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CLASSIFICATION OF HOMOGENEOUS LOCALLY NILPOTENT DERIVATIONS OF $\mathbf{k}[X, Y, Z]$

PART I: POSITIVE GRADINGS OF POSITIVE TYPE

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Abstract. Let $\mathbf{k}$ be a field of characteristic zero, let a, b, c be relatively prime positive integers, and define a grading "$\mathfrak{g}$" on the polynomial ring $B = \mathbf{k}[X, Y, Z]$ by declaring that X, Y, Z are homogeneous of degrees a, b, c respectively. Consider the problem of classifying $\mathfrak{g}$-homogeneous locally nilpotent derivations of B. The present paper solves the case where $\mathfrak{g}$ has positive type, which means that a, b, c are not pairwise relatively prime. The case where a, b, c are pairwise relatively prime is solved in our subsequent paper [5].

Introduction

Consider the polynomial ring $B = \mathbf{k}^{[3]}$ in three variables over a field $\mathbf{k}$ of characteristic zero. Despite the remarkable recent developments due to Shestakov and Umirbaev [14], our understanding of the automorphisms of B is still very incomplete and, more generally, several fundamental questions regarding the structure of B are still open. Because of the close relation between locally nilpotent derivations and automorphisms, and for other reasons as well, it is believed that a classification of the locally nilpotent derivations of B could lead to a better understanding of the structure of this $\mathbf{k}$-algebra. It is known that describing locally nilpotent derivations of B reduces to describing their kernels (cf. 2.4(b)), so we ask: *Which subalgebras of B are kernels of locally nilpotent derivations?* Although the answer is currently out of reach, the homogeneous case of this problem can now be solved, at least to some extent. To explain this, we need some definitions.

Let R be a ring (by which we mean a commutative ring with a unity element). A derivation $D : R \to R$ is *locally nilpotent* if for each $x \in R$ there exists an integer $n > 0$ such that $D^n(x) = 0$. We will use the notations $\ker(D) = \{ x \in R \mid D(x) = 0 \}$ and

$$\text{KLND}(R) = \{ \ker(D) \mid D : R \to R \text{ is a locally nilpotent derivation and } D \neq 0 \}.$$

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If D is a derivation of a graded ring $R = \oplus_{i \in \mathbb{N}} R_i$, and if there exists an integer $e \in \mathbb{Z}$ such that $D(R_i) \subseteq R_{i+e}$ holds for all $i \in \mathbb{N}$, then we say that D is a *homogeneous* derivation. If "$\mathfrak{g}$" denotes a grading of the ring R, we write:

$$\begin{aligned}\text{KLND}(R, \mathfrak{g}) = \{\ \ker(D)\ \mid\ & D : R \to R \text{ is a locally nilpotent derivation, } D \neq 0 \\ & \text{and } D \text{ is homogeneous with respect to the grading } \mathfrak{g}\ \}.\end{aligned}$$

We now return to the polynomial ring $B = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is a field of characteristic zero. By a *positive grading* of B, we mean an $\mathbb{N}$-grading $B = \oplus_{i \in \mathbb{N}} B_i$ satisfying $B_0 = \mathbf{k}$. In this paper and the forthcoming [5], our aim is to solve:

Problem. Let $\mathbf{k}$ be a field of characteristic zero. For each positive grading $\mathfrak{g}$ of $B = \mathbf{k}^{[3]}$, describe the set $\text{KLND}(B, \mathfrak{g})$.

The problem splits naturally into cases, according to the type of $\mathfrak{g}$ (for a positive grading $\mathfrak{g}$ of $\mathbf{k}^{[3]}$, $\text{type}(\mathfrak{g}) \in \{0, 1, 2, 3\}$; cf. 1.5 for the definition). The present paper solves the case $\text{type}(\mathfrak{g}) > 0$ of the problem; the case $\text{type}(\mathfrak{g}) = 0$ is solved in [5], under the assumption that $\mathbf{k}$ is algebraically closed.

This dichotomy is very natural, because the methods as well as the results are different in the two cases. In the case $\text{type}(\mathfrak{g}) > 0$ our methods are entirely algebraic, the conclusions are valid over any field $\mathbf{k}$ of characteristic zero and we describe $\text{KLND}(B, \mathfrak{g})$ explicitely. Solving the case $\text{type}(\mathfrak{g}) = 0$ requires the theory of algebraic surfaces and relies heavily on [7] and [8]; in that case it does not seem to be possible to give a description of $\text{KLND}(B, \mathfrak{g})$ which is completely explicit so, instead, we give a description in terms of the "local slice construction". In fact, [5] contains the most interesting part of the classification.

One consequence of the results of the two papers can be stated as follows:

Theorem. *Let $\mathbf{k}$ be a field of characteristic zero and let $\mathfrak{g}$ be a positive grading of $B = \mathbf{k}^{[3]}$ (if $\text{type}(\mathfrak{g}) = 0$, assume that $\mathbf{k}$ is algebraically closed). Then, given any two elements of $\text{KLND}(B, \mathfrak{g})$, there exists a finite sequence of local slice constructions which transforms one into the other.*

This answers a question first raised in [11]. The case $\text{type}(\mathfrak{g}) > 0$ of this statement is result 4.5 of the present paper, while the case $\text{type}(\mathfrak{g}) = 0$ requires a large portion of [5].

For easy reference as well as for establishing notation, the first two sections of this paper gather facts and definitions about gradings and locally nilpotent derivations. The third section gives an explicit list of all rings belonging to $\text{KLND}(B, \mathfrak{g})$, in the case $\text{type}(\mathfrak{g}) > 0$, and Section 4 proves the case $\text{type}(\mathfrak{g}) > 0$ of the theorem stated above. Sections 1, 2 and 4 also serve the purpose of collecting definitions and facts that will be needed in [5].

Notations. If R is a ring, R^* is its group of units; if $r \in R$ then R_r denotes the localization of R at the multiplicative set $\{1, r, r^2, \dots\}$. The notation $B = A^{[n]}$ means that B is isomorphic as an A-algebra to the polynomial ring in n variables over A.

1. Preliminaries on positive gradings of $\mathbf{k}^{[n]}$

Throughout this section, let $\mathbf{k}$ be a field of characteristic zero, $n \geq 1$ an integer and $B = \mathbf{k}^{[n]}$.

A *coordinate system* of B is an ordered n-tuple $(x_1, \dots, x_n)$ of elements of B satisfying $B = \mathbf{k}[x_1, \dots, x_n]$. A *variable* of B is an element $f \in B$ for which there exist $x_2, \dots, x_n \in B$ such that $(f, x_2, \dots, x_n)$ is a coordinate system of B.

In this paper, by a *positive grading* of B we mean an $\mathbb{N}$-grading of B, say $B = \oplus_{i \in \mathbb{N}} B_i$, satisfying the condition $B_0 = \mathbf{k}$. We will use symbols such as $\mathfrak{g}$ or $\mathfrak{g}'$ to represent gradings of B; specifically, if $B = \oplus_{i \in \mathbb{N}} B_i$ is a grading of B then we let $\mathfrak{g} = \{B_i\}_{i \in \mathbb{N}}$ and speak of the grading $\mathfrak{g}$ of B.

If B is endowed with a grading $\mathfrak{g}$, by a $\mathfrak{g}$-*homogeneous* coordinate system of B (or simply a *homogeneous* coordinate system of B) we mean a coordinate system $(x_1, \dots, x_n)$ of B with the property that each x_i is $\mathfrak{g}$-homogeneous. It is not a priori obvious that for any $\mathfrak{g}$ there exists a $\mathfrak{g}$-homogeneous coordinate system, but this is indeed the case whenever $\mathfrak{g}$ is a positive grading. More precisely:

1.1. Lemma. *Suppose that $B = \mathbf{k}^{[n]}$ is endowed with a positive grading $\mathfrak{g}$ and let $p \in \{0, 1, \dots, n-1\}$. If $f_1, \dots, f_p$ are $\mathfrak{g}$-homogeneous elements of B satisfying*

$$B = \mathbf{k}[f_1, \dots, f_p]^{[n-p]},$$

then there exist $\mathfrak{g}$-homogeneous $f_{p+1}, \dots, f_n \in B$ satisfying $B = \mathbf{k}[f_1, \dots, f_n]$.

The proof of 1.1 is left to the reader. It follows:

1.2. Corollary. *Suppose that B is endowed with a positive grading $\mathfrak{g}$. Then there exists a $\mathfrak{g}$-homogeneous coordinate system of B. Moreover, if $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ are two $\mathfrak{g}$-homogeneous coordinate systems of B, then $(\deg(x'_1), \dots, \deg(x'_n))$ is a permutation of $(\deg(x_1), \dots, \deg(x_n))$.*

Proof. The existence of a homogeneous coordinate system of B is the case $p = 0$ of 1.1. The invariance of $(\deg(x_1), \dots, \deg(x_n))$ up to order is an exercise left to the reader.

1.3. Given a positive grading $\mathfrak{g} = \{B_i\}_{i \in \mathbb{N}}$ of B, the greatest common divisor of the set $\{\, i \in \mathbb{N} \mid B_i \neq 0\}$ is denoted $\gcd\operatorname{supp}(\mathfrak{g})$. Let $d = \gcd\operatorname{supp}(\mathfrak{g})$ and define another positive grading $\mathfrak{g}'$ of B by $B = \oplus_{i \in \mathbb{N}} B'_i$, $B'_i = B_{di}$. We call $\mathfrak{g}'$ the *normalization* of $\mathfrak{g}$. Observe that $\gcd\operatorname{supp}(\mathfrak{g}') = 1$. (*Remark.* It is clear that $\text{KLND}(B, \mathfrak{g}) = \text{KLND}(B, \mathfrak{g}')$ so, when seeking a description of $\text{KLND}(B, \mathfrak{g})$, there is no loss of generality in assuming that $\gcd\operatorname{supp}(\mathfrak{g}) = 1$.)

1.4. Let $\mathfrak{g} = \{B_i\}_{i \in \mathbb{N}}$ be a positive grading of B such that $\gcd\operatorname{supp}(\mathfrak{g}) = 1$ and let $\gamma = (x_1, \dots, x_n)$ be a homogeneous coordinate system of B. Then the triple $(B, \mathfrak{g}, \gamma)$ determines the following data:

- The ordered n-tuple $a(\mathfrak{g}, \gamma) = (a_1, \dots, a_n) = (\deg(x_1), \dots, \deg(x_n))$;
- the ordered n-tuple $\alpha(\mathfrak{g}, \gamma) = (\alpha_1, \dots, \alpha_n)$, where for each $i \in \{1, \dots, n\}$

$$\alpha_i = \gcd\left\{\ a_j \ \mid \ j \in \{1, \dots, n\} \setminus \{i\}\ \right\};$$

- the positive integer $\nu = \prod_{i=1}^{n} \alpha_i$;
- the cardinality c of the set $\{ i \mid \alpha_i > 1 \}$;
- the subring $R = \bigoplus_{i \in \nu\mathbb{N}} B_i$ of B.

These objects have the following properties:

(a) $a_1, \dots, a_n$ are relatively prime

(b) $\alpha_1, \dots, \alpha_n$ are pairwise relatively prime, $\gcd(\alpha_i, a_i) = 1$, and $i \neq j \implies \alpha_i \mid a_j$

(c) the unordered n-tuples $[a_1, \dots, a_n]$ and $[\alpha_1, \dots, \alpha_n]$, the integers ν and c and the ring R are independent of the choice of γ, i.e., are uniquely determined by $(B, \mathfrak{g})$.

Comments. Assertion (a) is equivalent to the assumption $\gcd \operatorname{supp}(\mathfrak{g}) = 1$, and assertion (b) follows immediately. By 1.2, the unordered n-tuple $[a_1, \dots, a_n]$ is independent of the choice of γ. It follows that $[\alpha_1, \dots, \alpha_n]$ is independent of the choice of γ, and consequently so are ν, c and R.

1.5. Definition. Let $\mathfrak{g}$ be a positive grading of B. We define an integer $\operatorname{type}(\mathfrak{g}) \in \{0, 1, \dots, n\}$ as follows.

- If $\gcd \operatorname{supp}(\mathfrak{g}) = 1$, consider the integer $c \in \{0, 1, \dots, n\}$ determined by $\mathfrak{g}$ as in 1.4, and define $\operatorname{type}(\mathfrak{g}) = c$.
- If $\gcd \operatorname{supp}(\mathfrak{g}) \neq 1$, define $\operatorname{type}(\mathfrak{g}) = \operatorname{type}(\mathfrak{g}')$ where $\mathfrak{g}'$ is the normalization of $\mathfrak{g}$ (cf. 1.3).

1.6. In this paragraph, suppose that B is endowed with a positive grading $\mathfrak{g}$ such that $\gcd \operatorname{supp}(\mathfrak{g}) = 1$, and consider the subring R of B determined by $(B, \mathfrak{g})$ as in 1.4. The following facts will be used in Section 3.

1.6.1. Lemma. *Let h be a homogeneous element of B. If there exists a homogeneous coordinate system $(x_1, \dots, x_n)$ of B satisfying $x_i \nmid h$ for all $i \in \{1, \dots, n\}$, then $h \in R$.*

Proof. Let $\gamma = (x_1, \dots, x_n)$ be as in the statement and consider $(a_1, \dots, a_n) = a(\mathfrak{g}, \gamma)$ and $(\alpha_1, \dots, \alpha_n) = \alpha(\mathfrak{g}, \gamma)$ (cf. 1.4). Express h as a polynomial in $x_1, \dots, x_n$. Since $x_1 \nmid h$, some term of h has the form $\lambda x_2^{e_2} \cdots x_n^{e_n}$ (for some $e_i \in \mathbb{N}$ and $\lambda \in \mathbf{k}^*$). Since $\deg(x_2^{e_2} \cdots x_n^{e_n}) = a_2 e_2 + \cdots + a_n e_n$ is divisible by α_1, we have $\alpha_1 \mid \deg h$. Similarly, $\alpha_i \mid \deg h$ for all $i \in \{1, \dots, n\}$, so $\nu \mid \deg h$ and hence $h \in R$.

1.6.2. Lemma. *Let $\gamma = (x_1, \dots, x_n)$ be a homogeneous coordinate system of B and consider $(\alpha_1, \dots, \alpha_n) = \alpha(\mathfrak{g}, \gamma)$. Then $R = \mathbf{k}[x_1^{\alpha_1}, \dots, x_n^{\alpha_n}]$.*

Proof. Also consider $(a_1, \dots, a_n) = a(\mathfrak{g}, \gamma)$. Since $\deg(x_i^{\alpha_i}) = a_i \alpha_i$ is divisible by ν, we have $\mathbf{k}[x_1^{\alpha_1}, \dots, x_n^{\alpha_n}] \subseteq R$. Conversely, consider a monomial $x_1^{e_1} \cdots x_n^{e_n}$ whose degree is in $\nu\mathbb{N}$: $a_1 e_1 + \cdots + a_n e_n = m\nu$. Since $a_2, \dots, a_n$ and ν are divisible by α_1, so is $a_1 e_1$; since $\gcd(a_1, \alpha_1) = 1$, we must have $\alpha_1 \mid e_1$. Similarly, $\alpha_i \mid e_i$ for all $i \in \{1, \dots, n\}$, so $x_1^{e_1} \cdots x_n^{e_n} \in \mathbf{k}[x_1^{\alpha_1}, \dots, x_n^{\alpha_n}]$. Thus $R \subseteq \mathbf{k}[x_1^{\alpha_1}, \dots, x_n^{\alpha_n}]$.

1.6.3. Definition. Let $\mathcal{H}$ be the set of homogeneous prime elements of B which do not belong to R. (Note that $\mathcal{H}$ is completely determined by $(B, \mathfrak{g})$.)

1.6.4. Lemma. *Let $\gamma = (x_1, \dots, x_n)$ be a homogeneous coordinate system of B and consider $(\alpha_1, \dots, \alpha_n) = \alpha(\mathfrak{g}, \gamma)$. Then*

$$\mathcal{H} = \{ \lambda x_i \mid \lambda \in \mathbf{k}^* \text{ and } i \in \{1, \dots, n\} \text{ is such that } \alpha_i > 1 \} . \tag{1}$$

Proof. It is clear from 1.6.2 that the set on the right hand side of (1) is included in $\mathcal{H}$. Conversely, let h be an element of $\mathcal{H}$. If $\gamma = (x_1, \dots, x_n)$ is a homogeneous coordinate system of B then $h \notin R$ and 1.6.1 imply that $x_i \mid h$ for some i; then $h = \lambda x_i$ for some $\lambda \in \mathbf{k}^*$. It follows that $x_i \notin R$; since $x_i^{\alpha_i} \in R$ by 1.6.2, we obtain $\alpha_i > 1$.

1.6.5. Corollary. *Suppose that* $\operatorname{type}(\mathfrak{g}) = n$ *and let $(x_1, \dots, x_n)$ be a homogeneous coordinate system of B. Then every homogeneous coordinate system of B is a permutation of $(\lambda_1 x_1, \dots, \lambda_n x_n)$ for suitable $\lambda_1, \dots, \lambda_n \in \mathbf{k}^*$.*

Proof. Since $\operatorname{type}(\mathfrak{g}) = n$, 1.6.4 implies that $\mathcal{H} = \{ \lambda x_i \mid \lambda \in \mathbf{k}^* \text{ and } i \in \{1, \dots, n\}\}$ for any homogeneous coordinate system $(x_1, \dots, x_n)$ of B. Since $\mathcal{H}$ is independent of a choice of a coordinate system, the desired conclusion follows.

2. Preliminaries on locally nilpotent derivations

2.1. *Let B be a domain containing $\mathbb{Q}$, $D : B \to B$ a nonzero locally nilpotent derivation and $A = \ker D$.*

- *(a) A is a factorially closed subring of B (that is, the conditions $x, y \in B \setminus \{0\}$ and $xy \in A$ imply $x, y \in A$); consequently, $A^* = B^*$ and if k is any field contained in B then D is a k-derivation.*
- *(b) There exists an element $s \in B$ satisfying $Ds \neq 0$ and $D^2 s = 0$, and any such s has the following property: if we write $a = Ds$, then $B_a = A_a[s] = {A_a}^{[1]}$. In particular, if s is any element of B satisfying $Ds \in B^*$, then $B = A[s] = A^{[1]}$.*
- *(c) Define $\deg_D(x) = \max \{ n \in \mathbb{N} \mid D^n x \neq 0\}$ for $x \in B \setminus \{0\}$, and $\deg_D(0) = -\infty$. Then the map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function, i.e., the following hold for all $x, y \in B$: (i) $\deg_D x = -\infty \Leftrightarrow x = 0$; (ii) $\deg_D(xy) = \deg_D x + \deg_D y$; (iii) $\deg_D(x + y) \leq \max(\deg_D x, \deg_D y)$.*

For the above, see [9] or the forthcoming [10]. As an immediate consequence of Theorem 2.4 of [6], we have:

2.2. *Let R be a UFD containing $\mathbb{Q}$, let $S = R[X, Y] = R^{[2]}$ and let $K = \operatorname{Frac} R$. For a subring A of S, the following are equivalent:*

- *(a) A is the kernel of some nonzero locally nilpotent R-derivation of S.*
- *(b) $A = R[P]$ for some $P \in S$ which satisfies $\gcd_S(P_X, P_Y) = 1$ and which is a variable of $K[X, Y]$.*

2.3. Notation. Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. Given a two-generated subalgebra A of B, we define a $\mathbf{k}$-derivation $\Delta_A : B \to B$ as follows. Given a coordinate system $\gamma = (X, Y, Z)$ of B and $f, g \in B$ such that $A = \mathbf{k}[f, g]$, define $\Delta^\gamma_{(f,g)} : B \to B$ by the jacobian determinant

$$\Delta^\gamma_{(f,g)}(h) = \left| \frac{\partial(f, g, h)}{\partial(X, Y, Z)} \right| \qquad \text{(for each } h \in B).$$

Note that if γ' is another coordinate system of B and if $f', g' \in B$ are such that $A = \mathbf{k}[f', g']$ then $\Delta^{\gamma'}_{(f',g')} = \lambda \Delta^{\gamma}_{(f,g)}$ for some $\lambda \in \mathbf{k}^*$. Define $\Delta_A : B \to B$ to be any one of the derivations $\Delta^{\gamma}_{(f,g)}$, so that Δ_A is determined up to multiplication by a unit. We will sometimes write $\Delta_{(f,g)}$ in place of Δ_A.

2.4. *Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. If $A \in \text{KLND}(B)$ then:*

- *(a) $A = \mathbf{k}^{[2]}$*
- *(b) The set of locally nilpotent derivations $D : B \to B$ with $\ker(D) = A$ is $\{\, a\Delta_A \mid a \in A \setminus \{0\}\}$.*
- *(c) Given a positive grading $\mathfrak{g}$ of B, the condition $A \in \text{KLND}(B, \mathfrak{g})$ is equivalent to the existence of $\mathfrak{g}$-homogeneous elements $f, g \in B$ satisfying $A = \mathbf{k}[f, g]$. Moreover, if $A \in \text{KLND}(B, \mathfrak{g})$ then Δ_A is $\mathfrak{g}$-homogeneous.*

Comments. The case $\mathbf{k} = \mathbb{C}$ of assertion (a) was proved by Miyanishi in [13]; then one can use [12] to deduce the general case. Assertion (b) is result 2.6 of [2]. Assertion (c) follows from (a), (b) and 1.2.

The following is a consequence of Theorem 3.7 of [3]:

2.5. *Let $\mathbf{k}$ be a field of characteristic zero and suppose that $B = \mathbf{k}^{[3]}$ is endowed with an $\mathbb{N}$-grading $\mathfrak{g} = \{B_i\}_{i\in\mathbb{N}}$ such that $\gcd \operatorname{supp}(\mathfrak{g}) = 1$. If $A \in \text{KLND}(B, \mathfrak{g})$ is such that $\gcd\{\, i \in \mathbb{N} \mid A \cap B_i \neq 0\} \neq 1$, then there exists a homogeneous coordinate system (X_0, X_1, X_2) of B such that $X_0 \in A$.*

3. Solution of the case $\operatorname{type}(\mathfrak{g}) > 0$

Throughout this section we fix a pair $(B, \mathfrak{g})$, where $B = \mathbf{k}^{[3]}$, $\mathbf{k}$ is any field of characteristic zero and $\mathfrak{g} = \{B_i\}_{i\in\mathbb{N}}$ is a positive grading of B satisfying

$$\gcd \operatorname{supp}(\mathfrak{g}) = 1 \quad \text{and} \quad \operatorname{type}(\mathfrak{g}) > 0. \tag{2}$$

Our aim is to describe the set $\text{KLND}(B, \mathfrak{g})$. The preliminary result 3.3 states that, given any $A \in \text{KLND}(B, \mathfrak{g})$, there exists a homogeneous coordinate system of B with respect to which A has a simple form. Then paragraph 3.4 gives an explicit description of $\text{KLND}(B, \mathfrak{g})$; there, we fix a homogeneous coordinate system γ of B and all elements of $\text{KLND}(B, \mathfrak{g})$ are described in terms of γ.

3.1. Notations. Choosing a $\mathfrak{g}$-homogeneous coordinate system $\gamma = (X_0, X_1, X_2)$ of B determines the following ordered triples:

- $(a_0, a_1, a_2) = a(\mathfrak{g}, \gamma) = (\deg(X_0), \deg(X_1), \deg(X_2))$;
- $(\alpha_0, \alpha_1, \alpha_2) = \alpha(\mathfrak{g}, \gamma) = (\gcd(a_1, a_2), \gcd(a_0, a_2), \gcd(a_0, a_1))$.

As noted in 1.4, the integer $\nu = \alpha_0\alpha_1\alpha_2$ is independent of the choice of γ, and so is the subring $R = \bigoplus_{i\in\nu\mathbb{N}} B_i$ of B. Note that (2) implies that $\nu > 1$.

3.2. Lemma. *Let $A \in \text{KLND}(B, \mathfrak{g})$. Then there exists a homogeneous coordinate system (Y_0, Y_1, Y_2) of B such that $Y_0 \in A$.*

Proof. If $A \not\subseteq R$ then some homogeneous element h of A satisfies $h \notin R$. Then, by 1.6.1, any homogeneous coordinate system $\gamma = (Y_0, Y_1, Y_2)$ of B satisfies $Y_i \mid h$ for some i; clearly, we may choose γ so that $Y_0 \mid h$. Since (by 2.1) A is factorially closed in B, $Y_0 \in A$.

If $A \subseteq R$ then $\gcd\{\, i \in \mathbb{N} \mid A \cap B_i \neq \{0\}\}$ is divisible by $\nu > 1$, so this gcd is not equal to 1. In this case the result follows from 2.5.

The proof of 3.2 is short, but note that it relies on Theorem 3.7 of [3] (numbered 2.5 here), which is not an easy result.

3.3. Theorem. *Let* $A \in \text{KLND}(B, \mathfrak{g})$. *Then there exists a homogeneous coordinate system* $\gamma = (Y_0, Y_1, Y_2)$ *of* B *such that one of the following conditions holds:*

(a) $A = \mathbf{k}[Y_0, Y_1]$.

(b) $\gcd(\deg Y_0, \deg Y_1) = 1$ *and* $A = \mathbf{k}[Y_0, Y_0^e Y_2 + \psi(Y_0, Y_1)]$, *for some* $e \in \mathbb{N}$ *and some* $\psi(Y_0, Y_1) \in \mathbf{k}[Y_0, Y_1]$ *such that* $Y_0^e Y_2 + \psi(Y_0, Y_1)$ *is homogeneous and irreducible.*

(c) $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$ *and* $A = k[Y_0, P]$ *for some homogeneous* $P \in B$ *which satisfies* $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$ *and which is a variable of* $\mathbf{k}(Y_0)[Y_1, Y_2]$.

Proof. By 3.2, we may choose a homogeneous coordinate system (Y_0, Y_1, Y_2) of B such that $Y_0 \in A$. We show that one of $\gamma = (Y_0, Y_1, Y_2)$, $\gamma' = (Y_0, Y_2, Y_1)$ satisfies one of conditions (a–c).

Applying 2.2 to $B = \mathbf{k}[Y_0, Y_1, Y_2] = \mathbf{k}[Y_0]^{[2]}$ gives $A = \mathbf{k}[Y_0, P]$ for some $P \in B$ which is a variable of $\mathbf{k}(Y_0)[Y_1, Y_2] = \mathbf{k}(Y_0)^{[2]}$ and satisfies $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$. Note that P must be irreducible in B, because it is irreducible in A and A is factorially closed in B; by 1.1, and since $A = \mathbf{k}[Y_0]^{[1]}$ where Y_0 is homogeneous, we may (and we shall) choose P to be homogeneous. If $P \notin R$ then by 1.6.1 we have $Y_i \mid P$ for some i, so $A = \mathbf{k}[Y_0, Y_i]$ (where $i \neq 0$), so one of γ, γ' satisfies condition (a). From now-on, assume that $P \in R$. Let us prove:

$$\mathbf{k}(Y_0)[P, Q] = \mathbf{k}(Y_0)[Y_1, Y_2] \text{ for some homogeneous and irreducible } Q \in B. \tag{3}$$

By 2.4, the $\mathbf{k}$-derivation $\Delta_A : B \to B$ is locally nilpotent, has kernel A and is $\mathfrak{g}$-homogeneous. Write $K = \mathbf{k}(Y_0)$. As P is a variable of $K[Y_1, Y_2]$, there exists $Q' \in K[Y_1, Y_2]$ satisfying $K[P, Q'] = K[Y_1, Y_2]$; clearly, we may arrange that $Q' \in B$. Note that $\Delta_A(Q') = \Delta^{\gamma}_{(Y_0, P)}(Q')$ is equal to the jacobian determinant of (P, Q') with respect to (Y_1, Y_2), which belongs to K^* because $K[P, Q'] = K[Y_1, Y_2]$. Thus $\Delta_A(Q') \in \mathbf{k}[Y_0] \setminus \{0\}$. As Δ_A is homogeneous, some homogeneous component Q'_i of Q' satisfies $\Delta_A(Q'_i) \in \mathbf{k}[Y_0] \setminus \{0\}$. If $\deg_\Delta$ denotes the degree function determined by Δ_A as in 2.1, we have $\deg_\Delta(Q'_i) = 1$ so a unique prime factor Q of Q'_i satisfies $\deg_\Delta(Q) = 1$. Then $Q \in B$ is homogeneous and irreducible, and satisfies $\Delta_A(Q) \in \mathbf{k}[Y_0] \setminus \{0\}$. Extend Δ_A to a locally nilpotent derivation D of $K[Y_1, Y_2]$. We have $\ker D = K[P]$ and $D(Q) \in K^*$, so $K[Y_1, Y_2] = K[P, Q]$ by 2.1(b), proving (3).

Let Q be as in (3). There are two cases to consider.

Case $Q \notin R$. Then by 1.6.1 we have $Y_i \mid Q$ for some i, so $Q = Y_i$. Replacing if necessary (Y_0, Y_1, Y_2) by (Y_0, Y_2, Y_1), we may assume that $i = 1$; so $\mathbf{k}(Y_0)[Y_1, P] = \mathbf{k}(Y_0)[Y_1, Y_2]$ and consequently $P = \varphi(Y_0)Y_2 + \psi(Y_0, Y_1)$ where $\varphi(Y_0) \in \mathbf{k}(Y_0)^*$ and $\psi(Y_0, Y_1) \in \mathbf{k}(Y_0)[Y_1]$. In fact we have $\varphi(Y_0) \in \mathbf{k}[Y_0] \setminus \{0\}$ and $\psi(Y_0, Y_1) \in \mathbf{k}[Y_0, Y_1]$, because $P \in B$. Since P is homogeneous, $\varphi(Y_0) = \lambda Y_0^e$ for some $e \in \mathbb{N}$ and $\lambda \in \mathbf{k}^*$; multiplying P and ψ by λ^{-1}, we arrange that $P = Y_0^e Y_2 + \psi(Y_0, Y_1)$. If $\psi(Y_0, Y_1) = 0$ then $e = 0$ (because P is irreducible in B) and condition (a) is satisfied (after interchanging Y_1 and Y_2). If $\psi(Y_0, Y_1) \neq 0$ then, since $Y_0^e Y_2 + \psi(Y_0, Y_1)$ is homogeneous, $\deg(Y_2)$ is divisible by $\gcd(\deg Y_0, \deg Y_1)$; it follows that $\gcd(\deg Y_0, \deg Y_1) = 1$, because the degrees of Y_0, Y_1, Y_2 are relatively prime; so condition (b) is satisfied.
Case $Q \in R$. Then, since $P \in R$, $\mathbf{k}(Y_0)[P, Q] = \mathbf{k}(Y_0)[Y_1, Y_2]$ implies that $(\operatorname{Frac} R)(Y_0) = \mathbf{k}(Y_0, Y_1, Y_2)$. In view of 1.6.2, this implies that $\alpha_1 = 1 = \alpha_2$, where $\alpha_1 = \gcd(\deg(Y_0), \deg(Y_2))$ and $\alpha_2 = \gcd(\deg(Y_0), \deg(Y_1))$, so condition (c) holds.

3.4. Recall that a positive grading $\mathfrak{g}$ satisfying (2) is fixed throughout Section 3. We now fix a $\mathfrak{g}$-homogeneous coordinate system $\gamma = (X_0, X_1, X_2)$ and describe all elements of $\textsc{klnd}(B, \mathfrak{g})$ in terms of γ. We shall assume throughout 3.4 that γ has been chosen so that

$$\alpha_0 \geq \alpha_1 \geq \alpha_2, \tag{4}$$

where $(\alpha_0, \alpha_1, \alpha_2) = \alpha(\mathfrak{g}, \gamma)$. There is no loss of generality because if γ is an arbitrary homogeneous coordinate system of B then some permutation of it satisfies (4). We shall also use the notation $(a_0, a_1, a_2) = a(\mathfrak{g}, \gamma)$ as well as:

3.4.1. Notation. Let $W(B, \mathfrak{g}, \gamma)$ be the set of all pairs $(u, v) \in B \times B$ satisfying:

$$(X_0, u, v) \textit{ is a } \mathfrak{g}\textit{-homogeneous coordinate system of } B.$$

3.4.2. Theorem. *If the setup is as in 3.4, the elements of* $\textsc{klnd}(B, \mathfrak{g})$ *are:*

(a) $k[X_0, P]$ *for all homogeneous* $P \in B$ *such that* $\gcd_B(P_{X_1}, P_{X_2}) = 1$ *and such that* P *is a variable of* $\mathbf{k}(X_0)[X_1, X_2]$.

(b) $\mathbf{k}[u, u^e v + \psi(u, X_0)]$, *for all* $(u, v) \in W(B, \mathfrak{g}, \gamma)$, $e \in \mathbb{N}$ *and* $\psi(u, X_0) \in \mathbf{k}[u, X_0]$ *such that* $u^e v + \psi(u, X_0)$ *is homogeneous and irreducible.*

Proof. Let A be one of the rings displayed in part (a) (resp. part (b)) of the statement. Applying 2.2 to $B = \mathbf{k}[X_0]^{[2]}$ (resp. to $B = \mathbf{k}[u]^{[2]}$) shows that $A \in \textsc{klnd}(B)$. Moreover, $A = \mathbf{k}[f, g]$ for some $\mathfrak{g}$-homogeneous f, g, so $A \in \textsc{klnd}(B, \mathfrak{g})$ by 2.4. Thus all rings displayed in (a), (b) belong to $\textsc{klnd}(B, \mathfrak{g})$.

Conversely, let $A \in \textsc{klnd}(B, \mathfrak{g})$; we have to show that A occurs among the rings displayed in (a), (b). If $X_0 \in A$ then 2.2 implies that $A = \mathbf{k}[X_0, P']$ for some $P' \in B$ such that $\gcd_B(P'_{X_1}, P'_{X_2}) = 1$ and such that P' is a variable of $\mathbf{k}(X_0)[X_1, X_2]$; since $A = \mathbf{k}^{[2]}$ is positively graded, 1.1 implies that $A = \mathbf{k}[X_0, P]$ for some homogeneous P; then $P = \lambda P' + f(X_0)$ for some $\lambda \in \mathbf{k}^*$ and $f(X_0) \in \mathbf{k}[X_0]$; consequently $\gcd_B(P_{X_1}, P_{X_2}) = 1$ and P is a variable of $\mathbf{k}(X_0)[X_1, X_2]$, i.e., A is of the type described in (a). We assume from now-on that

$$X_0 \notin A. \tag{5}$$

By 3.3, there exists a homogeneous coordinate system $\gamma' = (Y_0, Y_1, Y_2)$ of B such that one of the following holds:

(a′) $A = \mathbf{k}[Y_0, Y_1]$.

(b′) $\gcd(\deg Y_0, \deg Y_1) = 1$ and $A = \mathbf{k}[Y_0, Y_0^e Y_2 + \psi(Y_0, Y_1)]$, for some $e \in \mathbb{N}$ and some $\psi(Y_0, Y_1) \in \mathbf{k}[Y_0, Y_1]$ such that $Y_0^e Y_2 + \psi(Y_0, Y_1)$ is homogeneous and irreducible.

(c′) $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$ and $A = k[Y_0, P]$ for some homogeneous $P \in B$ which satisfies $\gcd_B(P_{Y_1}, P_{Y_2}) = 1$ and which is a variable of $\mathbf{k}(Y_0)[Y_1, Y_2]$.

By (4), we have $\alpha_0 > 1$; so applying 1.6.4 to γ and γ' implies that there exists $i \in \{0, 1, 2\}$ satisfying

$$X_0 = \lambda Y_i \text{ for some } \lambda \in \mathbf{k}^*. \tag{6}$$

Then $\alpha'_i > 1$, where $(\alpha'_0, \alpha'_1, \alpha'_2) = \alpha(\mathfrak{g}, \gamma')$, so i also satisfies

given distinct elements j, k of $\{0, 1, 2\}$, (7)

$$\gcd(\deg Y_j, \deg Y_k) = 1 \quad \Longrightarrow \quad i \in \{j, k\}.$$

As one of (a′–c′) holds, we have $Y_0 \in A$ and hence $i \neq 0$ by (5) and (6). Thus:

$$i \in \{1, 2\}. \tag{8}$$

If (c′) holds then $\gcd(\deg Y_0, \deg Y_1) = 1 = \gcd(\deg Y_0, \deg Y_2)$ and (7) imply that $i = 0$, contradicting (8). So (c′) cannot hold.

If (a′) holds then $Y_0, Y_1 \in A$ and $X_0 \notin A$ imply that $i = 2$, so (X_0, Y_0, Y_1) is a homogeneous coordinate system of B and $(Y_0, Y_1) \in W(B, \mathfrak{g}, \gamma)$. Thus $A = \mathbf{k}[u, v]$ for some $(u, v) \in W(B, \mathfrak{g}, \gamma)$, so A occurs in (b) with $e = 0$ and $\psi = 0$.

If (b′) holds then $\gcd(\deg Y_0, \deg Y_1) = 1$, (7) and (8) imply that $i = 1$. Thus (X_0, Y_0, Y_2) is a homogeneous coordinate system of B and $(Y_0, Y_2) \in W(B, \mathfrak{g}, \gamma)$. Since $A = \mathbf{k}[Y_0, Y_0^e Y_2 + \psi(Y_0, Y_1)]$, A occurs in (b). So the elements of $\text{KLND}(B, \mathfrak{g})$ are precisely the rings listed in (a), (b).

The above result, which is valid whenever $\text{type}(\mathfrak{g}) > 0$, can be made more explicit. This is what we now do, beginning with the case $\text{type}(\mathfrak{g}) = 1$. Recall that the triple $(B, \mathfrak{g}, \gamma)$ has been fixed at the beginning of 3.4. Replacing γ by a permutation of it, we may arrange (if $\text{type}(\mathfrak{g}) = 1$) that γ satisfies condition (9) in the following:

3.4.3. Lemma. *Assume that* $\text{type}(\mathfrak{g}) = 1$ *and that* γ *satisfies*

$$\alpha_0 > 1, \qquad \alpha_1 = 1 = \alpha_2 \qquad \textit{and} \qquad a_1 \leq a_2. \tag{9}$$

Define $W_0(B, \mathfrak{g}, \gamma) = \{\, (u, v) \in W(B, \mathfrak{g}, \gamma) \mid \deg u \leq \deg v \}$, *then clearly*

$$W(B, \mathfrak{g}, \gamma) = W_0(B, \mathfrak{g}, \gamma) \cup \{\, (v, u) \mid (u, v) \in W_0(B, \mathfrak{g}, \gamma) \}\,.$$

Moreover, $W_0(B, \mathfrak{g}, \gamma)$ *is described explicitely as follows:*

(a) *If $a_1 = a_2$ then $W_0(B, \mathfrak{g}, \gamma)$ is the set of pairs $(u, v) \in B \times B$ of the form*

$$u = L(X_1, X_2) + f(X_0), \quad v = M(X_1, X_2) + g(X_0),$$

where $L, M \in \mathbf{k}[X_1, X_2]$ are linearly independent linear forms, $f, g \in \mathbf{k}[X_0]$, and where u, v are required to be $\mathfrak{g}$-homogeneous.

(b) *If $a_1 \neq a_2$ then $W_0(B, \mathfrak{g}, \gamma)$ is the set of pairs $(u, v) \in B \times B$ of the form*

$$u = \lambda_1 X_1 + f(X_0), \quad v = \lambda_2 X_2 + g(X_0, X_1),$$

where $\lambda_1, \lambda_2 \in \mathbf{k}^*$, $f \in \mathbf{k}[X_0]$, $g \in \mathbf{k}[X_0, X_1]$, and where u, v are required to be $\mathfrak{g}$-homogeneous.

The straightforward verification of 3.4.3 is left to the reader. We stress:

3.4.4. *Results 3.4.2 and 3.4.3 together give an explicit description of* $\text{KLND}(B, \mathfrak{g})$ *in the case* $\text{type}(\mathfrak{g}) = 1$.

We now show that the statement of 3.4.2 takes a simpler form (3.4.5) when $\text{type}(\mathfrak{g}) \geq 2$, and an even simpler form (3.4.6) when $\text{type}(\mathfrak{g}) = 3$.

3.4.5. Corollary. *If* $\text{type}(\mathfrak{g}) \geq 2$ *then the elements of* $\text{KLND}(B, \mathfrak{g})$ *are:*

(a) $\mathbf{k}[X_0, X_1]$

(b) $\mathbf{k}[X_0, X_0^e X_2 + \psi(X_0, X_1)]$, *for all $e \in \mathbb{N}$ and $\psi(X_0, X_1) \in \mathbf{k}[X_0, X_1]$ such that $X_0^e X_2 + \psi(X_0, X_1)$ is homogeneous and irreducible in B.*

(c) $\mathbf{k}[X_1, X_1^e X_2 + \psi(X_0, X_1)]$ *for all $e \in \mathbb{N}$ and $\psi(X_0, X_1) \in \mathbf{k}[X_0, X_1]$ such that $X_1^e X_2 + \psi(X_0, X_1)$ is homogeneous and irreducible in B.*

Proof. The rings which appear in parts (a), (b) (resp. in part (c)) belong to $\text{KLND}(B, \mathfrak{g})$ by 3.4.2(a) (resp. by 3.4.2(b)). Conversely, let $A \in \text{KLND}(B, \mathfrak{g})$; we have to show that A occurs among the rings displayed in (a–c). By 3.4.2, one of the following holds:

(a′) $A = k[X_0, P]$ for some homogeneous $P \in B$ such that $\gcd_B(P_{X_1}, P_{X_2}) = 1$ and such that P is a variable of $\mathbf{k}(X_0)[X_1, X_2]$.

(b′) $A = \mathbf{k}[u, u^e v + \psi_1(u, X_0)]$, for some $(u, v) \in W(B, \mathfrak{g}, \gamma)$, $e \in \mathbb{N}$ and $\psi_1(u, X_0) \in \mathbf{k}[u, X_0]$ such that $u^e v + \psi_1(u, X_0)$ is homogeneous and irreducible.

Suppose that (a′) holds. Let $K = \mathbf{k}(X_0)$ and write $P = \sum_{i,j \geq 0} c_{ij} X_1^i X_2^j$ with $c_{ij} \in \mathbf{k}[X_0]$. We examine two cases.

Case $c_{10} \neq 0$. Then we claim that $P = \lambda X_1$ for some $\lambda \in \mathbf{k}^*$ (so $A = \mathbf{k}[X_0, X_1]$). Indeed, if this is not the case then P contains monomials $X_0^i X_1$ and $X_0^j X_2^k$ for some $i, j, k \in \mathbb{N}$. Then $\deg(X_0^i X_1) = \deg(X_0^j X_2^k)$ implies that $a_1 = \deg(X_1)$ belongs to the ideal $(\deg(X_0), \deg(X_2)) = (\alpha_1)$ of $\mathbb{Z}$, which is impossible because $\alpha_1 > 1$ (by (4) and $\text{type}(\mathfrak{g}) \geq 2$) and $\gcd(\alpha_1, a_1) = 1$ (by 1.4).

Case $c_{10} = 0$. Then $P_{X_1}(0, X_2) = 0$, and since P is a variable of $K[X_1, X_2]$ it follows that $P_{X_2}(0, X_2) \in K^*$; consequently $\deg_{X_2}(P) = 1$, so $P = X_0^e X_2 + \psi(X_0, X_1)$ and A occurs in (b).

Now suppose that (b′) holds. Taking into account $\text{type}(\mathfrak{g}) \geq 2$ and (4) we see that any element (u, v) of $W(B, \mathfrak{g}, \gamma)$ is either $(\lambda_1 X_1, \lambda_2 X_2 + f(X_0, X_1))$ or $(\lambda_2 X_2 + f(X_0, X_1), \lambda_1 X_1)$, for some $\lambda_1, \lambda_2 \in \mathbf{k}^*$ and $f(X_0, X_1) \in \mathbf{k}[X_0, X_1]$. In the first case we have

$$A = \mathbf{k}[u, u^e v + \psi_1(u, X_0)] = \mathbf{k}[X_1, (\lambda_1 X_1)^e(\lambda_2 X_2 + f(X_0, X_1)) + \psi_1(\lambda_1 X_1, X_0)],$$

so A occurs in (c). In the second case we have $\deg(u) = a_2$ and $deg(v) = a_1$, and this implies that $\psi_1 = 0$. Indeed, if $\psi_1 \neq 0$ then $\deg(u^e v) = \deg(u^i X_0^j)$ for some $i, j \in \mathbb{N}$, so $a_1 = \deg(v) \in (\deg(u), \deg(X_0)) = (a_2, a_0) = (\alpha_1)$, which is impossible because $\alpha_1 > 1$ and $\gcd(\alpha_1, a_1) = 1$. Since $\psi_1 = 0$ we also have $e = 0$, so

$$A = \mathbf{k}[u, v] = \mathbf{k}[\lambda_2 X_2 + f(X_0, X_1), X_1]$$

occurs in (c).

3.4.6. Corollary. *If* $\text{type}(\mathfrak{g}) = 3$ *then*

$$\text{KLND}(B, \mathfrak{g}) = \big\{\mathbf{k}[X_0, X_1], \mathbf{k}[X_0, X_2], \mathbf{k}[X_1, X_2]\big\}.$$

Proof. By 3.3, each element of $\text{KLND}(B, \mathfrak{g})$ is $\mathbf{k}[Y_0, Y_1]$ for some homogeneous coordinate system (Y_0, Y_1, Y_2) of B. By 1.6.4, the only homogeneous coordinate systems of B are the triples $(\lambda_i X_i, \lambda_j X_j, \lambda_k X_k)$ where the λ's belong to $\mathbf{k}^*$ and where (i, j, k) is a permutation of $(0, 1, 2)$.

4. Local slice construction

This section has two objectives: to recall the definition and some basic properties of the "local slice construction", and to prove the case $\text{type}(\mathfrak{g}) > 0$ of the theorem stated in the introduction. Note that the concepts discussed in this section, and in particular the graph $\underline{\text{KLND}}(B, \mathfrak{g})$ defined in 4.3, will play an important role in [5].

Throughout this section we fix a pair $(B, \mathfrak{g})$ where $B = \mathbf{k}^{[3]}$, $\mathbf{k}$ is a field of characteristic zero and $\mathfrak{g}$ is any positive grading of B (we do not assume that $\text{type}(\mathfrak{g})$ is positive).

Given $A \in \text{KLND}(B, \mathfrak{g})$, define the following subset of $B^3 = B \times B \times B$:

$$\Sigma(A) = \big\{(f, g, s) \in B^3 \ \mid \ f, g, s \text{ are } \mathfrak{g}\text{-homogeneous, } A = \mathbf{k}[f, g] \\ \text{and } \Delta_A(s) = \lambda f^i g \text{ for some } i \in \mathbb{N} \text{ and } \lambda \in \mathbf{k}^*\big\}$$

where $\Delta_A : B \to B$ was defined in 2.3.

4.1. Proposition and definition. *Let* $A \in \text{KLND}(B, \mathfrak{g})$*, suppose that* $\Sigma(A) \neq \varnothing$ *and consider an element* $\sigma = (f, g, s) \in \Sigma(A)$.

(a) *There exists an irreducible polynomial* $\Phi(X,Y) \in \mathbf{k}[X,Y]$ *such that* $\Phi(f,s) \in gB$. *Moreover,* Φ *is unique up to multiplication by a unit.*
(b) *Consider the* $\mathbf{k}$*-derivation* $\Delta_{(f,h)} : B \to B$, *where* $h \in B$ *is defined by* $h = \Phi(f,s)/g$. *Then* $\Delta_{(f,h)}$ *is locally nilpotent and homogeneous. Moreover, the kernel of* $\Delta_{(f,h)}$ *is an element of* $\text{KLND}(B,\mathfrak{g})$ *distinct from* A, *and is equal to* $\mathbf{k}[f,h']$ *for some homogeneous* $h' \in B$.

Define $\text{LSC}(A,\sigma) = \ker \Delta_{(f,h)}$. *The element* $\text{LSC}(A,\sigma)$ *of* $\text{KLND}(B,\mathfrak{g}) \setminus \{A\}$ *is said to be obtained from* A *by "local slice construction" (see the remark after the proof).*

Proof. From Theorem 2 of [11], one obtains assertion (a) and the claim that $\Delta_{(f,h)}$ is locally nilpotent. It is easy to see that $\Phi(X,Y)$ is homogeneous with respect to the grading of $\mathbf{k}[X,Y]$ defined by declaring that X,Y are homogeneous of degrees $\deg_{\mathfrak{g}}(f)$ and $\deg_{\mathfrak{g}}(s)$ respectively; so h is homogeneous, and so is $\Delta_{(f,h)}$. In the proof of [11], Theorem 2, we also find that $\ker \Delta_{(f,h)}$ is neither A nor B, so we see that $\ker \Delta_{(f,h)}$ is an element of $\text{KLND}(B,\mathfrak{g})$ distinct from A. So the only assertion which requires a proof is the claim that $\ker \Delta_{(f,h)} = \mathbf{k}[f,h']$ for some homogeneous $h' \in B$. Let $A' = \ker \Delta_{(f,h)}$. We have $f \in A'$ and we know from 2.4 that $A' = \mathbf{k}^{[2]}$; it suffices to show that f is a variable of A', because if this is the case then $A' = \mathbf{k}[f]^{[1]}$ together with 1.1 imply the desired conclusion.

Write $K = \mathbf{k}(f)$. Localizing $\Delta_{(f,g)} : B \to B$ at the multiplicative set $S = \mathbf{k}[f] \setminus \{0\}$ gives a locally nilpotent derivation $\delta : S^{-1}B \to S^{-1}B$ with kernel $S^{-1}A = K[g]$, so:

$$\ker \delta = K[\delta s] = K^{[1]}.$$

By result 2.6.2 of [4], it follows that $\big(S^{-1}B, K\big)$ is a Danielewski surface (cf. Def. 2.1 of [4]); in particular, 2.8 of [4] implies that each element of $\text{KLND}(S^{-1}B)$ is a $K^{[1]}$. As $S \subset \ker \Delta_{(f,h)}$, the localization of $\Delta_{(f,h)}$ at S is a locally nilpotent derivation $\delta' : S^{-1}B \to S^{-1}B$ with kernel $S^{-1}A'$; so $S^{-1}A' \in \text{KLND}(S^{-1}B)$ and hence

$$S^{-1}A' = K^{[1]}. \tag{10}$$

We have $A' = \mathbf{k}[u,v] = \mathbf{k}^{[2]}$ (for some u,v), and condition (10) means that the generic fiber of f is an affine line: $\mathbf{k}(f)[u,v] = \mathbf{k}(f)^{[1]}$. As is well-known, this implies that f is a variable of A', so we are done.

Remark. The local slice construction was first introduced by Freudenburg in [11] as a process for modifying a given locally nilpotent derivation of $B = \mathbf{k}^{[3]}$ so as to obtain another one. The version that we present in 4.1 is suitable in the presence of a positive grading. Note that the local slice construction was generalised and clarified in [4] so as to be applicable to an arbitrary integral domain B of characteristic zero. The case where B is an affine UFD can be formulated in a way which is very close to the spirit of 4.1 and of [11], see 9.12 of [1].

Here is another claim whose proof can be found in [11]:

4.2. *Let* A *and* A' *be elements of* $\text{KLND}(B,\mathfrak{g})$. *If there exists* $\sigma \in \Sigma(A)$ *such that* $\text{LSC}(A,\sigma) = A'$, *then there exists* $\sigma' \in \Sigma(A')$ *such that* $\text{LSC}(A',\sigma') = A$.

4.3. Definition. We define $\underline{\text{KLND}}(B, \mathfrak{g})$ to be the graph with vertex set $\text{KLND}(B, \mathfrak{g})$ and in which distinct vertices A, A' are joined by an edge if and only if there exists $\sigma \in \Sigma(A)$ such that $\text{LSC}(A, \sigma) = A'$. (To be precise: edges are not oriented, each edge has distinct endpoints and there is at most one edge joining any given pair of vertices.)

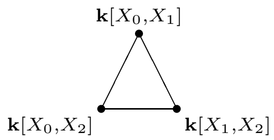
Clearly, a description of the graph $\underline{\text{KLND}}(B, \mathfrak{g})$ would constitute a solution of the problem stated in the introduction. As a partial solution we shall obtain that $\underline{\text{KLND}}(B, \mathfrak{g})$ is connected: the case $\text{type}(\mathfrak{g}) > 0$ is 4.5, below, and the case $\text{type}(\mathfrak{g}) = 0$ is proved in the forthcoming [5] (see the theorem stated in the introduction of the present paper).

4.4. We continue to consider a pair $(B, \mathfrak{g})$ where $B = \mathbf{k}^{[3]}$, $\mathbf{k}$ is a field of characteristic zero and $\mathfrak{g}$ is a positive grading of B. Since $\underline{\text{KLND}}(B, \mathfrak{g}) = \underline{\text{KLND}}(B, \mathfrak{g}')$ where $\mathfrak{g}'$ is the normalization of $\mathfrak{g}$ (cf. 1.3), there is no loss of generality in assuming

$$\gcd \operatorname{supp}(\mathfrak{g}) = 1,$$

and indeed this assumption is in effect throughout paragraph 4.4.

4.4.1. *Let (X_0, X_1, X_2) be a homogeneous coordinate system of B. Then*



(11)

is a subgraph of $\underline{\text{KLND}}(B, \mathfrak{g})$.

Proof. Let $A = \mathbf{k}[X_0, X_1]$. Then $\sigma = (X_0, X_1, X_1X_2)$ belongs to $\Sigma(A)$, because $\Delta_{(X_0,X_1)}(X_1X_2) = X_1$. Let $A' = \text{LSC}(A, \sigma)$. The polynomial $\Phi(X, Y)$ in part (1) of 4.1 is $\Phi(X, Y) = Y$ and $h = X_2$, so $\mathbf{k}[X_0, X_2] \subseteq A'$, so $A' = \mathbf{k}[X_0, X_2]$. This shows that the vertices $\mathbf{k}[X_0, X_1]$ and $\mathbf{k}[X_0, X_2]$ of $\underline{\text{KLND}}(B, \mathfrak{g})$ are joined by an edge. Similarly, $\mathbf{k}[X_0, X_1]$ and $\mathbf{k}[X_1, X_2]$ are joined by an edge, and so are $\mathbf{k}[X_0, X_2]$ and $\mathbf{k}[X_1, X_2]$.

4.4.2. *Suppose that* $\text{type}(\mathfrak{g}) = 3$ *and let (X_0, X_1, X_2) be a homogeneous coordinate system of B. Then $\underline{\text{KLND}}(B, \mathfrak{g})$ is the triangle displayed in* (11).

Proof. Follows immediately from 3.4.6 and 4.4.1.

4.4.3. *Suppose that* $\text{type}(\mathfrak{g}) = 2$ *and choose a homogeneous coordinate system $\gamma = (X_0, X_1, X_2)$ of B satisfying $\alpha_0 \geq \alpha_1 \geq \alpha_2$, where $(\alpha_0, \alpha_1, \alpha_2) = \alpha(\mathfrak{g}, \gamma)$. If A is any vertex of $\underline{\text{KLND}}(B, \mathfrak{g})$ other than $\mathbf{k}[X_0, X_1]$, then A is joined to $\mathbf{k}[X_0, X_1]$ by an edge.*

Proof. Condition (4) of 3.4 is satisfied, so A is one of the rings displayed in conditions (b–c) of 3.4.5. So there exists $i \in \{0, 1\}$ such that $A = \mathbf{k}[X_i, X_i^e X_2 + \psi(X_0, X_1)]$, for some $e \in \mathbb{N}$ and $\psi(X_0, X_1) \in \mathbf{k}[X_0, X_1]$ such that $X_i^e X_2 + \psi(X_0, X_1)$ is homogeneous and irreducible in B. Define j by $\{i, j\} = \{0, 1\}$ and let $P = X_i^e X_2 + \psi(X_0, X_1)$. Let $\sigma = (X_i, X_j, X_jP)$ and note that $\Delta_{(X_i,X_j)}(X_jP) = \pm\frac{\partial}{\partial X_2}(X_jP) = \pm X_i^e X_j$, so $\sigma \in \Sigma(A_2)$ where $A_2 = \mathbf{k}[X_0, X_1] = \mathbf{k}[X_i, X_j]$. We compute $\text{LSC}(A_2, \sigma)$ (refer to 4.1 for the notation). We have $\Phi(X, Y) = Y$, so $h = P$ and $\mathbf{k}[X_i, P] \subseteq \text{LSC}(A_2, \sigma)$. It follows that $\text{LSC}(A_2, \sigma) = A$, so A is joined to A_2 by an edge.

4.4.4. *Let $A \in \text{KLND}(B, \mathfrak{g})$ and suppose that (X_0, X_1, X_2) is a homogeneous coordinate system of B satisfying $X_0 \in A$. Then there exists a path $(A_0, \dots, A_n)$ in $\underline{\text{KLND}}(B, \mathfrak{g})$ with the following properties:*

(a) $A_0 = A$ *and* $A_n = \mathbf{k}[X_0, X_j]$ *for some* $j \in \{1, 2\}$.
(b) *For each* $i = 0, \dots, n$, $A_i = \mathbf{k}[X_0, P_i]$ *for some homogeneous* $P_i \in B$.
(c) $\operatorname{tdeg}(P_0) > \cdots > \operatorname{tdeg}(P_{n-1}) \geq \operatorname{tdeg}(P_n)$, *where* $\operatorname{tdeg} : \mathbf{k}(X_0)[X_1, X_2] \to \mathbb{N} \cup \{-\infty\}$ *is the usual total degree for polynomials in two variables:* $\operatorname{tdeg}(X_1) = 1 = \operatorname{tdeg}(X_2)$ *and* $\operatorname{tdeg}(f) = 0$ *for every* $f \in \mathbf{k}(X_0)^*$.
(d) $\mathbf{k}(X_0)[P_{i-1}, P_i] = \mathbf{k}(X_0)[X_1, X_2]$ *for every* $i \in \{1, \dots, n\}$.

Proof. This is essentially section 3 of [11], modulo minor adaptations.

4.4.5. *Let (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) be $\mathfrak{g}$-homogeneous coordinate systems of B. Then $\mathbf{k}[X_0, X_1]$ and $\mathbf{k}[Y_0, Y_1]$ belong to the same connected component of $\underline{\text{KLND}}(B, \mathfrak{g})$.*

Proof. By 4.4.2 and 4.4.3, $\underline{\text{KLND}}(B, \mathfrak{g})$ is connected whenever $\operatorname{type}(\mathfrak{g}) \geq 2$. So it suffices to consider the cases where $\operatorname{type}(\mathfrak{g}) \in \{0, 1\}$. Write $\gamma = (X_0, X_1, X_2)$ and $\gamma' = (Y_0, Y_1, Y_2)$.

Suppose that $\operatorname{type}(\mathfrak{g}) = 1$. By 4.4.1, we may permute each of γ, γ', so we may arrange that $a(\mathfrak{g}, \gamma) = a(\mathfrak{g}, \gamma')$ and that condition (9) of 3.4.3 is satisfied. Writing $(a_0, a_1, a_2) = a(\mathfrak{g}, \gamma) = a(\mathfrak{g}, \gamma')$ and $(\alpha_0, \alpha_1, \alpha_2) = \alpha(\mathfrak{g}, \gamma) = \alpha(\mathfrak{g}, \gamma')$, we have $\alpha_0 > 1 = \alpha_1 = \alpha_2$ and $a_1 \leq a_2$. By 1.6.4, there exists $\lambda \in \mathbf{k}^*$ such that $Y_0 = \lambda X_0$. Thus (X_0, Y_1, Y_2) is a homogeneous coordinate system of B and consequently $(Y_1, Y_2) \in W(B, \mathfrak{g}, \gamma)$ (cf. 3.4.1). As $\deg(Y_1) = a_1 \leq a_2 = \deg(Y_2)$, we have $(Y_1, Y_2) \in W_0(B, \mathfrak{g}, \gamma)$ and 3.4.3 gives $Y_1 = L(X_1, X_2) + f(X_0)$ where $L \in \mathbf{k}[X_1, X_2]$ is a nonzero linear form and $f(X_0) \in \mathbf{k}[X_0]$; thus, for some $i \in \{1, 2\}$, (X_0, X_i, Y_1) is a homogeneous coordinate system of B; by 4.4.1, $\mathbf{k}[Y_0, Y_1] = \mathbf{k}[X_0, Y_1]$ and $\mathbf{k}[X_0, X_i]$ are joined by an edge so, again by 4.4.1, $\mathbf{k}[Y_0, Y_1]$ and $\mathbf{k}[X_0, X_1]$ belong to the same connected component of $\underline{\text{KLND}}(B, \mathfrak{g})$.

From now-on, suppose that $\operatorname{type}(\mathfrak{g}) = 0$. Here we arrange that $a_0 \leq a_1 \leq a_2$, where $(a_0, a_1, a_2) = a(\mathfrak{g}, \gamma) = a(\mathfrak{g}, \gamma')$. We consider two cases.

Case $(a_0, a_1, a_2) = (1, 1, 1)$. Note that γ and γ' are two bases of the $\mathbf{k}$-vector space V spanned by X_0, X_1, X_2. As (X_0, X_1) spans a 2-dimensional subspace of V, we may choose i such that (X_0, X_1, Y_i) is a basis of V; as (X_1, Y_i) spans a 2-dimensional subspace of V, we may choose j such that (X_1, Y_i, Y_j) is a basis of V. Applying 4.4.1 to the homogeneous coordinate system (X_0, X_1, Y_i) (resp. (X_1, Y_i, Y_j)) of B shows that $\mathbf{k}[X_0, X_1]$ and $\mathbf{k}[X_1, Y_i]$ (resp. $\mathbf{k}[X_1, Y_i]$ and $\mathbf{k}[Y_i, Y_j]$) belong to the same connected component of $\underline{\text{KLND}}(B, \mathfrak{g})$. So $\mathbf{k}[X_0, X_1]$ and $\mathbf{k}[Y_i, Y_j]$ belong to the same connected component of $\underline{\text{KLND}}(B, \mathfrak{g})$ and we are done in this case.

Case $(a_0, a_1, a_2) \neq (1, 1, 1)$. Here we have $a_0 \leq a_1 < a_2$, because $a_2 > 1$ and $\gcd(a_1, a_2) = 1$. So $\max(\deg(Y_0), \deg(Y_1)) < \deg(X_2)$, which implies that $Y_0, Y_1 \in \mathbf{k}[X_0, X_1]$; and $\max(\deg(X_0), \deg(X_1)) < \deg(Y_2)$, which implies that $X_0, X_1 \in \mathbf{k}[Y_0, Y_1]$. So we have $\mathbf{k}[Y_0, Y_1] = \mathbf{k}[X_0, X_1]$ and the claim is true.

4.5. Theorem. *Let $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is a field of characteristic zero. If $\mathfrak{g}$ is a positive grading of B such that* $\text{type}(\mathfrak{g}) > 0$*, then* $\underline{\text{KLND}}(B, \mathfrak{g})$ *is connected.*

Proof. We may assume that $\gcd \operatorname{supp}(\mathfrak{g}) = 1$. Consider $A, A' \in \text{KLND}(B, \mathfrak{g})$. By 3.2, there exists homogeneous coordinate systems (X_0, X_1, X_2) and (Y_0, Y_1, Y_2) of B such that $X_0 \in A$ and $Y_0 \in A'$. By 4.4.4, there is a path in $\underline{\text{KLND}}(B, \mathfrak{g})$ going from A to $\mathbf{k}[X_0, X_1]$, and one going from A' to $\mathbf{k}[Y_0, Y_1]$. By 4.4.5, we are done.

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