion (1).

By 2.14, if $\mathcal{P}$ and $\mathcal{P}'$ are weighted pairs of type $(\star)$ and satisfying $\mathcal{P} \approx \mathcal{P}'$, then $\mathcal{P}$ and $\mathcal{P}'$ are identical, except possibly for the weight of the distinguished vertex. So assertion (2) follows from assertion (1) together with 2.12. $\square$

Remark. By part (2) of 2.15, the claim contained in 2.8 is true.

Some graph theory

2.16. **Definition.** A weighted graph G is called a *chain of type (Z)* if it is a linear chain of the form:

$$G = \overset{0}{\bullet} - \overset{x}{\bullet} - \overset{\omega_1}{\bullet} \cdots - \overset{\omega_q}{\bullet} \tag{5}$$

where $q \geq 0$, x is any integer and $\omega_1, \dots, \omega_q \in \mathbb{Z}$ are such that $\omega_i \leq -2$ for all i. We define the *bideterminant* of G to be the ordered pair

$$\operatorname{bidet} G = \Big(\det\big(\overset{\omega_1}{\bullet} - \overset{\omega_2}{\bullet} - \cdots - \overset{\omega_q}{\bullet} \big),\ \det\big(\overset{\omega_2}{\bullet} - \cdots - \overset{\omega_q}{\bullet} \big) \Big) \in \mathbb{N}^2,$$

where $\det(\overset{\omega_1}{\bullet} - \overset{\omega_2}{\bullet} - \cdots - \overset{\omega_q}{\bullet})$ and $\det(\overset{\omega_2}{\bullet} - \cdots - \overset{\omega_q}{\bullet})$ are determinants of weighted graphs (see [6]-1.10), and where we adopt the convention that $\operatorname{bidet} G = (1, 0)$ if $q = 0$, and that $\operatorname{bidet} G = (-\omega_1, 1)$ if $q = 1$. Since the ordered q-tuple $(\omega_1, \dots, \omega_q) \in \mathbb{Z}^q$ is uniquely determined by G, so is $\operatorname{bidet} G$. So every chain of type (Z) has a bideterminant.

2.17. **Lemma.** *The set of chains of type (Z) equivalent to the chain* (5) *is:*

$$\Big\{ \overset{0}{\bullet} - \overset{y}{\bullet} - \overset{\omega_1}{\bullet} \cdots - \overset{\omega_q}{\bullet} \ \Big|\ y \in \mathbb{Z} \Big\} \ \cup\ \Big\{ \overset{0}{\bullet} - \overset{y}{\bullet} - \overset{\omega_q}{\bullet} \cdots - \overset{\omega_1}{\bullet} \ \Big|\ y \in \mathbb{Z} \Big\}.$$

Proof. Immediate consequence of the classification given in [3]. Alternatively, but this requires some work, the assertion may be derived from section 3 of [5], which gives several results on chains of type (Z). □

The next result will be used in the proof of 2.19. If $\mathcal{G}$, $\mathcal{H}$ are weighted graphs, we write $\mathcal{G} \leftarrow \mathcal{H}$ or $\mathcal{H} \rightarrow \mathcal{G}$ to indicate that $\mathcal{H}$ is obtained by blowing-up $\mathcal{G}$ (or equivalently that $\mathcal{G}$ is obtained by blowing-down $\mathcal{H}$). The arrow should not be interpreted as a map, and the direction of the arrow reminds us of the geometric situation. Writing $\mathcal{G} \xleftarrow{\pi} \mathcal{H}$ allows us to speak of "the blowing-up π". We say that π is *subdivisional* if it is the blowing-up of $\mathcal{G}$ at an edge.

2.18. **Lemma.** *Let $\mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \cdots \leftarrow \mathcal{G}_n$ (where $n \geq 1$) be a sequence of blowings-up of weighted graphs and, for $i \in \{1, \dots, n\}$, let e_i be the vertex of $\mathcal{G}_i$ created by the blowing-up $\mathcal{G}_{i-1} \leftarrow \mathcal{G}_i$. Assume:*

(i) *$\mathcal{G}_0$ is the underlying weighted graph of a weighted pair of type ($\star\star$).*
(ii) *$(\mathcal{G}_n, e_n)$ is equivalent to a weighted pair of type ($\star\star$).*

Then $\mathcal{G}_0$ is a chain of type (Z).

Proof. Given a weighted graph $\mathcal{G}$, the notation $x \in \mathcal{G}$ means that x is a vertex of $\mathcal{G}$, $w(x, \mathcal{G})$ denotes the weight of the vertex x in $\mathcal{G}$ and the symbol $\mathcal{G} \ll 0$ means that for every $x \in \mathcal{G}$ we have $w(x, \mathcal{G}) < -1$.

Suppose that the result is false and consider a counterexample

$$\mathcal{G}_0 \xleftarrow{\pi_1} \mathcal{G}_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} \mathcal{G}_n \tag{6}$$

which minimizes n. Starting at $\mathcal{G}_n$, perform a sequence of blowings-down

$$\mathcal{G}_n = \mathcal{H}_0 \xrightarrow{\nu_1} \mathcal{H}_1 \xrightarrow{\nu_2} \cdots \xrightarrow{\nu_m} \mathcal{H}_m \tag{7}$$

in such a way that e_n is not contracted (so e_n is a vertex of $\mathcal{H}_m$) and $(\mathcal{H}_m, e_n)$ is a minimal weighted pair. Then $(\mathcal{H}_m, e_n) \approx (\mathcal{G}_n, e_n)$, so assumption (ii) and 2.14 imply that

$$(\mathcal{H}_m, e_n) \text{ is of type } (\star\star). \tag{8}$$

By (i), there exists a vertex v of $\mathcal{G}_0$ such that

$$(\mathcal{G}_0, v) \text{ is of type } (\star\star). \tag{9}$$

This and the fact that $\mathcal{G}_0$ is not a chain of type (Z) imply that $\mathcal{G}_0$ is not equivalent to a linear chain. Thus:

$$\text{No } \mathcal{G}_i \text{ or } \mathcal{H}_j \text{ is equivalent to a linear chain.} \tag{10}$$

Recall that, if $1 \le i < j \le n$, the vertex set of $\mathcal{G}_i$ is naturally embedded in that of $\mathcal{G}_j$. In particular, $e_1, \dots, e_n$ may be viewed as vertices of $\mathcal{G}_n$. We begin by showing:

$$\big\{\, i \mid w(e_i, \mathcal{G}_n) = -1 \,\big\} = \{n\} \tag{11}$$

or equivalently:

(12) For all i such that $1 < i \le n$, $\mathcal{G}_i$ is a blowing-up of $\mathcal{G}_{i-1}$ at e_{i-1} or at an edge incident to e_{i-1}.

If these conditions do not hold then we may change the order of the blowings-up in (6) in such a way that the vertex e_n is created at an earlier stage of the blowing-up process; more precisely, there exists a sequence

$$\mathcal{G}_0 = \mathcal{G}'_0 \xleftarrow{\pi'_1} \mathcal{G}'_1 \xleftarrow{\pi'_2} \cdots \xleftarrow{\pi'_n} \mathcal{G}'_n \tag{13}$$

satisfying the following condition: *Let e'_i denote the vertex of $\mathcal{G}'_i$ created by $\mathcal{G}'_{i-1} \leftarrow \mathcal{G}'_i$; then there exists an isomorphism of weighted graphs $\theta : \mathcal{G}_n \to \mathcal{G}'_n$ such that $\theta(e_n) = e'_p$, where $1 \le p < n$.* Identifying $\mathcal{G}_n$ with $\mathcal{G}'_n$, we may write:

$$(\mathcal{G}'_p, e'_p) \approx (\mathcal{G}'_n, e'_p) = (\mathcal{G}_n, e_n) \approx \text{ some weighted pair of type } (\star\star).$$

This means that $\mathcal{G}'_0 \leftarrow \cdots \leftarrow \mathcal{G}'_p$ is a counterexample, contradicting minimality of (6). This proves that (11) and (12) hold.

Since e_n is created by a blowing-up, $\mathcal{G}_n$ has at most two branches at e_n; in fact, it has exactly two. Indeed, (8) implies that some branch $\mathcal{B}$ of $\mathcal{G}_n$ at e_n satisfies:

$$\mathcal{B} \cup \{e_n\} \text{ can be contracted to } \overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{e_n}{\bullet} \tag{14}$$

and if there is no other branch then we get a contradiction with (10). Let $\mathcal{B}^*$ be the other branch of $\mathcal{G}_n$ at e_n. To avoid a contradiction with (10), we must have:

$$\mathcal{B}^* \cup \{e_n\} \text{ is not a linear chain.} \tag{15}$$

We claim:

$$\pi_1, \dots, \pi_n \text{ are subdivisional.} \tag{16}$$

We already know that π_n is subdivisional, because $\mathcal{G}_n$ has two branches at e_n. So, in view of (12), if (16) is false then some branch $\mathcal{B}'$ of $\mathcal{G}_n$ at e_n must satisfy:

$$\mathcal{B}' \cup \{e_n\} \text{ is a linear chain and } \mathcal{B}' \ll 0.$$

Then $\mathcal{B}' \neq \mathcal{B}$ by (14) and $\mathcal{B}' \neq \mathcal{B}^*$ by (15), which is absurd. This proves (16).

Let z be the unique vertex of weight zero in $\mathcal{G}_0 \setminus \{v\}$; let $\mathcal{A}$ and $\mathcal{A}'$ be the two branches of $\mathcal{G}_n$ at e_n, where $z \in \mathcal{A}$ (thus $\{\mathcal{A}, \mathcal{A}'\} = \{\mathcal{B}, \mathcal{B}^*\}$).

If $w(z, \mathcal{G}_n) = 0$ then z and v are neighbors in $\mathcal{G}_n$, so $z, v \in \mathcal{A}$. Then (11) gives

$$\big\{\, x \in \mathcal{G}_n \;\mid\; w(x, \mathcal{G}_n) \geq -1 \,\big\} \subseteq \{z, v, e_n\} \subseteq \mathcal{A} \cup \{e_n\},$$

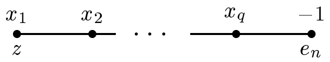
so $\mathcal{A}' \ll 0$, so $\mathcal{A}' \neq \mathcal{B}$ by (14), so $\mathcal{B} = \mathcal{A}$ and $z, v \in \mathcal{B}$. Then $\mathcal{B}$ contains at least two vertices and one of them has weight 0; this makes (14) impossible.

This proves that $w(z, \mathcal{G}_n) \neq 0$, so π_1 is the blowing-up of $\mathcal{G}_0$ at the edge $\{z, v\}$. Then (16) implies that $\mathcal{A} \cup \{e_n\}$ is a linear chain, so $\mathcal{A} \neq \mathcal{B}^*$ by (15). So $\mathcal{B} = \mathcal{A}$ and $z \in \mathcal{B}$.

To summarize, conditions (12) and (16), together with

$$z \in \mathcal{B} \text{ and } \pi_1 \text{ is the blowing-up of } \mathcal{G}_0 \text{ at the edge } \{z, v\},$$

imply that $\mathcal{B} \cup \{e_n\}$ has the following form:



where $q \geq 1$, $x_1 \leq -1$ and $x_j \leq -2$ for all $j \geq 2$. This makes (14) impossible, so we are done. $\square$

A characterization of elements of $\mathcal{M}$

It is known that, for a normal affine surface U, the implication

$$\text{(17)} \qquad \mathrm{ML}(U) = \mathbf{k} \implies \mathcal{G}_\infty[U] \textit{ contains a linear chain}$$

is true (smooth case: [2] or [1], which are based on earlier work of Gizatullin: [8], [9]; general case: [10]). However the converse is false, as shown by the example $U = \mathbb{A}^1_* \times \mathbb{A}^1$ (where $\mathbb{A}^1_*$ is $\mathbb{A}^1$ minus a point).

The next two results are improved versions of (17). Because of the confusion associated with this type of result ((17) is sometimes stated and proved as an "if and only if" statement, which is incorrect) and because all available proofs assume either that $\mathbf{k} = \mathbb{C}$ or that U is smooth, we decided to include the proofs of 2.19 and 2.20. Note that (17) is a consequence of 2.19.

2.19. **Proposition.** *Let $(U, \rho) \in \mathcal{M}^+$. If (3) is any minimal completion of (U, ρ), and if D denotes its boundary divisor, then $\mathcal{G}(D, S)$ is a chain of type (Z).*

Proof. Consider the weighted pair $\mathcal{P}(U, \rho, u) = (\mathcal{G}(D, S), H)$, where D and H are the boundary divisor and the horizontal component of the minimal completion (3) being considered. Let $B = \mathcal{O}_U(U)$, then 2.3 gives $B^* = \mathbf{k}^*$ and 2.12 implies that $\mathcal{P}(U, \rho, u)$ is of type $(\star\star)$.

Let Z be the unique component of D satisfying $Z \neq H$ and $Z^2 = 0$. The assumption $ML(U) = \mathbf{k}$ implies that U is rational, so the complete linear system $|Z|$ on S

is one-dimensional, base point free and has general member isomorphic to $\mathbb{P}^1$. The corresponding morphism $S \to \mathbb{P}^1$ is the $\bar{\rho}$ of (3).

In view of 2.3, the assumption $\mathrm{ML}(U) = \mathbf{k}$ implies that there exists a morphism $\rho' : U \to \mathbb{A}^1$ whose general fiber is an $\mathbb{A}^1$ and which is "genuinely different" from ρ, in the following sense: Extend ρ' to a rational map $S \dashrightarrow \mathbb{P}^1$ and consider the corresponding linear system Λ on S; then $\Lambda \neq |Z|$. Consider the minimal resolution of the base points[2] of Λ (where $n = 0$ if $\mathrm{Bs}(\Lambda) = \varnothing$):

$$S_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} S_0 = S \tag{18}$$

where each π_i is the blowing-up of S_{i-1} at a smooth point P_i. Write $E_i = \pi_i^{-1}(P_i) \subset S_i$ and let $u_n : U \hookrightarrow S_n$ be the open immersion obtained by factoring u through (18). Now ρ' extends to a morphism $\bar{\rho}' : S_n \to \mathbb{P}^1$ and

$$\begin{tikzcd} U \arrow[r, "u_n"] \arrow[d, "\rho'"'] & S_n \arrow[d, "\bar{\rho}'"] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd} \tag{19}$$

is a completion of (U, ρ').

Consider the case where $\mathrm{Bs}(\Lambda) = \varnothing$; then $n = 0$, $S_n = S$ and $u_n = u$, so (19) is a completion of (U, ρ') with boundary divisor D. If Z is ρ'-vertical then the fact that $Z^2 = 0$ implies that Z is *equal* to a fiber of $\bar{\rho}'$; then $Z \in \Lambda$, so $\Lambda \subseteq |Z|$ (because $|Z|$ is a complete linear system), so $\Lambda = |Z|$ (because $\dim |Z| = 1 = \dim \Lambda$), which is absurd. Hence, Z must be ρ'-horizontal, so every component of $D - Z$ is ρ'-vertical. Since Z has only one neighbor in $\mathcal{G}(D, S)$ (namely, H), $\mathrm{supp}(D - Z)$ is connected and consequently is included in a fiber of $\bar{\rho}'$; since some fiber of $\bar{\rho}'$ is entirely contained in D, $\mathrm{supp}(D - Z)$ is equal to a fiber. By properties of $\mathbb{P}^1$-fibrations, if the fiber $\mathrm{supp}(D - Z)$ is not irreducible then it contains a (-1)-curve other than H; this is not the case by minimality of (3), so $D - Z = H$ and $\mathcal{G}(D, S)$ is a chain of type (Z).

From now-on, suppose that $\mathrm{Bs}(\Lambda) \neq \varnothing$ and consider the weighted pair $\mathcal{P}(U, \rho', u_n)$ associated to the completion (19). Since $n \geq 1$ and (18) is minimal, E_n is the horizontal component of (19); so E_n is the distinguished vertex of $\mathcal{P}(U, \rho', u_n)$. We have $\mathcal{P}(U, \rho', u_n) \approx \mathcal{P}'$ by 2.15, where $\mathcal{P}'$ is the weighted pair associated to a minimal completion of (U, ρ'). Since $\mathcal{P}'$ is of type $(\star\star)$ by the first paragraph of the proof, the desired conclusion follows from 2.18. $\square$

2.20. **Theorem.** *Let U be a normal affine surface. Then $\mathrm{ML}(U) = \mathbf{k}$ if and only if the following conditions are satisfied:*

(a) *U is rational and completable by rational curves*
(b) *$\mathcal{G}_\infty[U]$ contains a chain of type (Z).*

Proof. It is well-known that $\mathrm{ML}(U) = \mathbf{k}$ implies condition (a), and 2.19 implies (b).

[2]Actually, Λ has at most one base point on S, but we don't need to know this for the proof.

Conversely, let U be a normal affine surface satisfying conditions (a) and (b) and let $B = \mathcal{O}_U(U)$. We shall show that $\text{KLND}(B)$ is an infinite set, which will prove the "if" part of 2.20.

We may choose a smooth-normal compactification $u : U \hookrightarrow S$ of U such that $\mathcal{G}(u)$ is a chain of type (Z). Let D be the SNC-divisor of S such that $S \setminus U = \text{supp}(D)$ and consider the components Z and Σ of D such that $\mathcal{G}(u) = \mathcal{G}(D, S)$ is the following:

$$\underset{Z}{\overset{0}{\bullet}} \text{---} \underset{\Sigma}{\overset{-1}{\bullet}} \text{---} \overset{\omega_1}{\bullet} \text{---} \cdots \text{---} \overset{\omega_q}{\bullet} \tag{20}$$

where $q \geq 0$ and $\omega_1, \dots, \omega_q \leq -2$ (by 2.17, it is indeed possible to arrange $\Sigma^2 = -1$; we choose a smooth-normal compactification u which has this property).

Let $P \in Z \setminus \Sigma$. We now proceed to construct a surjective morphism $\rho : U \to \mathbb{A}^1$ with general fiber $\mathbb{A}^1$ and which satisfies:

(21) If we extend ρ to a rational map $S \dashrightarrow \mathbb{P}^1$ and consider the corresponding linear system Λ on S, then $\text{Bs}(\Lambda) = \{P\}$.

In view of 2.3, this will imply that $\text{KLND}(B)$ is an infinite set.

By 2.17, the chains (20) and

$$\overset{0}{\bullet} \text{---} \overset{-2}{\bullet} \text{---} \overset{\omega_q}{\bullet} \text{---} \cdots \text{---} \overset{\omega_1}{\bullet} \tag{22}$$

are equivalent weighted graphs. In fact, Theorem 3.32 of [5] is more precise: It describes a sequence of blowings-up and blowings-down which transforms (20) into (22). From that description, and since we assumed that U is completable by rational curves (and hence all components of D are rational), it follows that there exists birational morphisms

$$S \xleftarrow{\pi} S^* \xrightarrow{\pi'} S'$$

of projective normal surfaces satisfying the following conditions:

(i) $\text{cent}(\pi) = \{P\}$ and $\text{cent}(\pi')$ is a smooth point of S'.
(ii) $\pi^{-1}(\text{supp}(D))$ is the support of an SNC-divisor D^* of S^* and $\text{supp}(D^*) = \text{exc}(\pi) \cup \text{exc}(\pi')$.
(iii) $\pi'(\text{supp}(D^*))$ is the support of an SNC-divisor D' of S' such that $\mathcal{G}(D', S')$ is the chain (22).

Let U' be the complement of $\text{supp}(D')$ in S', let Z' be the unique component of D' of self-intersection 0 and let Σ' be the unique component of D' satisfying $\Sigma' \neq Z'$ and $\Sigma' \cap Z' \neq \varnothing$ (refer to (22)). Since U (and hence S') is rational by assumption, it is well-known that the complete linear system $|Z'|$ on S' is one-dimensional, base point free, and has general member isomorphic to $\mathbb{P}^1$; hence $|Z'|$ gives rise to a $\mathbb{P}^1$-fibration $\bar{f} : S' \to \mathbb{P}^1$. Consider the point $P_\infty \in \mathbb{P}^1$ defined by $\bar{f}(Z') = \{P_\infty\}$. Since the fiber $\bar{f}^{-1}(P_\infty) = Z'$ is disjoint from U', we have $\bar{f}(U') \subseteq \mathbb{P}^1 \setminus \{P_\infty\}$, so $\bar{f}$ restricts to a morphism $f : U' \to \mathbb{A}^1$ whose general fiber is an $\mathbb{A}^1$. By properties of $\mathbb{P}^1$-fibrations, the fact that $\omega_1, \dots, \omega_q \leq -2$ implies that Z' is the only fiber of $\bar{f}$ which is entirely contained in $\text{supp}(D')$. Thus $f : U' \to \mathbb{A}^1$ is surjective. Observe that $\pi' \circ \pi^{-1}$ restricts to an isomorphism $\theta : U \to U'$ and define $\rho = f \circ \theta : U \to \mathbb{A}^1$. Then ρ is surjective and has general fiber $\mathbb{A}^1$.

Define Λ as in (21) and note that Λ corresponds to $|Z'|$ via $\pi' \circ \pi^{-1}$. Note that Σ' is a section of $|Z'|$ and let $\tilde{\Sigma}' \subset S^*$ denote the strict transform of Σ'. Since $\operatorname{supp}(D^*) = \operatorname{exc}(\pi) \cup \operatorname{exc}(\pi')$, and since $\tilde{\Sigma}'$ is obviously not shrunk by π', we have $\tilde{\Sigma}' \subseteq \operatorname{exc}(\pi)$. So $\pi(\tilde{\Sigma}') = \{P\}$, which proves (21) and the "if" part of 2.20. $\square$

3. Tableaux and affine rulings

3.1. **Definition.** A *tableau* is a matrix $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)$ whose entries are integers satisfying $c_i \geq p_i \geq 1$ and $\gcd(p_i, c_i) = 1$ for all $i = 1, \dots, k$. We allow $k = 0$, in which case we say that T is the *empty tableau* and write $T = \mathbf{1}$. The set of all tableaux is denoted $\mathcal{T}$. We define subsets $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_*$ of $\mathcal{T}$ as follows:

$$\begin{aligned} \mathcal{T}_1 &= \{\mathbf{1}\} \cup \left\{ \tbinom{p}{c} \mid p, c \in \mathbb{Z},\ 1 \leq p < c \text{ and } \gcd(p, c) = 1 \right\} \\ \mathcal{T}_2 &= \left\{ \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \mid a, p, c \in \mathbb{Z},\ a \geq 1,\ 1 \leq p < c \text{ and } \gcd(p, c) = 1 \right\} \\ \mathcal{T}_* &= \mathcal{T}_1 \cup \mathcal{T}_2. \end{aligned}$$

We will often identify the empty tableau with the column $\binom{0}{1}$. This allows us to write $\mathcal{T}_1 = \left\{ \binom{p}{c} \mid p, c \in \mathbb{Z},\ 0 \leq p < c \text{ and } \gcd(p, c) = 1 \right\}$, which turns out to be very convenient. This practice is in agreement with convention 2.29 of [6].

Remark. Although the notations $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_*$ are not used in [5] and [6], the elements of those sets play an important role in the cited papers. In particular, $\mathcal{T}_*$ is the set of tableaux satisfying *one of the conditions* (1–3) *of* [6]-2.16 (or equivalently *one of the conditions* (1–3) *of* [5]-5.6).

3.2. **Definition.** Given a tableau $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right) \in \mathcal{T}$, define

$$\mu(T) = \prod_{i=1}^{k} c_i$$

where in particular $\mu(\mathbf{1}) = 1$. The positive integer $\mu(T)$ is called the *multiplicity* of T.

3.3. **Definition.** Given $T \in \mathcal{T}_*$, define $\check{T} \in \mathcal{T}_*$ by:

(1) If $T = \binom{p}{c} \in \mathcal{T}_1$, then $\check{T} = \binom{p'}{c} \in \mathcal{T}_1$, where p' is defined by $0 \leq p' < c$ and $pp' \equiv 1 \pmod{c}$;

(2) If $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$, then $\check{T} = \left(\begin{smallmatrix} c-p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$.

Note that $\check{\mathbf{1}} = \mathbf{1}$ and $\big(\check{T}\big)\check{} = T$.

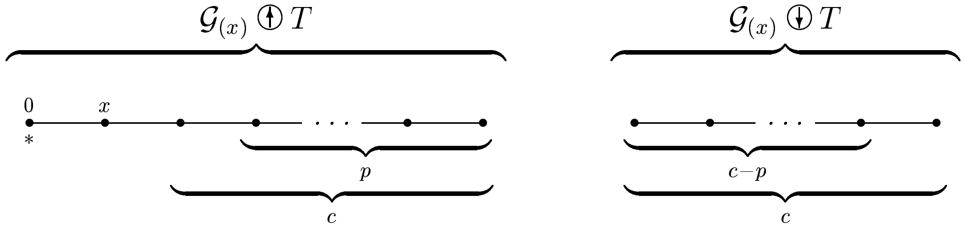
3.4. (See [6]-2.2 and [6]-2.3.) Let n be a positive integer. A *weighted n-tuple* is an ordered n-tuple $S = (\mathcal{G}, v_1, \dots, v_{n-1})$ where $\mathcal{G}$ is a weighted graph and $v_1, \dots, v_{n-1}$ are distinct vertices of $\mathcal{G}$. If $n = 1$ then S is simply a weighted graph; if $n = 2$ then it is a weighted pair (2.9). Given $x \in \mathbb{Z}$, let $\mathcal{G}_{(x)}$ be the weighted triple $(\mathcal{G}, v_1, v_2)$ where $\mathcal{G}$ is the weighted graph: $\overset{0}{\underset{v_1}{\bullet}} \!\!-\!\!\!-\!\!\!-\!\! \overset{x}{\bullet} \!\!-\!\!\!-\!\!\!-\!\! \overset{0}{\underset{v_2}{\bullet}}$.

3.5. Let S be a weighted n-tuple with $n \geq 2$ and let $T \in \mathcal{T}$ be a tableau. By definitions 2.8–2.10 of [6], T may be regarded as a compact "recipe" for performing a finite sequence of blowings-up on S; the graph obtained at the end of this blowing-up process is denoted ST and is a weighted n-tuple (with the same n as S). If we delete from ST the vertex of weight -1 created by the last blowing-up of the sequence, we obtain a subgraph $S \ominus T$ of ST, and $S \ominus T$ is a weighted $(n-1)$-tuple. One also defines subgraphs $S \textcircled{$\uparrow$} T$ and $S \textcircled{$\downarrow$} T$ of $S \ominus T$, refer to [6]-2.10 for details. Note in particular:

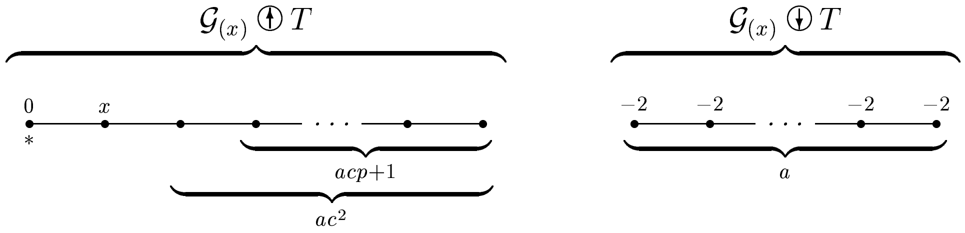
- $S \ominus T$ is the disjoint union of $S \textcircled{$\uparrow$} T$ and $S \textcircled{$\downarrow$} T$;
- $S \textcircled{$\uparrow$} T$ is a weighted $(n-1)$-tuple, with the same sequence of distinguished vertices as $S \ominus T$.
- $S \textcircled{$\downarrow$} T$ is a weighted graph; in fact, it is a (possibly empty) admissible chain.

In particular, $\mathcal{G}_{(x)} \ominus T$ and $\mathcal{G}_{(x)} \textcircled{$\uparrow$} T$ are weighted pairs with the same distinguished vertex, $\mathcal{G}_{(x)} \textcircled{$\downarrow$} T$ is a weighted graph (in fact an admissible chain) and $\mathcal{G}_{(x)} \ominus T$ is the disjoint union of $\mathcal{G}_{(x)} \textcircled{$\uparrow$} T$ and $\mathcal{G}_{(x)} \textcircled{$\downarrow$} T$. The following statement is an immediate consequence of [6]-2.12:

3.5.1. **Lemma.** *If $x \in \mathbb{Z}$ and $T \in \mathcal{T}_*$, the weighted pair $\mathcal{G}_{(x)} \ominus T$ is as follows.*
If $T = \binom{p}{c} \in \mathcal{T}_1$, then:

(23)


If $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$, then:

(24)


In the above pictures, the distinguished vertex is marked with a "$*$" and the integers under the braces are the determinants *of the indicated linear chains (see* [6]*-1.10). Also, whenever the weight of a vertex is not indicated, that weight is strictly less than -1.*

3.6. Various subsets of $\mathbb{Z}^+ \times \mathcal{T} \times \mathcal{T}$ are considered in [5] and [6] (where $\mathbb{Z}^+$ denotes the set of positive integers). In particular (see [5]-5.24 or [6]-2.24):

$$\mathbb{T}_0(\ddagger) = \big\{ (m, T_1, T_2) \in \mathbb{Z}^+ \times \mathcal{T}_* \times \mathcal{T}_* \ \mid \ \text{each connected component of the weighted graph } (\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2 \text{ is equivalent to an admissible chain} \big\}.$$

An explicit description of the elements of $\mathbb{T}_0(\ddagger)$ is given in [5]-5.41. The set $\mathbb{T}_0(\ddagger)$ is used for classifying affine rulings, as we will see in 3.8 and 3.9, below.

Affine rulings

We begin by defining the class of surfaces "satisfying the condition (‡)." By [6]-1.20 and 1.21, this class contains all weighted projective planes.

3.7. Consider the following condition on an algebraic surface X:

(†) X is a complete normal rational surface, X is affine-ruled and rank(Pic X_s) $= 1$,

where X_s denotes the smooth locus of X. Also consider the stronger condition:

(‡) X satisfies (†) and every singular point of X is a cyclic quotient singularity.

If X is a complete normal rational surface, [6]-2.21 defines an *affine ruling* of X to be a linear system on X satisfying certain conditions. Suppose that X satisfies (‡).

- The notion of a *basic* affine ruling of X is defined in [6]-2.22.1.
- If Λ is an affine ruling of X then a **nonempty** subset Λ_* of Λ is defined in [6]-2.22.3.

While writing the present paper we realised that statement 3.7.1 is true and can be used as a definition of Λ_*.

3.7.1. *Let Λ be an affine ruling of a surface X satisfying (‡). Then, for any element F of Λ, the following conditions are equivalent:*

(1) $F \in \Lambda_*$
(2) *Some element of the set $\mathcal{G}_\infty[\, X \setminus \operatorname{supp}(F)\,]$ is a chain of type (Z).*
(3) *Some element of the set $\mathcal{G}_\infty[\, X \setminus \operatorname{supp}(F)\,]$ is a linear chain.*

In the same circle of ideas, we note:

3.7.2. *Let Λ be an affine ruling of a surface X satisfying (‡), let $F_1, \dots, F_r$ be distinct elements of Λ and let $U = X \setminus \operatorname{supp}(F_1 + \cdots + F_r)$. Then the following are equivalent:*

(1) *Some element of the set $\mathcal{G}_\infty[U]$ is a chain of type (Z).*
(2) *$r = 1$ and $F_1 \in \Lambda_*$.*

Facts 3.7.1 and 3.7.2 are implicitely contained in section 1 of [5], so we omit their proofs.

3.8. Consider triples (X, Λ, F) where X is a surface satisfying (‡), Λ is a basic affine ruling of X and $F \in \Lambda_*$.

- Given a triple (X, Λ, F), [6]-2.23 defines an element disc(X, Λ, F) of $\mathbb{T}_0(‡)$, called the *discrete part* of (X, Λ, F). (See 3.6 for $\mathbb{T}_0(‡)$.)
- Two triples are *equivalent*, $(X, \Lambda, F) \sim (X', \Lambda', F')$, when there exists an isomorphism $X \to X'$ which transforms Λ into Λ' and F into F'. The equivalence class of (X, Λ, F) is denoted $[X, \Lambda, F]$ and $\mathbb{S}_0(‡)$ is the set of equivalence classes.
- If $(X, \Lambda, F) \sim (X', \Lambda', F')$, then (X, Λ, F) and (X', Λ', F') have the same discrete part; so we may speak of the discrete part of the equivalence class $[X, \Lambda, F]$

of (X, Λ, F), and we have a set map

$$\mathrm{disc}: \mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger) \qquad [X, \Lambda, F] \mapsto \text{discrete part of } [X, \Lambda, F].$$

The following may be regarded as a classification of triples (X, Λ, F):

3.9. **Theorem.** *The map* $\mathrm{disc}: \mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$ *is bijective.*

Proof. This is part of [5]-5.25. □

3.10. If X is a surface satisfying $(\ddagger)$ then [5]-5.2 defines a subset $\mathbb{T}_0(X)$ of $\mathbb{T}_0(\ddagger)$ by

$$\mathbb{T}_0(X) = \big\{ \operatorname{disc}(X, \Lambda, F) \mid \Lambda \text{ is a basic affine ruling of } X \text{ and } F \in \Lambda_* \big\}.$$

4. A result about tableaux

If a_0, a_1, a_2 are pairwise relatively prime positive integers, [6]-2.26 defines four subsets

$$\mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2),\ \mathbb{T}_{\mathrm{II}.1}(a_0, a_1, a_2),\ \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2) \ \text{ and } \ \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$$

of $\mathbb{T}_0(\ddagger)$. Results 4.2 and 4.3, below, are concerned with $\mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2)$, $\mathbb{T}_{\mathrm{II}.1}(a_0, a_1, a_2)$ and $\mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2)$.

4.1. **Lemma.** *Let a, b, c be positive integers such that $c > 1$ and $\gcd(b, c) = 1$. Then there exists an integer y satisfying*

$$by \equiv 1 \pmod{c}, \quad \gcd(y, a) = 1 \quad \textit{and} \quad 0 < y < ac.$$

Proof. Write $a = \alpha\alpha'$, where α and α' are the positive integers satisfying

$$\gcd(\alpha, c) = 1$$

$$\text{For every prime number } p, \quad p \mid \alpha' \implies p \mid c.$$

Since $\gcd(b, c) = 1$, there exists an integer y_0 satisfying $by_0 \equiv 1 \pmod{c}$. Since $\gcd(\alpha, c) = 1$, the Chinese Remainder Theorem implies that exactly one integer y satisfies $0 \le y < \alpha c$ and

$$y \equiv y_0 \pmod{c}$$
$$y \equiv 1 \pmod{\alpha}.$$

Then it is easy to see that y satisfies the desired conditions. □

4.2. **Lemma.** *Given any $T_2 \in \mathcal{T}_2$, there exist relatively prime positive integers a_0, a_1, a_2 and a tableau $T_1 \in \mathcal{T}_1$ such that*

$$\big(1, T_1, T_2\big) \in \mathbb{T}_{\mathrm{II}.1}(a_0, a_1, a_2).$$

Proof. We have $T_2 = \left(\begin{smallmatrix} p_2 & 1 \\ c_2 & a_2 \end{smallmatrix}\right)$, where p_2, c_2 and a_2 are integers such that

$$c_2 > p_2 \ge 1, \quad \gcd(p_2, c_2) = 1 \quad \text{and} \quad a_2 \ge 1.$$

By 4.1, we may choose an integer a_1 satisfying

$$(c_2 - p_2)a_1 \equiv 1 \pmod{c_2} \tag{25}$$

$$\gcd(a_1, a_2) = 1 \tag{26}$$

$$0 < a_1 < a_2c_2. \tag{27}$$

Then we may define an integer a_0 by

$$a_0 = a_2c_2 - a_1. \tag{28}$$

Note that $a_0 > 0$ by (27). We claim:

(29) *a_0, a_1, a_2 are pairwise relatively prime positive integers.*

In fact, we already know that $\gcd(a_1, a_2) = 1$. From (25), we obtain $\gcd(a_1, c_2) = 1$, so $\gcd(a_1, a_2c_2) = 1$. Thus $\gcd(a_0, a_1) = \gcd(a_2c_2 - a_1, a_1) = \gcd(a_2c_2, a_1) = 1$, and $\gcd(a_0, a_2) = \gcd(a_2c_2 - a_1, a_2) = \gcd(a_1, a_2) = 1$. Hence, statement (29) is true.

Let k be the unique integer such that

$$\frac{c_2}{a_1} \le k < \frac{c_2}{a_1} + 1 \tag{30}$$

and define integers x_0, x_1 and x_2 by

$$x_1 = ka_1 - c_2 \tag{31}$$

$$x_2 = \begin{cases} 0, & \text{if } a_2 = 1 \\ 1, & \text{if } a_2 > 1. \end{cases} \tag{32}$$

$$x_0 = k + x_2 - 1. \tag{33}$$

Then, in the terminology of 7.2 of [6], we claim that

(34) *(x_0, x_1, x_2) is the unique solution to* $\mathrm{Eq}(a_0, a_1, a_2)$.

Explicitely, (34) means that (x_0, x_1, x_2) satisfies the three conditions:

$$a_0 = a_1a_2x_0 - a_2x_1 - a_1x_2 \tag{35}$$

$$0 \le x_1 < a_1 \tag{36}$$

$$0 \le x_2 < a_2. \tag{37}$$

Indeed, (37) is obvious from (32), (36) follows from (31) and (30), and the calculation:

$$\begin{aligned} a_1a_2x_0 - a_2x_1 - a_1x_2 &= a_1a_2(k + x_2 - 1) - a_2(ka_1 - c_2) - a_1x_2 && \text{by (33) and (31)} \\ &= a_1a_2(x_2 - 1) + a_2c_2 - a_1x_2 \\ &= a_1a_2(x_2 - 1) + (a_0 + a_1) - a_1x_2 && \text{by (28)} \\ &= a_0 + a_1(a_2 - 1)(x_2 - 1) = a_0 \end{aligned}$$

shows that (35) holds. So (34) is true. We define two more integers by:

$$x_1' = \frac{a_1(c_2 - p_2) - 1}{c_2} \tag{38}$$

$$x_1'' = kx_1' - c_2 + p_2 \tag{39}$$

Note that (25) implies that x_1' is an integer, so x_1'' is an integer as well. We claim:

(40) *x_1' and x_1'' are the integers determined by the solution (x_0, x_1, x_2) to $\mathrm{Eq}(a_0, a_1, a_2)$, as described in 7.2 of* [6].

Explicitely, (40) means that the following conditions hold:

$$x_1 x_1' \equiv 1 \pmod{a_1} \quad \text{and} \quad 0 \le x_1' < a_1 \tag{41}$$

$$\begin{vmatrix} x_1 & x_1'' \\ a_1 & x_1' \end{vmatrix} = 1. \tag{42}$$

We check that these hold. Clearly, $0 \le x_1' < a_1$ follows immediately from (38). By (31) and (38),

$$\begin{aligned} x_1 x_1' = (ka_1 - c_2)x_1' \equiv (-c_2)x_1' = (-c_2)\,\frac{a_1(c_2 - p_2) - 1}{c_2} \\ = 1 - a_1(c_2 - p_2) \equiv 1 \pmod{a_1}, \end{aligned}$$

so (41) holds. By (31) and (39),

$$\begin{vmatrix} x_1 & x_1'' \\ a_1 & x_1' \end{vmatrix} = \begin{vmatrix} ka_1 - c_2 & kx_1' - c_2 + p_2 \\ a_1 & x_1' \end{vmatrix} = \begin{vmatrix} -c_2 & -c_2 + p_2 \\ a_1 & x_1' \end{vmatrix} = \begin{vmatrix} a_1 & x_1' \\ c_2 & c_2 - p_2 \end{vmatrix},$$

and the last determinant is equal to 1 by (38), so (42) holds as well. So statement (40) is true. Next, we verify that

$$\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 - x_1 - x_1' + x_1'' \\ a_1 - x_1 \end{pmatrix} + (x_0 - x_2)\begin{pmatrix} a_1 - x_1' \\ a_1 \end{pmatrix}. \tag{43}$$

To see this, note that the right hand side of (43) is equal to:

$$\begin{aligned} \begin{pmatrix} (x_0 - x_2 + 1)(a_1 - x_1') - x_1 + x_1'' \\ (x_0 - x_2 + 1)a_1 - x_1 \end{pmatrix} &= \begin{pmatrix} k(a_1 - x_1') - x_1 + x_1'' \\ ka_1 - x_1 \end{pmatrix} && \text{by (33)} \\ &= \begin{pmatrix} (ka_1 - x_1) + (x_1'' - kx_1') \\ ka_1 - x_1 \end{pmatrix} \\ &= \begin{pmatrix} c_2 + (p_2 - c_2) \\ c_2 \end{pmatrix} && \text{by (31) and (39)} \end{aligned}$$

so (43) is true.

Finally, we note that (28) gives $\frac{a_0+a_1+a_2}{a_2} = \frac{a_2c_2+a_2}{a_2} = c_2 + 1$, hence:

(44) $\dfrac{a_0 + a_1 + a_2}{a_2}$ *is a natural number strictly greater than* 2.

We may now apply Proposition 7.4 of [6]. By (44) and the cited result, the set $\mathbb{T}_{\mathrm{II.1}}(a_0, a_1, a_2)$ has exactly one element. The same result, together with (34), (40) and (43), implies that the unique element of $\mathbb{T}_{\mathrm{II.1}}(a_0, a_1, a_2)$ is the triple:

$$\left(1, \begin{pmatrix} x_1' \\ a_1 \end{pmatrix}, \begin{pmatrix} p_2 & 1 \\ c_2 & a_2 \end{pmatrix}\right). \tag{45}$$

This completes the proof of 4.2. $\square$

Remark. In the above argument, a_1 may be equal to 1 (this is the case iff $c_2 - p_2 = 1$). If $a_1 = 1$, then $x_1' = 0$ and the column $\left(\begin{smallmatrix} x_1' \\ a_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ which appears in (45) should be interpreted as the empty tableau. See 3.1.

4.3. **Proposition.** *Given any $T_1 \in \mathcal{T}_*$, there exist relatively prime positive integers a_0, a_1, a_2 and a tableau $T_2 \in \mathcal{T}_1$ such that*

$$\big(1, T_1, T_2\big) \in \mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2).$$

Proof. If $T_1 \in \mathcal{T}_1$ then $T_1 = \left(\begin{smallmatrix} x_1 \\ a_1 \end{smallmatrix}\right)$ where $x_1, a_1 \in \mathbb{Z}$, $0 \le x_1 < a_1$ and $\gcd(x_1, a_1) = 1$. Define $a_0 = a_1 - x_1$ and $a_2 = 1$, then it is immediate that

$$a_0, a_1, a_2 \textit{ are pairwise relatively prime positive integers}$$

and that

$$(1, x_1, 0) \textit{ is the unique solution to } \mathrm{Eq}(a_0, a_1, a_2)$$

(see (34–37) for meaning). Then Proposition 7.3 of [6] implies that

$$(1, T_1, \mathbf{1}) \in \mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2),$$

so the assertion is true in this case.

If $T_1 \in \mathcal{T}_2$ then, by 4.2, we have $(1, T_2, T_1) \in \mathbb{T}_{\mathrm{II}.1}(a_0, a_2, a_1)$ for some tableau $T_2 \in \mathcal{T}_1$ and some pairwise relatively prime positive integers a_0, a_2, a_1; then

$$(1, T_1, T_2) \in \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2)$$

and we are done. □

We now give the geometric interpretation of 4.3. Recall from 3.10 that each surface X satisfying (‡) determines a subset $\mathbb{T}_0(X)$ of $\mathbb{T}_0(‡)$. In the case where X is a weighted projective plane, an explicit description of $\mathbb{T}_0(X)$ is given in [6]. In particular, [6]-7.1 implies that

$$\mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2) \subseteq \mathbb{T}_0\big(\mathbb{P}(a_0, a_1, a_2)\big)$$

for any pairwise relatively prime positive integers a_0, a_1, a_2. Thus result 4.3 has the following

4.4. **Corollary.** *Given any $T_1 \in \mathcal{T}_*$, there exist a weighted projective plane $\mathbb{P}$ and a tableau $T_2 \in \mathcal{T}_1$ such that*

$$\big(1, T_1, T_2\big) \in \mathbb{T}_0(\mathbb{P}).$$

5. The map $\mathfrak{F}$

5.1. Given a triple (X, Λ, F) as in 3.8, consider the open subset $U = X \setminus \mathrm{supp}(F)$ of X and use the affine ruling Λ to define a morphism $\rho : U \to \mathbb{A}^1$ whose general fiber is an affine line (the fibers of ρ are the elements of the linear system $\Lambda|_U$). Although ρ is not unique, it is clear that the equivalence class $[U, \rho] \in \overline{\mathcal{N}}^+$ (see 2.2) is uniquely determined by (X, Λ, F) and that, furthermore, $[X, \Lambda, F] \mapsto [U, \rho]$ is a well-defined map. We call this map the "restriction", and write

$$\begin{array}{ccc} \mathbb{S}_0(‡) & \xrightarrow{\text{res}} & \overline{\mathcal{N}}^+ \\ [X, \Lambda, F] & \longmapsto & [U, \rho]. \end{array}$$

We will see in 6.4 that the image of res is exactly $\overline{\mathcal{M}}_0^+$.

5.2. Recall from 3.9 that disc : $\mathbb{S}_0(\ddagger) \to \mathbb{T}_0(\ddagger)$ is bijective. We now consider the composition

$$\mathbb{T}_0(\ddagger) \xrightarrow{\text{disc}^{-1}} \mathbb{S}_0(\ddagger) \xrightarrow{\text{res}} \overline{\mathcal{N}}^+,$$

which we denote

$$\mathfrak{F} : \mathbb{T}_0(\ddagger) \longrightarrow \overline{\mathcal{N}}^+.$$

So the fact that $\mathfrak{F}$ is well-defined depends on 3.9, which is a nontrivial fact.

5.3. **Proposition.** *Given two elements $\tau = (m, T_1, T_2)$ and $\tau' = (m', T_1', T_2')$ of $\mathbb{T}_0(\ddagger)$,*

$$\mathfrak{F}(\tau) = \mathfrak{F}(\tau') \iff T_1 = T_1'.$$

Proof. Let $\tau = (m, T_1, T_2) \in \mathbb{T}_0(\ddagger)$ (we will not consider a τ') and let $[U, \rho] = \mathfrak{F}(\tau)$. We will show that $[U, \rho]$ is independent of T_2 and m (see (49) and (50)), and this proves "$\Longleftarrow$". We will show that T_1 can be recovered from $[U, \rho]$ (see (57) and (58)), which proves " $\Longrightarrow$ ".

Let us explain how $\mathfrak{F}$ constructs $[U, \rho]$ from τ. We consider each of the two steps:

$$\tau \overset{\text{disc}^{-1}}{\longmapsto} [X, \Lambda, F] \overset{\text{res}}{\longmapsto} [U, \rho].$$

(i) The first step is to construct $[X, \Lambda, F]$ from τ. This explanation is adapted from the more general [5]-5.29 (more general because $\mathbb{T}_0(\ddagger) \subset \mathbb{T}(\dagger)$). Consider the ruled surface $\mathbb{F}_m$, its ruling Λ_m and directrix $\Sigma_m \subset \mathbb{F}_m$. Choose a blowing-up

$$\left(\tilde{X} \xrightarrow{\pi} \mathbb{F}_m, P_1, P_2\right) \tag{46}$$

of $\mathbb{F}_m$ according to (T_1, T_2) (see [5]-5.26 for definition) and let Z_1 and Z_2 be the distinct elements of Λ_m satisfying $P_i \in Z_i$. In particular, recall that $\tilde{X}$ is a smooth projective surface, π is a birational morphism, $\text{cent}(\pi) \subseteq \{P_1, P_2\}$ and for each $i \in \{1, 2\}$ we have:

- $\pi^{-1}(P_i)$ has at most one (-1)-component
- $\overline{\text{HN}}(\pi, Z_i) = T_i$ (see [5]-4.1 for $\overline{\text{HN}}$).

For each $i \in \{1, 2\}$, define

$$E_i = \begin{cases} \pi^{-1}(Z_i), & \text{if } P_i \notin \text{cent}(\pi), \\ \text{the } (-1)\text{-curve in } \pi^{-1}(P_i), & \text{if } P_i \in \text{cent}(\pi). \end{cases}$$

Then $E_1, E_2 \subset \tilde{X}$ are irreducible curves. Let Δ be the SNC-divisor of $\tilde{X}$ defined by

$$\pi^{-1}(Z_1 \cup \Sigma_m \cup Z_2) = \text{supp}(E_1 + \Delta + E_2) \quad \text{and} \quad E_1, E_2 \not\subseteq \text{supp}(\Delta). \tag{47}$$

(Δ is called D in [5]-5.29, but D has a different meaning in the present argument). Then, as claimed in [5]-5.29, we have $\mathcal{G}(\Delta, \tilde{X}) = (\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ and consequently Δ has a negative definite intersection matrix. So there exists a complete normal surface X and a birational morphism $\gamma : \tilde{X} \to X$ with $\text{exc}(\gamma) = \text{supp}(\Delta)$. By [5]-5.29 we also obtain that X satisfies condition ($\dagger$) of 3.7; that Λ_m determines an affine ruling Λ of X via the birational morphisms $\mathbb{F}_m \xleftarrow{\pi} \tilde{X} \xrightarrow{\gamma} X$; and that $\gamma(E_2)$ is the support of an element F of Λ_* (in [5]-5.29, F is called F_2).

In our special case ($\tau \in \mathbb{T}_0(\ddagger)$, so $T_2 \in \mathcal{T}_*$), result [5]-5.11 implies that X satisfies ($\ddagger$) and that Λ is a *basic* affine ruling of X; so we have constructed a triple (X, Λ, F) as in 3.8, i.e., $[X, \Lambda, F] \in \mathbb{S}_0(\ddagger)$. Moreover, [5]-5.29 asserts that $\operatorname{disc}(X, \Lambda, F) = \tau$. (*Remark:* It is not trivial that $[X, \Lambda, F]$ is independent of the choice of (46), but this is essentially the fact (3.9) that disc is injective.)

(ii) The second step is to define (U, ρ) from (X, Λ, F), by applying the restriction map 5.1. In particular, U is defined as

$$U = X \setminus C_2,$$

where for each $i \in \{1, 2\}$ we define the irreducible curve $C_i = \gamma(E_i) \subset X$.

We have explained how $\mathfrak{F}$ constructs $[U, \rho]$ from τ. Now some remarks.

First, we describe the divisor Δ of $\tilde{X}$. As noted before, $\mathcal{G}(\Delta, \tilde{X})$ may be identified with $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$. Now the weighted graph $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ has connected components $(\mathcal{G}_{(-m)} \oplus T_1) \oplus T_2$, $\mathcal{G}_{(-m)} \oplus T_1$ and $(\mathcal{G}_{(-m)} \oplus T_1) \oplus T_2$, where the last two may be empty (see [6]-2.10 for $\ominus$, $\oplus$ and $\oplus$). These correspond respectively to the three connected components $\mathcal{C}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\operatorname{supp}(\Delta)$, where $\mathcal{C}$ is the one which contains the proper transform of Σ_m with respect to π, and where $\mathcal{A}_1$ and $\mathcal{A}_2$ are (possibly empty) admissible chains satisfying $\mathcal{A}_i \subset \pi^{-1}(P_i)$. Also, E_i meets $\mathcal{C}$ and if $\mathcal{A}_i \neq \varnothing$ then E_i meets $\mathcal{A}_i$.

In the above, we obtained U from $\tilde{X}$ by first shrinking $\mathcal{A}_1$, $\mathcal{C}$ and $\mathcal{A}_2$ and then removing $C_2 = \gamma(E_2) = \gamma(\mathcal{C} \cup E_2 \cup \mathcal{A}_2)$. Alternatively, *we obtain the same* $[U, \rho]$ *if we shrink only* $\mathcal{A}_1$ *and then remove (the image of)* $\mathcal{C} \cup E_2 \cup \mathcal{A}_2$. More precisely: Starting from $\tilde{X}$, let S be the normal surface obtained by shrinking $\mathcal{A}_1$ to a point and let $\sigma : \tilde{X} \to S$ be the corresponding birational morphism. Then $\sigma(\mathcal{C} \cup E_2 \cup \mathcal{A}_2)$ is the support of an SNC-divisor D of S, and $S \setminus \operatorname{supp}(D) = U$; let $u : U \hookrightarrow S$ denote the inclusion map, then u is a smooth-normal compactification of U. Note that

$$\pi^{-1}(P_2) \subset \pi^{-1}(\Sigma_m \cup Z_2) \subset \mathcal{C} \cup E_2 \cup \mathcal{A}_2, \tag{48}$$

where the right-hand-side of (48) corresponds to the complement of U via σ. It follows that

$$[U, \rho] \textit{ does not depend on } T_2. \tag{49}$$

Indeed, by definition of a blowing-up of $\mathbb{F}_m$ according to (T_1, T_2), T_2 only affects the part of π which could be described as the "blowing-up at points infinitely near P_2". By (48), all that blowing-up occurs in the complement of U and so has no effect on $[U, \rho]$. So (49) is true. Similarly,

$$[U, \rho] \textit{ does not depend on } m. \tag{50}$$

This is because we can change the value of m by performing elementary operations: Blow-up $\mathbb{F}_m$ at a point of Z_2 and then shrink the proper transform of Z_2. By (48), these operations affect only the complement of U, so (50) is true.

Note that (49) and (50) prove the implication "$\Longleftarrow$" of 5.3. We now proceed to show that T_1 is completely determined by $[U,\rho]$. By (49) and (50), $[U,\rho]$ does not change if we replace (m,T_1,T_2) by $(1,T_1,\mathbf{1})$. So, from now-on, let us assume that

$$\tau = (1, T_1, \mathbf{1})$$

(where T_1 is the same as before). First, we go over the definition of ρ, in a slightly different way. We begin the construction of the commutative diagram

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[dd, "\rho"] & S \arrow[dd, "\bar\rho"] & \tilde X \arrow[l, "\sigma"'] \arrow[d, "\pi"] \\ & & \mathbb{F}_1 \arrow[d, "\rho_1"] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 & \mathbb{P}^1 \arrow[l, "\mathrm{id}"'] \end{tikzcd} \tag{51}$$

by choosing a morphism $\rho_1 : \mathbb{F}_1 \to \mathbb{P}^1$ compatible with the linear system Λ_1 (i.e., the fibers of ρ_1 are the elements of Λ_1). Since π shrinks $\operatorname{exc}(\sigma) = \mathcal{A}_1$ to a point, there exists a morphism $\bar\rho : S \to \mathbb{P}^1$ which makes the right hand side of (51) a commutative rectangle. Then simply define ρ to be the restriction of $\bar\rho$. So we see that

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "\rho"'] & S \arrow[d, "\bar\rho"] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd} \quad \textit{is a completion of } (U,\rho). \tag{52}$$

The boundary divisor of this completion is the D which we have already defined (see the paragraph before (48)). Moreover, if $\tilde\Sigma_1 \subset \tilde X$ denotes the proper transform of Σ_1 with respect to π, then $\sigma\big(\tilde\Sigma_1\big)$ is the ρ-horizontal component of D. Let us now argue that we have the following isomorphisms of weighted graphs:

$$\begin{aligned} \mathcal{G}(D,S) \;&\cong\; \mathcal{G}(\mathcal{C}\cup E_2\cup \mathcal{A}_2, \tilde X) \\ &=\; \mathcal{G}(\mathcal{C}+\tilde Z_2, \tilde X) \;\cong\; \text{underlying weighted graph of } \mathcal{G}_{(-1)} \oplus T_1, \end{aligned} \tag{53}$$

and

$$\mathcal{G}_{\mathrm{res}}[U] \;\cong\; \mathcal{G}(\mathcal{A}_1, \tilde X) \;\cong\; \mathcal{G}_{(-1)} \oplus T_1. \tag{54}$$

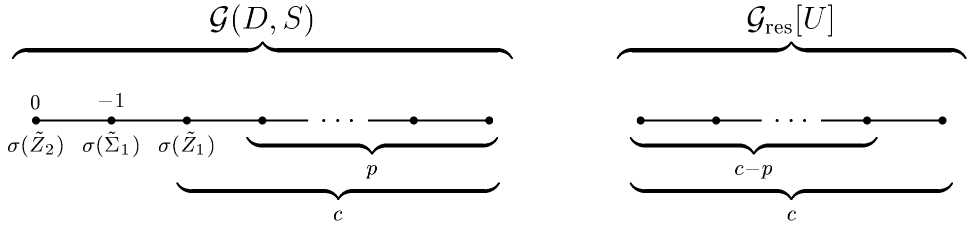
First, $\mathcal{G}(D,S)$ is isomorphic to $\mathcal{G}(\mathcal{C}\cup E_2\cup\mathcal{A}_2,\tilde X)$ because σ restricts to an isomorphism from an open neighborhood of $\mathcal{C}\cup E_2\cup\mathcal{A}_2$ to an open neighborhood of D. Then we have a genuine equality, $\mathcal{G}(\mathcal{C}\cup E_2\cup\mathcal{A}_2,\tilde X) = \mathcal{G}(\mathcal{C}+\tilde Z_2,\tilde X)$, because $T_2 = \mathbf{1}$ implies that $\mathcal{A}_2 = \varnothing$ and that $E_2 = \tilde Z_2$, where $\tilde Z_2$ is the proper transform of Z_2 with respect to π. We have already observed that $\mathcal{G}(\Delta,\tilde X)$ may be identified with $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2 = (\mathcal{G}_{(-m)} \ominus T_1) \ominus \mathbf{1}$; since the effect of "$\ominus\mathbf{1}$" is to remove the distinguished vertex of $\mathcal{G}_{(-m)} \ominus T_1$, it follows that $\mathcal{G}(\Delta+\tilde Z_2,\tilde X)$ may be identified with the underlying weighted graph of $\mathcal{G}_{(-1)} \ominus T_1$; or equivalently, that $\mathcal{G}(\mathcal{C}+\tilde Z_2,\tilde X)$ and $\mathcal{G}(\mathcal{A}_1,\tilde X)$ may be respectively identified with the two graphs:

$$\text{underlying weighted graph of } \mathcal{G}_{(-1)} \textcircled{$\uparrow$}\, T_1 \qquad \text{and} \qquad \mathcal{G}_{(-1)} \textcircled{$\downarrow$}\, T_1.$$

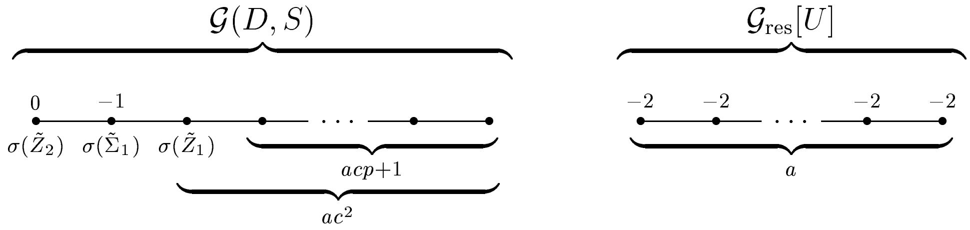
Finally, we may regard $\sigma^{-1}(U) \xrightarrow{\sigma} U$ as the minimal resolution of singularities of U; since $\mathrm{exc}(\sigma) = \mathcal{A}_1$, $\mathcal{G}_{\mathrm{res}}[U]$ may be identified with $\mathcal{G}(\mathcal{A}_1, \tilde{X})$. This explains (53) and (54).

So the description of $\mathcal{G}_{(-1)} \textcircled{$\uparrow$}\, T_1$ and $\mathcal{G}_{(-1)} \textcircled{$\downarrow$}\, T_1$ given in 3.5.1 may be regarded as a description of $\mathcal{G}(D, S)$ and $\mathcal{G}_{\mathrm{res}}[U]$. This gives:

If $T_1 = \binom{p}{c} \in \mathcal{T}_1$, then:

(55) 

If $T_1 = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$, then:

(56) 

In the above pictures, the integers under the braces are the determinants of the indicated linear chains and all weights which are not indicated are strictly less than -1. From (55) and (56), we see that $\sigma\big(\tilde{\Sigma}_1\big)$ is the only (-1)-component of D; since this is the horizontal component, (52) is in fact a *minimal* completion of (U, ρ), so:

$$\mathcal{G}_\infty[U, \rho] = \mathcal{G}(D, S)$$

(see 2.8 for $\mathcal{G}_\infty[U, \rho]$). From (55) and (56), it is obvious that $\mathcal{G}_\infty[U, \rho]$ is a chain of type (Z) with bideterminant (see 2.16):

$$\text{bidet}\, \mathcal{G}_\infty[U, \rho] = \begin{cases} (c, p), & \text{if } T_1 = \binom{p}{c} \in \mathcal{T}_1; \\ (ac^2, acp + 1), & \text{if } T_1 = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2. \end{cases} \tag{57}$$

It also follows that $\mathcal{G}_{\mathrm{res}}[U]$ is an admissible chain with determinant:

$$\det \mathcal{G}_{\mathrm{res}}[U] = \begin{cases} c, & \text{if } T_1 = \binom{p}{c} \in \mathcal{T}_1; \\ a, & \text{if } T_1 = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2. \end{cases} \tag{58}$$

Observe that the first component of $\text{bidet}\, \mathcal{G}_\infty[U, \rho]$ is equal to $\det \mathcal{G}_{\mathrm{res}}[U]$ if and only if $T_1 \in \mathcal{T}_1$. Thus, in view of (57) and (58), it is clear that T_1 can be recovered from $\mathcal{G}_\infty[U, \rho]$ and $\mathcal{G}_{\mathrm{res}}[U]$. So T_1 is completely determined by $[U, \rho]$, which proves the implication " $\Longrightarrow$ " of 5.3. $\square$

The disjoint union of the weighted graphs $\mathcal{G}_\infty[U,\rho]$ and $\mathcal{G}_{\text{res}}[U]$ occurred naturally in the above proof.

5.3.1. **Definition.** Given $(U,\rho) \in \mathcal{N}^+$, define the weighted graph

$$\mathcal{G}[U,\rho] = \mathcal{G}_\infty[U,\rho] \uplus \mathcal{G}_{\text{res}}[U]$$

where $\uplus$ means "disjoint union of weighted graphs", and where the weighted graphs $\mathcal{G}_{\text{res}}[U]$ and $\mathcal{G}_\infty[U,\rho]$ are defined in 2.1 and 2.8 respectively. Note that $\mathcal{G}[U,\rho]$ is uniquely determined by $[U,\rho]$.

For future use, we record the geometric interpretation of the weighted pair $\mathcal{G}_{(-1)} \ominus T$.

5.3.2. **Lemma.** *Let $T \in \mathcal{T}_*$ and let $[U,\rho] = \mathfrak{F}(1,T,\mathbf{1})$. Then $\mathcal{G}[U,\rho]$ has a unique vertex L of weight zero and*

$$\mathcal{G}_{(-1)} \ominus T = \big(\mathcal{G}[U,\rho], L\big).$$

More precisely, L belongs to the subgraph $\mathcal{G}_\infty[U,\rho]$ of $\mathcal{G}[U,\rho]$ and we have:

$$\mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} T = \big(\mathcal{G}_\infty[U,\rho], L\big) \quad \text{and} \quad \mathcal{G}_{(-1)} \mathbin{\textcircled{$\downarrow$}} T = \mathcal{G}_{\text{res}}[U].$$

Proof. Let $T_1 = T$ and note that $\tau = (1, T_1, \mathbf{1})$ belongs to $\mathbb{T}_0(\ddagger)$. Starting with this τ, go through the proof of 5.3. As noted in that proof, the underlying weighted graph of $\mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} T_1$ is isomorphic to $\mathcal{G}(D,S) = \mathcal{G}_\infty[U,\rho]$; comparing (23) with (55) and (24) with (56), we see that this isomorphism maps the distinguished vertex of $\mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} T_1$ to $\sigma(\tilde{Z}_2)$. So:

$$\mathcal{G}_{(-1)} \mathbin{\textcircled{$\uparrow$}} T_1 = \big(\mathcal{G}(D,S), \sigma(\tilde{Z}_2)\big) = \big(\mathcal{G}_\infty[U,\rho], \sigma(\tilde{Z}_2)\big).$$

It was also noted that $\mathcal{G}_{(-1)} \mathbin{\textcircled{$\downarrow$}} T_1 = \mathcal{G}_{\text{res}}[U]$, so

$$\mathcal{G}_{(-1)} \ominus T_1 = \big(\mathcal{G}_\infty[U,\rho] \uplus \mathcal{G}_{\text{res}}[U], \sigma(\tilde{Z}_2)\big) = \big(\mathcal{G}[U,\rho], \sigma(\tilde{Z}_2)\big).$$

Looking at (55) and (56), we see that $\sigma(\tilde{Z}_2)$ is the unique vertex of weight zero in $\mathcal{G}[U,\rho] = \mathcal{G}(D,S) \uplus \mathcal{G}_{\text{res}}[U]$; so $\sigma(\tilde{Z}_2)$ is the L of the statement and we are done. $\square$

5.4. **Proposition.** *Let $[U,\rho]$ be an element of the image of the map $\mathfrak{F}$. Then $U \in \mathcal{M}_0$ and U is an open subset of a weighted projective plane.*

Proof. We have $[U,\rho] = \mathfrak{F}(\tau')$ for some $\tau' = (m, T_1, T_2') \in \mathbb{T}_0(\ddagger)$. Then $T_1 \in \mathcal{T}_*$ and, by 4.4, there exist a weighted projective plane $\mathbb{P}$ and a tableau $T_2 \in \mathcal{T}_1$ such that

$$\big(1, T_1, T_2\big) \in \mathbb{T}_0(\mathbb{P}). \tag{59}$$

Write $\tau = (1, T_1, T_2)$, then $\mathfrak{F}(\tau) = [U,\rho]$ by 5.3. By (59) and the definition (3.10) of $\mathbb{T}_0(X)$, there exists a basic affine ruling Λ of $\mathbb{P}$ and an element F of Λ_* such that $\text{disc}(\mathbb{P},\Lambda,F) = \tau$. Recalling that $\mathfrak{F}$ is the composite

$$\mathbb{T}_0(\ddagger) \xrightarrow{\text{disc}^{-1}} \mathbb{S}_0(\ddagger) \xrightarrow{\text{res}} \overline{\mathcal{N}}^+,$$

we get $[U,\rho] = \mathfrak{F}(\tau) = \text{res}[\mathbb{P},\Lambda,F]$ and in particular $U \cong \mathbb{P} \setminus \text{supp}(F)$. Then U is the complement of a curve in $\mathbb{P}$, so U is affine. More precisely, $U = \mathbb{P} \setminus C_2$ (notation as in the proof of 5.3); since C_2 is irreducible and, by [6]-1.7, $\text{Pic}(\mathbb{P}_s) = \mathbb{Z}$ (where $\mathbb{P}_s$ is the smooth locus of $\mathbb{P}$), we have $\text{Pic}(U_s) = \mathbb{Z}/d\mathbb{Z}$ where $d = \deg(C_2)$, so $\text{Pic}(U_s)$ is a finite group. (*Remark:* We have $d = \mu(T_1)$ by the remark following [5]-5.37. See 3.2 for the definition of μ.)

The fact that $\mathcal{G}_\infty[U, \rho]$ is a chain of type (Z) was mentioned in the proof of 5.3, see in particular (57). Thus U satisfies conditions (a) and (b) of 2.20 and consequently $\mathrm{ML}(U) = \mathbf{k}$. Hence, $U \in \mathcal{M}_0$. □

6. The map $\mathfrak{f}$

6.1. **Definition.** If $T \in \mathcal{T}_*$ then it is clear that $(1, T, \mathbf{1}) \in \mathbb{T}_0(\ddagger)$ and 5.4 shows (in particular) that $\mathfrak{F}(1, T, \mathbf{1}) \in \overline{\mathcal{M}}_0^+$. So the following is a well-defined map:

$$\begin{array}{rcl} \mathfrak{f} : \mathcal{T}_* & \longrightarrow & \overline{\mathcal{M}}_0^+ \\ T & \longmapsto & \mathfrak{F}(1, T, \mathbf{1}). \end{array}$$

By 5.3, $\mathfrak{f}$ is injective and has the same image as $\mathfrak{F}$. We will show that $\mathfrak{f}$ is bijective.

Note that if U is any member of $\mathcal{M}$ then there exists a morphism $\rho : U \to \mathbb{A}^1$ whose general fiber is an $\mathbb{A}^1$, so the following paragraph applies to U.

6.2. Consider a pair (U, ρ) where U is a normal affine surface and $\rho : U \to \mathbb{A}^1$ is a morphism whose general fiber is an $\mathbb{A}^1$.

By 2.7, we may choose a minimal completion (3) of (U, ρ) whose horizontal component H satisfies $H^2 = -1$; let D be the boundary divisor of that minimal completion. Let $\sigma : \tilde{X} \to S$ be the minimal resolution of singularities of S and consider the commutative diagram (ignoring π for now):

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "\rho"'] & S \arrow[d, "\bar{\rho}"] & \tilde{X} \arrow[l, "\sigma"'] \arrow[r, "\pi"] & \mathbb{F}_1 \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 & & \end{tikzcd} \tag{60}$$

Let $\tilde{D}$ be the SNC-divisor of $\tilde{X}$ which satisfies $\mathrm{supp}\big(\tilde{D}\big) = \sigma^{-1}\big(\mathrm{supp}(D)\big)$. Note that, since $\mathrm{Sing}(S) = \mathrm{Sing}(U)$, σ restricts to an isomorphism going from an open neighborhood of $\tilde{D}$ to an open neighborhood of D. Let $\tilde{H} = \sigma^{-1}(H)$, then $\tilde{H}^2 = H^2 = -1$.

Let $\tilde{\rho} : \tilde{X} \to \mathbb{P}^1$ denote the composition $\tilde{X} \xrightarrow{\sigma} S \xrightarrow{\bar{\rho}} \mathbb{P}^1$. Since $\tilde{\rho}$ is a morphism whose general fiber is a $\mathbb{P}^1$, and since $\tilde{X}$ is smooth, rational and complete, we know that each reducible fiber of $\tilde{\rho}$ is a tree of projective lines which can be contracted to an irreducible curve, and that there exists such a contraction process which yields the surface $\mathbb{F}_1$. More precisely, let us regard fibers of $\tilde{\rho}$ as divisors, i.e., we consider scheme-theoretic fibers. Then each fiber F of $\tilde{\rho}$ satisfies $F \cdot \tilde{H} = 1$. Consequently, F has a unique component F° which meets $\tilde{H}$, and F° has multiplicity one in the divisor F. So we can write $F = F^\circ + F'$, where the support of F' is disjoint from $\tilde{H}$. Let $F_1, \dots, F_r$ be the reducible fibers of $\tilde{\rho}$ and write $F_i = F_i^\circ + F_i'$ for every i. Then there exists a birational morphism $\pi : \tilde{X} \to \mathbb{F}_1$ whose exceptional locus is $\mathrm{supp}(F_1' + \cdots + F_r')$. This completes the definition of diagram (60).

For each $i \in \{1, \dots, r\}$, write $F_i = \mathcal{D}_i + \mathcal{A}_i + \Gamma_i$, where $\mathcal{D}_i$ (resp. $\mathcal{A}_i$) contains all components of F_i which are part of $\tilde{D}$ (resp. which are part of $\mathrm{exc}(\sigma)$), and where Γ_i contains all other components. Note that if C is a component of $\mathrm{exc}(\sigma)$ then $\tilde{\rho}$ shrinks C to a point; since C must be disjoint from $\tilde{D}$ (because $\mathrm{Sing}\, S = \mathrm{Sing}\, U$), we have $C \cap \tilde{H} = \varnothing$, so $C \subseteq \mathrm{supp}(F_i')$ for some i and consequently:

$$\mathrm{exc}(\sigma) = \mathrm{supp}(\mathcal{A}_1 + \cdots + \mathcal{A}_r) \subseteq \mathrm{exc}(\pi).$$

From $\mathrm{exc}(\sigma) \subseteq \mathrm{exc}(\pi)$, we infer that if some component C of $\mathrm{exc}(\sigma)$ satisfies $C^2 \geq -1$, then $C^2 = -1$ and C is not a branching component of $\mathrm{exc}(\sigma)$; this is impossible, because we chose σ to be a *minimal* resolution of singularities. So:

(61) *Each component C of* $\mathrm{exc}(\sigma)$ *satisfies* $C^2 < -1$.

We claim:

(62) *For each* $i \in \{1, \dots, r\}$, $\mathrm{supp}(\Gamma_i)$ *contains all* (-1)*-components of* F_i.

Indeed, suppose that C is a component of F_i satisfying $C^2 = -1$. Then C cannot be part of $\mathcal{D}_i$, because the square in (60) is a *minimal* completion of (U, ρ); and it cannot be part of $\mathcal{A}_i$, because of (61). So C must be in Γ_i, which proves (62). We claim:

(63) *For each* $i \in \{1, \dots, r\}$, *we have* $\Gamma_i \neq 0$ *and* $\mathcal{D}_i \cdot \tilde{H} = 1$. *Moreover, each component C of* Γ_i *satisfies* $C \cap \tilde{H} = \varnothing$ *and* $C \cap \mathrm{supp}(\mathcal{D}_i) \neq \varnothing$.

Indeed, it is known that some component C_* of F_i' satisfies $C_*^2 = -1$; by (62), C_* is a component of Γ_i, so $\Gamma_i \neq 0$. If C is any component of Γ_i then $\sigma(C)$ is a complete curve, and hence is not contained in the affine surface U; thus C meets $\tilde{H} \cup \mathrm{supp}(\mathcal{D}_i)$. Applying this to $C = C_*$ gives $C_* \cap \mathrm{supp}(\mathcal{D}_i) \neq \varnothing$, because $C_* \cap \tilde{H} = \varnothing$. In particular, $\mathcal{D}_i \neq 0$. Since U is affine, $\tilde{D}$ is connected, so $\tilde{H} \cup \mathrm{supp}(\mathcal{D}_i)$ is connected and $\mathcal{D}_i \cdot \tilde{H} > 0$. Together with $F_i \cdot \tilde{H} = 1$, this implies that $\mathcal{D}_i \cdot \tilde{H} = 1$ and that no component of Γ_i meets $\tilde{H}$, which proves (63).

Consider the fibers $G_1, \dots, G_s$ of $\tilde{\rho}$ which are entirely contained in $\tilde{D}$. Note that each G_i is an irreducible curve such that $G_i^2 = 0$ (otherwise G_i would have a (-1)-component, which is impossible because the square in (60) is a minimal completion of (U, ρ)). Also, we have $s \geq 1$ because the composite $U \xrightarrow{\rho} \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ is of course not surjective. Next we show:

(64) *Every branching component of* $\mathrm{supp}\left(\tilde{H} + \sum_{i=1}^r F_i + \sum_{i=1}^s G_i\right)$ *is in fact a component of* $\tilde{D}$.

To simplify language and notation, let us identify $\tilde{H} + \sum_{i=1}^r F_i + \sum_{i=1}^s G_i$ with the corresponding dual graph. By contradiction, suppose that C is a branch point of this graph and is not a vertex of $\tilde{D}$. For some i, C is a component of F_i; then C is a branch point of $G_1 + \tilde{H} + F_i$. Since $G_1 + \tilde{H} + F_i$ shrinks to a linear chain, some branch $\mathcal{B}$ of $G_1 + \tilde{H} + F_i$ at C shrinks to the empty graph; since $\mathcal{B}$ cannot contain the vertex G_1 of weight zero, $\mathcal{B}$ is included in F_i. We may consider a vertex C_* of $\mathcal{B}$ of weight -1; by (62) and (63), some vertex of $\mathcal{D}_i$ is adjacent to C_*, so $\mathcal{B}$ contains a vertex of $\mathcal{D}_i$; since

$G_1 + \tilde{H} + \mathcal{D}_i$ is a connected graph which does not contain C, it follows that $\mathcal{B}$ contains the vertex G_1, a contradiction. This proves (64). Next, we have:

(65) *Let $i \in \{1, \dots, r\}$ and let $\mathcal{C}$ be a connected component of $\Gamma_i + \mathcal{A}_i$. Then $\mathcal{C}$ is a linear chain and exactly one component C of $\mathcal{C}$ is a component of Γ_i. Moreover, C is a "terminal vertex" of the linear chain $\mathcal{C}$.*

The fact that $\mathcal{C}$ is a linear chain follows immediately from (64). Since F_i is connected but $\mathcal{A}_i$ is disjoint from $\mathcal{D}_i$, it follows that at least one vertex C of $\mathcal{C}$ is a vertex of Γ_i. If C is not unique then, by (63), the connected set $\mathcal{C}$ meets the connected set $\tilde{D}$ in at least two points, which is impossible because $\tilde{D} + \mathcal{C}$ is contained in a tree and hence has no loops. By (64), C is not a branch point of the graph $\mathcal{D}_i + C + \mathcal{A}_i$; however, some vertex of $\mathcal{D}_i$ is adjacent to C by (63); so at most one vertex of $\mathcal{A}_i$ is adjacent to C, which proves (65). From this and (61), it follows:

(66) *Each connected component of* $\mathrm{exc}(\sigma)$ *is an admissible chain. Thus every singular point of U is a cyclic quotient singularity.*

Write $\Gamma_i = \sum_{j=1}^{\gamma_i} c_{ij} C_{ij}$, where $C_{i1}, \dots, C_{i\gamma_i}$ are the distinct components of Γ_i and where the c_{ij} are positive integers. Let H_i denote the free abelian group generated by the symbols $C_{i1}, \dots, C_{i\gamma_i}$ modulo the equation $\sum_{j=1}^{\gamma_i} c_{ij} C_{ij} = 0$. By a well-known argument,

$$\mathrm{Pic}(U_s) = \bigoplus_{i=1}^{r} H_i. \tag{67}$$

In fact, let G denote a fiber of $\tilde{\rho}$. Since $\mathrm{Pic}(\mathbb{F}_1)$ is the free abelian group generated by a fiber and the directrix, $\mathrm{Pic}(\tilde{X})$ is the free abelian group generated by G, $\tilde{H}$ and all components of $\mathrm{exc}(\pi)$. Now $\mathrm{Pic}(U_s)$ is $\mathrm{Pic}(\tilde{X})$ modulo the subgroup generated by the components of $\tilde{D}$ and those of $\mathrm{exc}(\sigma) = \mathrm{supp}(\mathcal{A}_1 + \dots + \mathcal{A}_r)$; this subgroup contains the linear equivalence class of G, because $\tilde{D}$ contains the fiber G_1. Also note that, by (63), we have $\mathrm{supp}(\Gamma_i) \subseteq \mathrm{exc}(\pi)$ for every $i \in \{1, \dots, r\}$. These remarks and a straightforward calculation give (67).

6.2.1. *If $U \in \mathcal{M}$ then $\tilde{D}$ is a linear chain, $s = 1$ and $r \leq 1$.*

Proof. The fact that $\tilde{D}$ is a linear chain follows from 2.19, which also implies that $s \leq 1$, hence $s = 1$. The number of vertices of (the dual graph of) $\tilde{D}$ which are adjacent to $\tilde{H}$ is at most two, and is precisely $r + s$ (because $\mathcal{D}_i \cdot \tilde{H} = 1$ for all i, by (63)), so $r \leq 1$. □

6.2.2. *If $\mathrm{Pic}(U_s)$ is a finite group then:*

1. *For each $i \in \{1, \dots, r\}$, $\mathrm{supp}(\Gamma_i)$ is an irreducible curve (i.e., $\gamma_i = 1$).*
2. *Each connected component of $\mathrm{exc}(\pi)$ has exactly one (-1)-component.*
3. $\mathrm{exc}(\pi)$ *has r connected components, namely: F_1', ..., F_r'.*
4. *For each $i \in \{1, \dots, r\}$, $\mathcal{A}_i$ is connected (and hence is an admissible chain).*
5. *No two singular points of U belong to the same fiber of ρ.*

Proof. Assertion (1) follows immediately from (67). Consider a connected component $\mathcal{C}$ of $\mathrm{exc}(\pi)$. Since $\mathrm{exc}(\pi) = \mathrm{supp}(F'_1 + \cdots + F'_r)$, and since distinct fibers of $\tilde{\rho}$ are disjoint, there exists i such that $\mathcal{C} \subseteq F'_i$. Now at least one component C of $\mathcal{C}$ satisfies $C^2 = -1$; by (62) we have $C \subseteq \mathrm{supp}(\Gamma_i)$, so in fact $C = \mathrm{supp}(\Gamma_i)$ by assertion (1). So $\mathcal{C}$ has only one (-1)-component, which proves assertion (2); and if $\mathcal{C}$ and $\mathcal{C}'$ are connected components of $\mathrm{exc}(\pi)$ such that $\mathcal{C} \cup \mathcal{C}' \subseteq F'_i$, then $\mathrm{supp}(\Gamma_i) \subseteq \mathcal{C} \cap \mathcal{C}'$, so $\mathcal{C} = \mathcal{C}'$ and assertion (3) is true. Assertion (4) follows from (65) and the fact that $\mathrm{supp}(\Gamma_i)$ is irreducible. Assertion (5) follows from (4). $\square$

6.2.3. *If $U \in \mathcal{M}_0$ (and ρ is as before), there exists $T \in \mathcal{T}_*$ such that $\mathfrak{f}(T) = [U, \rho]$.*

Proof. Consider the ruling Λ_1 of $\mathbb{F}_1$ and its directrix Σ_1 (and note that $\Sigma_1 = \pi(\tilde{H})$). Define distinct elements Z_1, Z_2 of Λ_1 and points $P_i \in Z_i \setminus \Sigma_1$ $(i = 1, 2)$ as follows:

- Set $Z_2 = \pi(G_1)$ and pick any point P_2 of $Z_2 \setminus \Sigma_1$ (the curves $G_1, \dots, G_s$ are defined before (64) and we have $s = 1$ by 6.2.1).
- If π is not an isomorphism then $r = 1$ by 6.2.1, so we may define Z_1 and P_1 by $Z_1 = \pi(F_1)$ and $\mathrm{cent}(\pi) = \{P_1\}$; if π is an isomorphism, let Z_1 be any element of Λ_1 other than Z_2 and let P_1 be any point of $Z_1 \setminus \Sigma_1$.

Then:

$$\mathrm{cent}(\pi) \subseteq \{P_1\} \textit{ and, for each } i \in \{1,2\},\ \pi^{-1}(P_i) \textit{ has at most one } (-1)\textit{-component.}$$

So, by [5]-4.1, the tableau $T_i = \overline{\mathrm{HN}}(\pi, Z_i) \in \mathcal{T}$ is defined for each $i \in \{1, 2\}$. Note that $T_2 = \mathbf{1}$ (the empty tableau), since $\mathrm{cent}(\pi) \cap Z_2 = \varnothing$. By definition [5]-5.26, it is obvious that

$$\left(\tilde{X} \xrightarrow{\pi} \mathbb{F}_1, P_1, P_2\right) \textit{ is a blowing-up of } \mathbb{F}_1 \textit{ with respect to } (T_1, T_2). \tag{68}$$

We now argue that $T_1 \in \mathcal{T}_*$. It is a general fact that whenever (68) is true then so is

$$\mathcal{G}(\Delta, \tilde{X}) = (\mathcal{G}_{(-1)} \ominus T_1) \ominus T_2, \tag{69}$$

where Δ is the SNC-divisor of $\tilde{X}$ defined exactly as in (47), in the proof of 5.3. In fact the definition of Δ implies:

$$\mathrm{supp}(\Delta) = \begin{cases} \mathrm{supp}(\tilde{H} + \mathcal{D}_1) \cup \mathrm{supp}(\mathcal{A}_1), & \text{if } \pi \text{ is not an isomorphism,} \\ \tilde{H}, & \text{if } \pi \text{ is an isomorphism} \end{cases}$$

and recall that $\mathcal{A}_1$ is an admissible chain by 6.2.2. So in all cases we have $\mathrm{supp}(\Delta) = \mathcal{C} \cup \mathcal{A}$, where $\mathcal{C}$ and $\mathcal{A}$ are disjoint, $\mathcal{C} \subseteq \mathrm{supp}(\tilde{D})$ and $\mathcal{A}$ is a (possibly empty) admissible chain. Since $\tilde{D}$ is a linear chain by 6.2.1, $\mathcal{C}$ is a linear chain and Δ satisfies: *Every connected component of Δ is a linear chain and every (irreducible) component C of Δ other than $\tilde{H}$ satisfies $C^2 < -1$.* By (69), this translates into

> $(\mathcal{G}_{(-1)} \ominus T_1) \ominus T_2$ *has no branch point and every weight in it, except that of the middle vertex of $\mathcal{G}_{(-1)}$, is strictly less than -1.*

Then part (2) of result [5]-5.7 implies that

$$T_1 \in \mathcal{T}_*.$$

Let $\tau = (1, T_1, T_2) = (1, T_1, \mathbf{1})$ and note that $\tau \in \mathbb{T}_0(\ddagger)$. Then we claim that $\mathfrak{F}(\tau) = [U, \rho]$. Indeed, this is clear from consideration of the beginning of the proof of 5.3, up to (48) (the notation is compatible). Thus $\mathfrak{f}(T_1) = [U, \rho]$. $\square$

6.3. **Theorem.** *The map $\mathfrak{f} : \mathcal{T}_* \to \overline{\mathcal{M}}_0^+$ is bijective.*

Proof. $\mathfrak{f}$ is injective by 5.3 and surjective by 6.2.3. $\square$

6.4. **Corollary.** *The three maps*

$$\mathrm{res} : \mathbb{S}_0(\ddagger) \longrightarrow \overline{\mathcal{N}}^+, \qquad \mathfrak{F} : \mathbb{T}_0(\ddagger) \longrightarrow \overline{\mathcal{N}}^+ \quad \textit{and} \quad \mathfrak{f} : \mathcal{T}_* \longrightarrow \overline{\mathcal{M}}_0^+$$

have the same image, namely, $\overline{\mathcal{M}}_0^+$.

Proof. Immediate. $\square$

6.5. **Theorem.** *Let U and U' be surfaces belonging to the class $\mathcal{M}_0$. If $\mathcal{G}_\infty[U] = \mathcal{G}_\infty[U']$ and $\mathcal{G}_{\mathrm{res}}[U] = \mathcal{G}_{\mathrm{res}}[U']$ then $U \cong U'$.*

Proof. By 2.20 and 2.17, we may pick an element of $\mathcal{G}_\infty[U]$ of the form

$$\overset{0}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\omega_1}{\bullet}\!\!-\!\!\!-\ \cdots\ -\!\!\!-\!\!\overset{\omega_q}{\bullet} \tag{70}$$

where $q \geq 0$ and $\forall_i\ \omega_i \leq -2$. Consequently, we can find a smooth-normal compactification $u : U \hookrightarrow S$ of U such that the weighted graph $\mathcal{G}(u)$ is identical to (70). Let L be the unique 0-component of $S \setminus U$. Then the complete linear system $|L|$ on S is one-dimensional, free of base points and has the following property:

The general member $G \in |L|$ satisfies $G \cong \mathbb{P}^1$ and $G \cap U = \mathbb{A}^1$.

We may consider a morphism $\bar{\rho} : S \to \mathbb{P}^1$ whose fibers are the elements of $|L|$. Restricting $\bar{\rho}$ to U gives a morphism $\rho : U \to \mathbb{A}^1$ whose general fiber is an $\mathbb{A}^1$, i.e., $(U, \rho) \in \mathcal{M}_0^+$. Then the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & S \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\rho}} \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

is a minimal completion of (U, ρ) and consequently the weighted graph $\mathcal{G}_\infty[U, \rho]$ (defined in 2.8) is (70).

Since $\mathcal{G}_\infty[U] = \mathcal{G}_\infty[U']$, the weighted graph (70) is also an element of $\mathcal{G}_\infty[U']$; so the above argument shows that there exists a morphism $\rho' : U' \to \mathbb{A}^1$ such that $(U', \rho') \in \mathcal{M}_0^+$ and such that $\mathcal{G}_\infty[U', \rho']$ is (70).

Since $\mathfrak{f}$ is bijective, we may consider $T, T' \in \mathcal{T}_*$ such that $f(T) = [U, \rho]$ and $f(T') = [U', \rho']$; then by 5.3.2

$$\mathcal{G}_{(-1)} \ominus T = \big(\mathcal{G}[U, \rho],\ L\big) \qquad \text{and} \qquad \mathcal{G}_{(-1)} \ominus T' = \big(\mathcal{G}[U', \rho'],\ L'\big)$$

where L (resp. L') is the unique vertex of weight zero in $\mathcal{G}[U,\rho]$ (resp. in $\mathcal{G}[U',\rho']$). Since both $\mathcal{G}_\infty[U,\rho]$ and $\mathcal{G}_\infty[U',\rho']$ are identical to (70), and since $\mathcal{G}_{\text{res}}[U] = \mathcal{G}_{\text{res}}[U']$ by assumption, it follows that $\mathcal{G}[U,\rho]$ is identical to $\mathcal{G}[U',\rho']$. So

$$\mathcal{G}_{(-1)} \ominus T = \mathcal{G}_{(-1)} \ominus T'.$$

By [5]-5.14, it follows that $T = T'$. Thus $[U,\rho] = \mathfrak{f}(T) = \mathfrak{f}(T') = [U',\rho']$ and in particular $U \cong U'$. $\square$

The map $\bar{\mathfrak{f}}$

6.6. **Definition.** Let $\bar{\mathfrak{f}} : \mathcal{T}_* \to \overline{\mathcal{M}}_0$ be the composition

$$\mathcal{T}_* \xrightarrow{\mathfrak{f}} \overline{\mathcal{M}}_0^+ \xrightarrow{\text{proj}} \overline{\mathcal{M}}_0$$

where the projection map $\overline{\mathcal{M}}_0^+ \xrightarrow{\text{proj}} \overline{\mathcal{M}}_0$ was defined in 2.2. Note that $\bar{\mathfrak{f}}$ is surjective.

6.7. **Theorem.** *The inverse image by $\bar{\mathfrak{f}}$ of any element of $\overline{\mathcal{M}}_0$ is $\{T, \check{T}\}$ for some $T \in \mathcal{T}_*$.*

Proof. We first show that each $T \in \mathcal{T}_*$ satisfies $\bar{\mathfrak{f}}(T) = \bar{\mathfrak{f}}(\check{T})$. Let $[U,\rho] = \mathfrak{f}(T)$ and $[U',\rho'] = \mathfrak{f}(\check{T})$. By 5.3.2, $\mathcal{G}_{(-1)} \ominus T = \big(\mathcal{G}[U,\rho],\, L\big)$ and $\mathcal{G}_{(-1)} \ominus \check{T} = \big(\mathcal{G}[U',\rho'],\, L'\big)$, where L (resp. L') is the unique vertex of weight zero in $\mathcal{G}[U,\rho]$ (resp. in $\mathcal{G}[U',\rho']$). Using the notation of [5]-3.36 and result [5]-5.14,

$$\big(\mathcal{G}[U',\rho'],\, L'\big) = \mathcal{G}_{(-1)} \ominus \check{T} = \big(\mathcal{G}_{(-1)} \ominus T\big)^t = \big(\mathcal{G}[U,\rho],\, L\big)^t.$$

Whenever $\mathcal{P}$ and $\mathcal{P}'$ are weighted pairs satisfying $\mathcal{P}' = \mathcal{P}^t$, the underlying weighted graphs of $\mathcal{P}$ and $\mathcal{P}'$ are equivalent. Thus $\mathcal{G}[U,\rho]$ and $\mathcal{G}[U',\rho']$ are equivalent weighted graphs; it easily follows that $\mathcal{G}_\infty[U,\rho] \sim \mathcal{G}_\infty[U',\rho']$ and $\mathcal{G}_{\text{res}}[U] \sim \mathcal{G}_{\text{res}}[U']$. Since $\mathcal{G}_\infty[U,\rho] \in \mathcal{G}_\infty[U]$ and $\mathcal{G}_\infty[U',\rho'] \in \mathcal{G}_\infty[U']$, we get $\mathcal{G}_\infty[U] = \mathcal{G}_\infty[U']$; since $\mathcal{G}_{\text{res}}[U]$ and $\mathcal{G}_{\text{res}}[U']$ are equivalent admissible chains, they are identical. Then 6.5 gives $U \cong U'$, hence $\bar{\mathfrak{f}}(T) = [U] = [U'] = \bar{\mathfrak{f}}(\check{T})$.

Conversely, consider $T, T' \in \mathcal{T}_*$ such that $\bar{\mathfrak{f}}(T) = \bar{\mathfrak{f}}(T')$; we show that $T' \in \{T, \check{T}\}$. Let $[U,\rho] = \mathfrak{f}(T)$ and $[U,\rho'] = \mathfrak{f}(T')$ (with the same U). By 5.3.2, $\mathcal{G}_{(-1)} \ominus T = \big(\mathcal{G}[U,\rho],\, L\big)$ and $\mathcal{G}_{(-1)} \ominus T' = \big(\mathcal{G}[U,\rho'],\, L'\big)$, where L (resp. L') is the unique vertex of weight zero in $\mathcal{G}[U,\rho]$ (resp. in $\mathcal{G}[U,\rho']$). By 2.19, the connected component $\mathcal{G}_\infty[U,\rho]$ of $\mathcal{G}[U,\rho]$ has the form

(71)
$$\overset{0}{\bullet}\!-\!\!\!-\!\overset{-1}{\bullet}\!-\!\!\!-\!\overset{\omega_1}{\bullet}\!-\!\!\!-\ \cdots\ -\!\!\!-\!\overset{\omega_q}{\bullet}$$

where $q \geq 0$ and (for all i) $\omega_i \leq -2$. Since $\mathcal{G}_\infty[U,\rho]$ and $\mathcal{G}_\infty[U,\rho']$ belong to $\mathcal{G}_\infty[U]$, they are equivalent and 2.17 implies that $\mathcal{G}_\infty[U,\rho']$ is either (71) or

(72)
$$\overset{0}{\bullet}\!-\!\!\!-\!\overset{-1}{\bullet}\!-\!\!\!-\!\overset{\omega_q}{\bullet}\!-\!\!\!-\ \cdots\ -\!\!\!-\!\overset{\omega_1}{\bullet}$$

Consequently, $\big(\mathcal{G}[U,\rho'],\, L'\big)$ is either $\big(\mathcal{G}[U,\rho],\, L\big)$ or $\big(\mathcal{G}[U,\rho],\, L\big)^t$. In the first case, we have

$$\mathcal{G}_{(-1)} \ominus T' = \big(\mathcal{G}[U,\rho'],\, L'\big) = \big(\mathcal{G}[U,\rho],\, L\big) = \mathcal{G}_{(-1)} \ominus T,$$

and [5]-5.14 gives $T' = T$; in the second case,

$$\mathcal{G}_{(-1)} \ominus T' = \big(\mathcal{G}[U, \rho'],\, L'\big) = \big(\mathcal{G}[U, \rho],\, L\big)^t = \big(\mathcal{G}_{(-1)} \ominus T\big)^t = \mathcal{G}_{(-1)} \ominus \check{T},$$

and [5]-5.14 gives $T' = \check{T}$. So $T' \in \{T, \check{T}\}$. □

6.8. **Corollary.** *Let $T \in \mathcal{T}_*$, let $[U] = \bar{\mathfrak{f}}(T)$ and write $U = \operatorname{Spec} R$. Consider the action of the group $\operatorname{Aut}_{\mathbf{k}}(R)$ on the set* KLND(R) *(see the introduction). Then:*

1. *The number of orbits is either one or two.*
2. *The action is transitive if and only if $T = \check{T}$.*

Proof. Consider the inverse image $E \subset \overline{\mathcal{M}}_0^+$ of $[U] \in \overline{\mathcal{M}}_0$ by the projection $\overline{\mathcal{M}}_0^+ \to \overline{\mathcal{M}}_0$. Then by definition 6.6 of $\bar{\mathfrak{f}}$ we have $\mathfrak{f}^{-1}(E) = \bar{\mathfrak{f}}^{-1}\big([U]\big)$, which is equal to $\{T, \check{T}\}$ by 6.7. So, by 6.3, $\mathfrak{f}$ restricts to a bijection $\{T, \check{T}\} \to E$. The result follows from 2.4, which gives a bijection between E and the set of orbits. □

The following gathers some useful facts concerning the surjection $\bar{\mathfrak{f}} : \mathcal{T}_* \to \overline{\mathcal{M}}_0$.

6.9. **Corollary.** *Let $T \in \mathcal{T}_*$ and let $[U] = \bar{\mathfrak{f}}(T)$.*

1. *The Picard group of $U_s = U \setminus \operatorname{Sing}(U)$ is $\mathbb{Z}/\mu(T)\mathbb{Z}$.*
2. *$\mathcal{G}_{\text{res}}[U] = \mathcal{G}_{(-1)} \oplus T$*
3. *The underlying weighted graph of the weighted pair $\mathcal{G}_{(-1)} \oplus T$ is an element of $\mathcal{G}_\infty[U]$.*
4. *The surface U is smooth if and only if $T = \mathbf{1}$ or $T = \left(\begin{smallmatrix} p & 1 \\ c & 1 \end{smallmatrix}\right)$ where $c > p \geq 1$ and $\gcd(p, c) = 1$.*
5. *U is symmetric at infinity if and only if $\check{T} = T$ (symmetry at infinity is defined in the introduction).*
6. *If $T = \mathbf{1}$, then $U = \mathbb{A}^2$; if $T = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right)$, then U is $\mathbb{P}^2$ minus a conic. These are the only members of $\mathcal{M}_0$ which are smooth and symmetric at infinity.*

Proof. The point of this result is to have a statement which does not involve a morphism $\rho : U \to \mathbb{A}^1$. However, the proof consists in applying earlier results to $[U, \rho] = \mathfrak{f}(T)$.

Assertion (1) was noted in the proof of 5.4.

Assertions (2) and (3) reiterate 5.3.2.

By assertion (2), U is smooth iff $\mathcal{G}_{(-1)} \oplus T$ is the empty graph, which is the case iff the entry of T in the lower right position is a 1. This proves assertion (4).

To prove assertion (5), note that U is symmetric at infinity iff $\big(\mathcal{G}[U, \rho], L\big)^t = \big(\mathcal{G}[U, \rho], L\big)$. By 5.3.2, this is equivalent to $\big(\mathcal{G}_{(-1)} \ominus T\big)^t = \mathcal{G}_{(-1)} \ominus T$; and by [5]-5.14, this is equivalent to $\check{T} = T$.

We leave the verification of (6) to the reader. □

To complement assertions (2) and (3) of 6.9, note that the graphs $\mathcal{G}_{(-1)} \oplus T$ and $\mathcal{G}_{(-1)} \oplus T$ are explicitely described in 3.5.1.

6.10. **Corollary.** *If $U \in \mathcal{M}_0$ is singular, then U has only one singular point and it is a cyclic quotient singularity.*

Proof. This follows from part (2) of 6.9, because $\mathcal{G}_{(-1)} \oplus T$ is an admissible chain. □

If $c \geq 2$ is an integer, let $\varphi(c)$ denote the number of integers x satisfying $\gcd(x,c)=1$ and $0<x<c$. Then:

6.11. **Corollary.** *Let $c>2$ be an integer. Then, up to isomorphism, there exist exactly $\varphi(c)/2$ smooth surfaces in $\mathcal{M}_0$ having a Picard group of order c.*

Proof. Follows immediately from 6.9 and 6.7. □

7. Embeddings of surfaces

7.1. **Theorem.** *Given $T \in \mathcal{T}_*$, define $(a_0,a_1,a_2) \in \mathbb{Z}^3$ and $f \in \mathbf{k}[X_0,X_1,X_2]$ as follows.*

(1) *If $T = \binom{p}{c} \in \mathcal{T}_1$, let $(a_0,a_1,a_2)=(c-p,c,1)$ and $f = X_1$.*

(2) *If $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$, there exists $a_2 \in \mathbb{Z}$ satisfying:*

$$(c-p)a_2 \equiv 1 \pmod{c}, \quad \gcd(a_2,a)=1 \quad \textit{and} \quad 0<a_2<ac.$$

Choose any such a_2 and define $a_0 = ac-a_2$, $a_1 = a$ and $f = X_0X_2 + X_1^c$.

Then a_0,a_1,a_2 are pairwise relatively prime positive integers and f is homogeneous with respect to the grading of $\mathbf{k}[X_0,X_1,X_2]$ given by $\deg X_i = a_i$ ($i=0,1,2$). Moreover, the surface $\bar{\mathfrak{f}}(T)$ is the complement of the curve $f=0$ in $\mathbb{P}(a_0,a_1,a_2)$.

Proof. It is immediate that a_0,a_1,a_2 are pairwise relatively prime positive integers and that f is homogeneous with respect to the given grading. Write $R = \mathbf{k}[X_0,X_1,X_2]$ and always regard R as a graded algebra (so an "automorphism of R" is an automorphism of graded algebra). Let $\mathbb{P} = \mathbb{P}(a_0,a_1,a_2)$ and, given a homogeneous element $g \in R$, let $V(g) \subset \mathbb{P}$ denote the zero-set of g.

The proofs of 4.3 and 4.2 tell us that, if we define (a_0,a_1,a_2) as in the above statement, then there exists a tableau $T_2 \in \mathcal{T}_1$ such that

$$\big(1,T,T_2\big) \in \begin{cases} \mathbb{T}_{\mathrm{I}}(a_0,a_1,a_2) & \text{if } T \in \mathcal{T}_1 \\ \mathbb{T}_{\mathrm{II}.2}(a_0,a_1,a_2) & \text{if } T \in \mathcal{T}_2. \end{cases}$$

Write $\tau = \big(1,T,T_2\big)$.

Consider the case where $T = \binom{p}{c} \in \mathcal{T}_1$; by [6]-7.3 and [6]-3.1, the set $\mathbb{T}_{\mathrm{I}}(a_0,a_1,a_2)$ is a singleton and its unique element is $\operatorname{disc}[\mathbb{P},\Lambda_0,F]$, where Λ_0 is a basic affine ruling of $\mathbb{P}$ defined in [6]-3.1 and F is an element of $(\Lambda_0)_*$ satisfying $\operatorname{supp}(F) = V(X_1)$. So, by definition of $\mathfrak{F}$, we have $\mathfrak{F}(\tau) = \operatorname{res}[\mathbb{P},\Lambda_0,F] = [\mathbb{P}\setminus V(X_1),\rho]$ for some $\rho : \mathbb{P}\setminus V(X_1) \to \mathbb{A}^1$. Since $\mathfrak{f}(T) = \mathfrak{F}(\tau)$, we obtain $\bar{\mathfrak{f}}(T) = [\mathbb{P}\setminus V(X_1)]$.

Consider the other case: $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right) \in \mathcal{T}_2$. By [6]-7.1 we have $\mathbb{T}_{\mathrm{II}.2}(a_0,a_1,a_2) \subset \mathbb{T}_0(\mathbb{P})$ so $\tau = \operatorname{disc}(\mathbb{P},\Lambda,F)$ for some basic affine ruling Λ of $\mathbb{P}$ and some element F of Λ_*. As in the first case, it follows that $\bar{\mathfrak{f}}(T) = [\mathbb{P}\setminus \operatorname{supp} F]$. Since every element of Λ has irreducible support by [5]-1.8, $\operatorname{supp}(F)$ is an irreducible curve $C \subset \mathbb{P}$; then $\deg C = \mu(T) = ac = a_1c$ by 6.9. Let f be a homogeneous prime element of R such that $V(f) = C$. To finish the proof it suffices to show that

(73) $\theta(f) = X_0X_2 + X_1^c$, *for some automorphism θ of the graded algebra R.*

By [5]-1.15, Λ has at most two multiple members; so we may pick $F' \in \Lambda$ such that $F' \neq F$ and $\{F, F'\}$ contains all multiple members of Λ. Let f' be a homogeneous prime element of R such that $\operatorname{supp}(F') = V(f')$. We claim:

(74) $\theta'(f') = X_2$, *for some automorphism* θ' *of the graded algebra* R.

To see this, note that the fact that Λ is a *basic* affine ruling of $\mathbb{P}$ immediately implies that $F' \in \Lambda_*$ and $\operatorname{disc}(\mathbb{P}, \Lambda, F') = (1, T_2, T)$ (see part (2) of [6]-2.26, for instance). Since $(1, T, T_2) \in \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2)$, it is obvious that $(1, T_2, T) \in \mathbb{T}_{\mathrm{II}.1}(a_0, a_2, a_1)$; then [6]-5.1 asserts that $(1, T_2, T)$ is not minimal in $\mathbb{T}(\ddagger)$ and that its immediate predecessor (call it τ^-) is some element of $\mathbb{T}_{\mathrm{I}}(a_0, a_2, a_1)$; thus

$$\operatorname{disc}(\mathbb{P}, \Lambda, F') > \tau^- \in \mathbb{T}_{\mathrm{I}}(a_0, a_2, a_1).$$

By [5]-5.13, there exists an affine ruling Λ^- of $\mathbb{P}$ and an element F^- of Λ^-_* such that $\operatorname{disc}(\mathbb{P}, \Lambda^-, F^-) = \tau^-$ and $\operatorname{supp}(F^-) = \operatorname{supp}(F')$. On the other hand, we obtain (by [6]-7.3 and [6]-3.1) that the set $\mathbb{T}_{\mathrm{I}}(a_0, a_2, a_1)$ is a singleton and that its unique element τ^- is $\operatorname{disc}(\mathbb{P}, \Lambda_0, F'')$, where $\operatorname{supp}(F'') = V(X_2)$. Thus $\operatorname{disc}(\mathbb{P}, \Lambda_0, F'') = \operatorname{disc}(\mathbb{P}, \Lambda^-, F^-)$ and by 3.9 we obtain $[\mathbb{P}, \Lambda_0, F''] = [\mathbb{P}, \Lambda^-, F^-]$. In particular, there exists an automorphism of $\mathbb{P}$ which carries $\operatorname{supp}(F'') = V(X_2)$ onto $\operatorname{supp}(F^-) = \operatorname{supp}(F')$. This proves (74).

So we may assume that $f' = X_2$. Using [5]-1.15 again, we obtain that $\mathbb{P} \setminus (\operatorname{supp}(F) \cup \operatorname{supp}(F'))$ is isomorphic to $\mathbb{P}^2$ minus two lines. Since $\gcd(\deg(f), \deg(f')) = \gcd(a_1 c, a_2) = 1$, Theorem 3.5 of [4] implies that

$$\mathbf{k}[X_2, f] = \ker D,$$

for some homogeneous locally nilpotent derivation $D : R \to R$. Here, we may choose D in such a way that $D(R) \not\subseteq X_2 R$; then $\bar{D} \neq 0$, where $\bar{D} : \bar{R} \to \bar{R}$ is the locally nilpotent derivation "$D \bmod X_2$", $\bar{R} = R/X_2 R \cong \mathbf{k}[X_0, X_1]$. By Rentschler's Theorem [12], there exist u, v such that $\ker(\bar{D}) = \mathbf{k}[v]$ and $\mathbf{k}[u, v] = \mathbf{k}[X_0, X_1]$; moreover, v is homogeneous.

Note that $\deg(f) = a_1 c = a_0 + a_2 > a_2$, so $\deg(f) > \deg(X_2)$. Since f is irreducible, it follows that $X_2 \nmid f$. So $f(X_0, X_1, 0)$ is a nonconstant element of $\ker \bar{D}$ and consequently

(75) *There exists a homogeneous variable* v *of* $\mathbf{k}[X_0, X_1]$ *such that* $f(X_0, X_1, 0)$ *is a power of* v.

We may write $f = g(X_0, X_1) + X_2 H(X_0, X_1, X_2)$ where $g \in \mathbf{k}[X_0, X_1]$ and $H \in \mathbf{k}[X_0, X_1, X_2]$ are homogeneous. We have $g = v^n$ by (75), so $f = v^n + X_2 H(X_0, X_1, X_2)$.

Given $p, q \in R$, let $p \sim q$ mean that p, q are associates. Note that $v \sim X_0 + \lambda X_1^m$ or $v \sim X_1 + \lambda X_0^m$ (where $\lambda \in \mathbf{k}$ and $m > 0$). We have $\mathbf{k}[v, X_1, X_2] = R$ in the first case and $\mathbf{k}[X_0, v, X_2] = R$ in the second case. So we may assume that one of the following holds:

(i) $f = X_0^n + X_2 H(X_0, X_1, X_2)$, or
(ii) $f = X_1^c + X_2 H(X_0, X_1, X_2)$.

Note that $H \neq 0$ by irreducibility of f, so $\deg(H) = a_0$.

In case (i), we have $a_0 \mid \deg(f) = a_1 c = a_0 + a_2$, so $a_0 \mid a_2$ and $a_0 = 1$. So $\deg(H) = 1$ and in particular H is a linear form in X_0, X_1, X_2. If $H \in \mathbf{k}[X_0, X_2]$, then f is a homogeneous prime element of $\mathbf{k}[X_0, X_2]$ and consequently $f \sim X_0$ or $f \sim X_2$ or $f \sim X_0^{a_2} + \lambda X_2^{a_0} = X_0^{a_2} + \lambda X_2$ (some $\lambda \in \mathbf{k}^*$); this is absurd because $\deg(f) = a_0 + a_2 > \max(a_0, a_2)$, so we conclude that $H \notin \mathbf{k}[X_0, X_2]$. Thus $\mathbf{k}[X_0, H, X_2] = R$, so $f = X_0^n + X_2 X_1$ up to an automorphism. It follows that $a_1 = a_0 = 1$ and that $n = c$, so the interchange of X_0 and X_1 is an automorphism and gives (73).

Consider (ii). If $H \in \mathbf{k}[X_1, X_2]$, then f is a homogeneous prime element of $\mathbf{k}[X_1, X_2]$, so $f \sim X_1$ or $f \sim X_2$ or $f \sim X_1^{a_2} + \lambda X_2^{a_1}$ (some $\lambda \in \mathbf{k}^*$); since $\deg(f) = a_1 c = a_0 + a_2$ and $c > 1$, we have $\deg(f) > \max(a_1, a_2)$, so we may rule out the first two cases and conclude that $f \sim X_1^{a_2} + \lambda X_2^{a_1}$ (some $\lambda \in \mathbf{k}^*$); then $a_2 = c$, which is impossible because $c > 1$ and $\gcd(c, a_2) = 1$. So $H \notin \mathbf{k}[X_1, X_2]$; since $\deg(H) = a_0$, we deduce that $H \sim X_0 + g(X_1, X_2)$ for some $g \in \mathbf{k}[X_1, X_2]$; then $\mathbf{k}[H, X_1, X_2] = R$ and, up to an automorphism, $f = X_1^c + X_2 X_0$, i.e., (73) holds and 7.1 is proved. □

We now address the questions:

- *Which open subsets of $\mathbb{P}(a_0, a_1, a_2)$ are members of $\mathcal{M}_0$?*
- *Given $U \in \mathcal{M}_0$ and (a_0, a_1, a_2), decide whether U can be embedded in $\mathbb{P}(a_0, a_1, a_2)$.*

Recall that all weighted projective planes belong to the class of surfaces satisfying the condition (‡) (see [6]-1.20 and 1.21).

7.2. **Definition.** Let X be a surface satisfying (‡). A *special curve* of X is an irreducible curve $C \subset X$ satisfying:

There exists an affine ruling Λ of X and a positive integer n such that $nC \in \Lambda_*$.

Remark. Let Λ be an affine ruling of a surface X satisfying (‡). Then, by [5]-1.8, every element of Λ has irreducible support. So, given any $F \in \Lambda_*$, $\operatorname{supp}(F)$ is a special curve of X.

7.2.1. **Lemma.** *Let C be a special curve of a surface X satisfying (‡). Then there exists a* basic *affine ruling Λ of X and a positive integer n such that $nC \in \Lambda_*$.*

Proof. Let Λ be an affine ruling of X such that $nC \in \Lambda_*$ for some $n > 0$. Set $(X_1, \Lambda_1, F_1) = (X_2, \Lambda_2, F_2) = (X, \Lambda, nC)$ and apply [5]-5.34. Part (1) of that result gives a basic affine ruling Λ_1' of X and an element F_1' of $(\Lambda_1')_*$ satisfying $\operatorname{supp}(F_1') = \operatorname{supp}(nC)$. □

7.3. **Proposition.** *For an open subset U of a surface X satisfying (‡), the following conditions are equivalent:*

(1) $U \in \mathcal{M}$
(2) $U \in \mathcal{M}_0$
(3) *$X \setminus U$ is a special curve of X.*

Proof. We first show that (3) implies (2). Suppose that $C = X \setminus U$ is a special curve of X. By 7.2.1, there exists a *basic* affine ruling Λ of X and an element F of Λ_* such that $\mathrm{supp}(F) = C$. Then $[X, \Lambda, F] \in \mathbb{S}_0(\ddagger)$ and consequently we may evaluate the map $\mathrm{res} : \mathbb{S}_0(\ddagger) \to \overline{\mathbb{N}}^+$ at $[X, \Lambda, F]$ (see 5.1). It is immediate that

$$\mathrm{res}[X, \Lambda, F] = [U, \rho]$$

for some $\rho : U \to \mathbb{A}^1$. By 6.4 we conclude that $U \in \mathcal{M}_0$, so (2) holds.

It is obvious that (2) implies (1). We prove that (1) implies (3). Assume that $U \in \mathcal{M}$. Then there exists a morphism $\rho : U \to \mathbb{A}^1$ whose general fiber is an affine line. By known properties of such morphisms, there holds:

(76) *There exists a nonempty open subset V of $\mathbb{A}^1$ such that $\rho^{-1}(V) \cong V \times \mathbb{A}^1$ and such that the composition $V \times \mathbb{A}^1 \to \rho^{-1}(V) \xrightarrow{\rho} V$ is the projection.*

Now ρ extends to a rational map $X \dashrightarrow \mathbb{P}^1$ which, in turn, determines a linear system Λ on X whose base locus is a finite subset of $X \setminus U$. By (76), Λ is an affine ruling of X and there exists a finite subset $\{M_1, \dots M_r\}$ of Λ satisfying

$$X \setminus \mathrm{supp}(M_1 + \cdots + M_r) \subseteq U \tag{77}$$

(the existence of the set $\{M_1, \dots M_r\}$ is asserted, e.g., by [5]-1.11). Note that, by [5]-1.8, every element of Λ has irreducible support; also, $X \setminus U$ is a nonempty union of curves, because U is affine and X is normal and complete. Consequently, if we choose the set $\{M_1, \dots M_r\}$ to be minimal with respect to property (77), then:

(78) *Equality holds in (77).*

By 2.20, the assumption that $U \in \mathcal{M}$ implies that some element of the set $\mathcal{G}_\infty[U]$ is a chain of type (Z). Then (78) and 3.7.2 give $r = 1$ and $M_1 \in \Lambda_*$, so (3) holds. $\square$

7.4. **Definition.** Given a surface X satisfying $(\ddagger)$, we define a subset $\mathbb{T}_\square(X)$ of $\mathcal{T}_*$ by:

$$\mathbb{T}_\square(X) = \big\{ T \in \mathcal{T}_* \ \mid\ \exists_{(m,T') \in \mathbb{Z}^+ \times \mathcal{T}_*} \text{ such that } (m, T, T') \in \mathbb{T}_0(X) \big\}.$$

See 3.10 for the definition of $\mathbb{T}_0(X)$. *Remark:* The following is easy to see:

$$(m, T, T') \in \mathbb{T}_0(X) \iff (m, T', T) \in \mathbb{T}_0(X).$$

7.5. **Theorem.** *Let $U \in \mathcal{M}_0$ and let $T \in \mathcal{T}_*$ be such that $\bar{\mathfrak{f}}(T) = [U]$. Given a surface X satisfying $(\ddagger)$, the following are equivalent:*

(1) *U is isomorphic to some open subset of X*
(2) *$\{T, \check{T}\} \cap \mathbb{T}_\square(X) \neq \varnothing$.*

Proof. Suppose that (1) holds. Then, by 7.3, $U = X \setminus C$ where C is a special curve of X. By 7.2.1, there exists a basic affine ruling Λ of X and an element F of Λ_* such that $\mathrm{supp}(F) = C$. It is clear that $[X, \Lambda, F] \in \mathbb{S}_0(\ddagger)$ and that $\mathrm{res}[X, \Lambda, F] = [U, \rho]$ for some $\rho : U \to \mathbb{A}^1$. So, if we define $(m, T_1, T_2) = \mathrm{disc}[X, \Lambda, F]$, we have $T_1 \in \mathbb{T}_\square(X)$ by definition of $\mathbb{T}_\square(X)$ and $\mathfrak{F}(m, T_1, T_2) = [U, \rho]$ by definition of $\mathfrak{F}$. The latter gives $\mathfrak{f}(T_1) = [U, \rho]$, so $\bar{\mathfrak{f}}(T_1) = [U] = \bar{\mathfrak{f}}(T)$ and consequently $T_1 \in \{T, \check{T}\}$, so (2) holds.

Conversely, suppose that (2) holds and pick an element T_1 of $\{T, \check{T}\} \cap \mathbb{T}_\square(X)$. Then, by definition of $\mathbb{T}_\square(X)$, we have $(m, T_1, T_2) \in \mathbb{T}_0(X)$ for some $(m, T_2) \in \mathbb{Z}^+ \times \mathcal{T}_*$; and from the definition of $\mathbb{T}_0(X)$ it follows that $(m, T_1, T_2) = \mathrm{disc}(X, \Lambda, F)$ for some basic affine ruling Λ of X and some element F of Λ_*. Then 5.3 and the definitions of $\mathfrak{f}$ and $\mathfrak{F}$ give

$$\mathfrak{f}(T_1) = \mathfrak{F}(m, T_1, T_2) = \mathrm{res}[X, \Lambda, F] = [X \setminus \mathrm{supp}(F), \rho]$$

for some $\rho : X \setminus \mathrm{supp}(F) \to \mathbb{A}^1$; consequently $\bar{\mathfrak{f}}(T_1) = [X \setminus \mathrm{supp}(F)]$. Since $T_1 \in \{T, \check{T}\}$, 6.7 gives $\bar{\mathfrak{f}}(T_1) = \bar{\mathfrak{f}}(T) = [U]$, so $U \cong X \setminus \mathrm{supp}(F)$. □

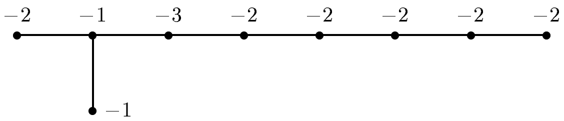
Let a_0, a_1, a_2 be pairwise relatively prime positive integers and consider $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2)$. In view of 7.5, the problem of deciding which members of $\mathcal{M}_0$ can be embedded in $\mathbb{P}$ reduces to describing the set $\mathbb{T}_\square(\mathbb{P})$. Now section 7 of [6] gives an explicit description of $\mathbb{T}_0(\mathbb{P})$, and it is easy to derive a description of $\mathbb{T}_\square(\mathbb{P})$. See 8.4.

8. Examples

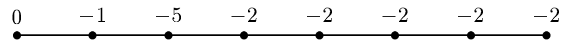
8.1. *Given a surface $U \in \mathcal{M}_0$, find a tableau $T \in \mathcal{T}_*$ such that $\bar{\mathfrak{f}}(T) = [U]$.*

8.1.1. **Example.** Let $U = \mathbb{P}^2 \setminus C$, where $C \subset \mathbb{P}^2$ is "Yoshihara's quintic" ([14], Prop. 3, cusp case). We show that $U \in \mathcal{M}_0$ and we find $T \in \mathcal{T}_*$ such that $\bar{\mathfrak{f}}(T) = [U]$.

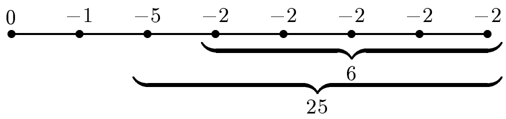
From the equation of the curve, one finds that C has a unique singular point P_0 (a cusp) and that if $P_1, P_2, \dots$ are the points of C i.n. P_0 then the multiplicity sequence $\{\mu(P_i, C)\}_{i=0}^{\infty}$ is $(2, 2, 2, 2, 2, 2, 1, 1, \dots)$. It follows from the genus formula that C is rational, so U is (rational and) completable by rational curves. A sequence of 8 blowings-up (at $P_0, \dots, P_7$) gives U as the complement of an SNC-divisor D of a smooth projective surface S, where $\mathcal{G}(D, S)$ is:



It is easily verified that the above weighted graph is equivalent to the following chain of type (Z):

(79)


so we get $\mathrm{ML}(U) = \mathbf{k}$ by 2.20; in fact we have $U \in \mathcal{M}_0$, because $\mathrm{Pic}(U) = \mathbb{Z}/5\mathbb{Z}$ is a finite group. We seek $T \in \mathcal{T}_*$ such that $\bar{\mathfrak{f}}(T) = [U]$. We have the two graphs:

(80)


$\in \mathcal{G}_\infty[U]$ and $\mathcal{G}_{\mathrm{res}}[U] = \boldsymbol{\varnothing}$

where $\boldsymbol{\varnothing}$ is the empty graph and the numbers under the braces are the determinants of the indicated subtrees. Compare (80) with (55) and (56), in the proof of 5.3. Since the determinant of the empty graph is 1, which is not equal to 25, we have $T \in \mathcal{T}_2$ by the last paragraph of the proof of 5.3. Writing $T = \left(\begin{smallmatrix} p & 1 \\ c & a \end{smallmatrix}\right)$ and comparing with (56), we find $a = 1$, $ac^2 = 25$ and $pc + 1 = 6$; thus $T = \left(\begin{smallmatrix} 1 & 1 \\ 5 & 1 \end{smallmatrix}\right)$.

Note that (79) is equivalent to

(81)


by 2.17, so (81) is another chain of type (Z) belonging to $\mathcal{G}_\infty[U]$. Making the above argument using (81) in place of (79) gives a different tableau, namely, $\left(\begin{smallmatrix}4&1\\5&1\end{smallmatrix}\right)$. There is no contradiction because this is $\check{T}$ and $\bar{\mathfrak{f}}(\check{T}) = \bar{\mathfrak{f}}(T)$ by 6.7.

Note that the number of orbits is two, in Theorem C, because $T \neq \check{T}$.

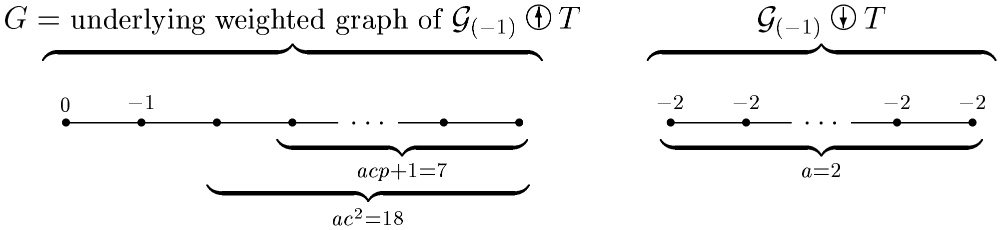
Finally, 7.1 implies that U is isomorphic to the complement of the curve $X_0X_2+X_1^5 = 0$ in $\mathbb{P}(1,1,4)$.

8.2. *Given a tableau $T \in \mathcal{T}_*$, describe the surface $U \in \mathcal{M}_0$ determined (up to isomorphism) by $\bar{\mathfrak{f}}(T) = [U]$.*

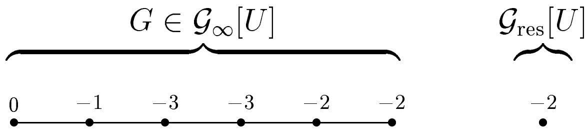
8.2.1. **Example.** Let $T = \left(\begin{smallmatrix}1&1\\3&2\end{smallmatrix}\right) \in \mathcal{T}_*$; let us describe the surface $[U] = \bar{\mathfrak{f}}(T)$.

By 7.1, U is isomorphic to the complement of the curve $X_0X_2 + X_1^3 = 0$ in $\mathbb{P}(1,2,5)$.

By 6.9, we may determine $\mathcal{G}_{\text{res}}[U]$ and an element of $\mathcal{G}_\infty[U]$ by computing $\mathcal{G}_{(-1)} \textcircled{$\downarrow$} T$ and $\mathcal{G}_{(-1)} \textcircled{$\uparrow$} T$. The quick way to do this is to use (56) (because $T \in \mathcal{T}_2$):

(82) 

where we used $T = \left(\begin{smallmatrix}p&1\\c&a\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&1\\3&2\end{smallmatrix}\right)$. Since the determinant of the right-hand side is 2, there is only one vertex in $\mathcal{G}_{\text{res}}[U] = \mathcal{G}_{(-1)} \textcircled{$\downarrow$} T$. Let G denote the underlying weighted graph of the weighted pair $\mathcal{G}_{(-1)} \textcircled{$\uparrow$} T$; then we know that $G \in \mathcal{G}_\infty[U]$. The subgraph of G obtained by deleting the two leftmost vertices is an admissible chain which is completely determined by the determinants 18 and 7, as explained in [6]-1.14. We find:

(83) 

We may continue the description of U by noting that $\text{Pic}(U_s) = \mathbb{Z}/6\mathbb{Z}$ (because $\mu(T) = 6$). By looking at the graph G in (83), we see that U is not symmetric at infinity (this also follows from $T \neq \check{T}$); so the number of orbits is 2, in Theorem C.

8.3. **Example.** Let us find all surfaces $U \in \mathcal{M}_0$ such that $\text{Pic}(U_s)$ has order 6. We use part (1) of 6.9. First, it is obvious that the elements of $\big\{\, T \in \mathcal{T}_* \mid \mu(T) = 6 \,\big\}$ are:

$$T_1 = \begin{pmatrix}1\\6\end{pmatrix},\ T_2 = \begin{pmatrix}5\\6\end{pmatrix},$$
$$T_3 = \begin{pmatrix}1&1\\6&1\end{pmatrix},\ T_4 = \begin{pmatrix}5&1\\6&1\end{pmatrix},\ T_5 = \begin{pmatrix}1&1\\3&2\end{pmatrix},\ T_6 = \begin{pmatrix}2&1\\3&2\end{pmatrix},\ T_7 = \begin{pmatrix}1&1\\2&3\end{pmatrix}.$$

Since $T_4 = \check{T}_3$ and $T_6 = \check{T}_5$, we get five (non-isomorphic) surfaces, say U_1, U_2, U_3, U_5, U_7, where $[U_i] = \bar{\mathfrak{f}}(T_i)$. Of these, U_3 is the only smooth one (6.9). As was shown in 8.2.1, each surface U_i can be described explicitely.

The last example describes $\mathbb{T}_\Box(\mathbb{P})$ for $\mathbb{P} = \mathbb{P}(1,1,1) = \mathbb{P}^2$. This is derived from the description of $\mathbb{T}_0(\mathbb{P}^2)$ given in Example 7.8 of [6].

8.4. **Example.** Define sequences $\{\xi_n\}_{n=0}^\infty$ and $\{u_n\}_{n=0}^\infty$ by:

$$\xi_n = 3\xi_{n-1} - \xi_{n-2}, \qquad \xi_0 = -4,\ \ \xi_1 = -1;$$
$$u_n = 3u_{n-1} - u_{n-2}, \quad u_0 = 1, \quad u_1 = 1.$$

Let $\{T_n\}_{n=1}^\infty$ be the sequence in $\mathcal{T}_*$ defined by $T_1 = \mathbf{1}$ (the empty tableau) and:

$$T_n = \begin{pmatrix} \xi_n & 1 \\ u_n & 1 \end{pmatrix} \qquad \text{for } n \geq 2.$$

Explicitely:

$$\{T_n\}_{n=1}^\infty = \left\{ \mathbf{1},\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},\ \begin{pmatrix} 4 & 1 \\ 5 & 1 \end{pmatrix},\ \begin{pmatrix} 11 & 1 \\ 13 & 1 \end{pmatrix},\ \begin{pmatrix} 29 & 1 \\ 34 & 1 \end{pmatrix},\ \begin{pmatrix} 76 & 1 \\ 89 & 1 \end{pmatrix},\ \dots \right\}.$$

Now it follows from [6]-7.8 that $\mathbb{T}_\Box(\mathbb{P}^2) = \bigcup_{n=1}^\infty \{T_n, \check{T}_n\}$. Consequently,

(84) $\bar{\mathfrak{f}}(T_1),\ \bar{\mathfrak{f}}(T_2),\ \bar{\mathfrak{f}}(T_3),\ \dots$ *are the members of* $\mathcal{M}_0$ *which can be embedded in* $\mathbb{P}^2$.

Comparing the list (84) with assertion (4) of 6.9, we see that many smooth surfaces belonging to $\mathcal{M}_0$ cannot be embedded in $\mathbb{P}^2$. Since $\mu(T_n) = u_n$ holds for all $n \geq 1$, the complement of $\bar{\mathfrak{f}}(T_n)$ in $\mathbb{P}^2$ is a curve of degree u_n, so:

(85) *There exists a special curve of* $\mathbb{P}^2$ *of degree* d *if and only if* d *is a term of* $\{u_n\}_{n=0}^\infty = \{1, 2, 5, 13, 34, 89, \dots\}$.

Moreover, the following fact can be derived from [6], but we don't give the proof here:

(86) *If two special curves of* $\mathbb{P}^2$ *have the same degree then some automorphism of* $\mathbb{P}^2$ *maps one onto the other.*

Remark. The special curve of $\mathbb{P}^2$ of degree 5 is Yoshihara's quintic (8.1.1), so the equation of a special curve can be much more complicated than $X_0X_1 + X_2^n = 0$. However, one can show that if "$f = 0$" is the equation of a special curve then the hypersurface "$f = 1$" in $\mathbb{A}^3$ is isomorphic to a hypersurface with equation "$xy = \varphi(z)$". Note that there may not exist an automorphism of $\mathbb{A}^3$ which maps one hypersurface onto the other. In other words, "$f = 1$" is a Danielewski surface which may be embedded in $\mathbb{A}^3$ in a non-standard way—the embedding is non-standard in the case of the quintic, and in fact in most cases.

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