ON KERNELS OF HOMOGENEOUS LOCALLY NILPOTENT DERIVATIONS OF $\mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]$

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Abstract. Let $B = \mathbf{k}[X_0, X_1, X_2]$ be the polynomial ring in three variables over a field $\mathbf{k}$ of characteristic zero, let ω_0, ω_1, ω_2 be pairwise relatively prime positive integers and let $B = \oplus_{i \in \mathbb{N}} B_i$ be the grading determined by $B_0 = \mathbf{k}$ and $X_i \in B_{\omega_i}$. Then the main result of the paper has the following corollary:
For elements f, g of B which are homogeneous, geometrically irreducible and not associates, the following are equivalent:

(1) *The homogeneous localization of B with respect to $\{1, fg, (fg)^2, \dots\}$ is a polynomial ring in one variable over a subring.*
(2) *$\mathbf{k}[f, g]$ is the kernel of a homogeneous locally nilpotent derivation $D : B \to B$.*

Moreover, if these equivalent conditions are satisfied then $\gcd(\deg f, \deg g) = 1$.

Note that this relates (1) the affine rulings of the weighted projective plane $\operatorname{Proj} B$ to (2) the homogeneous locally nilpotent derivations of B.

Consider the case "$n = 2$" of our main result, Theorem 2.2:

Corollary. *Let $B = \mathbf{k}[X_0, X_1, X_2]$ be the polynomial ring in three variables over a field $\mathbf{k}$ of characteristic zero, let ω_0, ω_1, ω_2 be pairwise relatively prime positive integers and let $B = \oplus_{i \in \mathbb{N}} B_i$ be the grading determined by $B_0 = \mathbf{k}$ and $X_i \in B_{\omega_i}$. For elements f, g of B which are homogeneous, geometrically irreducible and not associates, the following are equivalent:*

(1) *$B_{(fg)}$ is a polynomial ring in one variable over a subring.*
(2) *$\mathbf{k}[f, g]$ is the kernel of a homogeneous locally nilpotent derivation $D : B \to B$.*

Moreover, if these equivalent conditions are satisfied then $\gcd(\deg f, \deg g) = 1$.

Here, $B_{(fg)}$ is the homogeneous localization of B with respect to $\{1, fg, (fg)^2, \dots\}$. By "geometrically irreducible", we mean irreducible in $\bar{\mathbf{k}}[X_0, X_1, X_2]$, where $\bar{\mathbf{k}}$ is an algebraic closure of $\mathbf{k}$.

The reader should compare the above Corollary with 1.8. One notable difference is that the condition $\gcd(\deg f, \deg g) = 1$, which is part of the assumption of 1.8, is in the conclusion of the present result; we are also replacing the assumption $\gcd(\omega_0, \omega_1, \omega_2) = 1$ of 1.8 by the stronger "$\omega_0, \omega_1, \omega_2$ are *pairwise* relatively prime". The proof that $\gcd(\deg f, \deg g) = 1$ is one of the crucial steps of this paper; it is achieved by Theorem 2.1, in the form $\gcd\{i \mid A_i \neq 0\} = 1$.

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The fact that condition (1) of the Corollary implies $\gcd(\deg f, \deg g) = 1$ is needed in [4], which investigates the affine rulings of the weighted projective planes (see also the remark following 1.11). A proof of that implication is included in [4], but it relies on a considerable amount of machinery developed in [3, 4]; so we feel that it is appropriate to give a relatively self-contained proof, based on a different method.

Theorem 2.2 is also useful for establishing a precise correspondence between affine rulings and locally nilpotent derivations. That correspondence is used, in recent work, to relate the viewpoint of [3, 4] to that of [5].

1. Preliminaries

All rings are commutative and have a unity. For an A-algebra B, the notation $B = A^{[n]}$ (where n is a positive integer) means that B is A-isomorphic to the polynomial ring in n variables over A.

Given a nonzero graded ring $A = \oplus_{i \in \mathbb{Z}} A_i$, a *homogeneous multiplicatively closed subset* of A is a set $S \subseteq \cup_{i \in \mathbb{Z}}(A_i \setminus \{0\})$ closed under multiplication and such that $1 \in S$. Then $A_{(S)}$ denotes the homogeneous localization of A with respect to S, i.e., the component of degree zero of the graded ring $S^{-1}A$. If $a \in A_i \setminus \{0\}$ and $S = \{1, a, a^2, \dots\}$, we write $A_{(a)} = A_{(S)}$. By a *homogeneous subring* of A we mean a subring A' of A satisfying $A' = \sum(A' \cap A_i)$.

Let R be a domain. A derivation $\Delta : R \to R$ is *locally nilpotent* if for each $r \in R$ we have $\Delta^n(r) = 0$ for n sufficiently large; Δ is *irreducible* if the only principal ideal of R containing $\Delta(R)$ is R itself.

Facts 1.1–1.5 are needed in the proof of Theorem 2.1. The first one is due to W.V. Vasconcelos:

1.1 (Theorem 2.2 of [8])**.** *Let $B \supseteq R$ be an integral extension of domains containing $\mathbb{Q}$. Suppose that $\Delta : R \to R$ is a locally nilpotent derivation and that $D : B \to B$ is a derivation extending Δ. Then D is locally nilpotent.*

The next statement is a well-known consequence of a result of David Wright (Proposition 2.1 of [9]):

1.2. *Let $D : B \to B$ be a locally nilpotent derivation, where B is a domain containing $\mathbb{Q}$, and let $A = \ker D$. If $b \in B$ satisfies $Db \in A \setminus \{0\}$, then $B_a = A_a[b] = {A_a}^{[1]}$ where $a = Db$.*

Statements 1.3 and 1.4 are well-known:

1.3. *Let $D : B \to B$ be a nonzero derivation, where B is an integral domain satisfying the ascending chain condition on principal ideals. Then $D = bD'$, for some $b \in B$ and some irreducible derivation $D' : B \to B$.*

1.4. *Let B be an integral domain of characteristic zero, $D : B \to B$ a nonzero derivation and $b \in B \setminus \{0\}$. The derivation $bD : B \to B$ is locally nilpotent if and only if D is locally nilpotent and $b \in \ker D$.*

Lemma 1.5. *Let R be a $\mathbb{Z}$-graded integral domain containing $\mathbb{Q}$ and $\Delta : R \to R$ an irreducible, homogeneous locally nilpotent derivation. Suppose that $A = \ker \Delta$ is a UFD and that each homogeneous prime element of A is a prime element of R. Then every derivation $\Delta' : R \to R$ satisfying $\ker \Delta' \supseteq A$ has the form $\Delta' = \rho\Delta$ for some $\rho \in R$.*

Proof. If $A = R$ then $\Delta' = 0$ and the assertion is trivial. Assume that $A \neq R$.

Choose a homogeneous $t \in R$ such that $\Delta(t) \in A \setminus \{0\}$ and consider the multiplicatively closed set $S = \{1, \alpha, \alpha^2, \dots\} \subseteq A$ where $\alpha = \Delta(t)$. Then 1.2 gives $S^{-1}R = (S^{-1}A)[t] = (S^{-1}A)^{[1]}$ and $S^{-1}\Delta$ and $S^{-1}\Delta'$ are $(S^{-1}A)$-derivations going from $(S^{-1}A)[t]$ to itself. Thus $S^{-1}\Delta = \alpha \cdot \frac{d}{dt}$ and $S^{-1}\Delta' = \Delta'(t) \cdot \frac{d}{dt}$, so

$$\alpha\Delta' = \Delta'(t)\,\Delta. \tag{1}$$

Consider a factorization $\alpha = \lambda \prod_i p_i^{e_i}$ where $\lambda \in A^*$, $e_i \in \mathbb{N}$ and each p_i is a prime element of A. If some p_i divides $\Delta'(t)$ then we may cancel it both sides of equation (1); this yields

$$\alpha'\Delta' = \rho\,\Delta$$

where $\alpha' \mid \alpha$ in A, $\rho \in R$ and no prime factor p_i of α' divides ρ. In particular, $\alpha' \mid \rho\,\Delta r$ in R, for every $r \in R$.

If $\alpha' \notin A^*$ then $p_i \mid \alpha'$ for some i. Since p_i is a homogeneous prime element of A, our assumption implies that p_i is a prime element of R. By irreducibility of Δ, we may choose $r \in R$ such that $p_i \nmid \Delta r$; then $p_i \mid \rho$, a contradiction. Thus $\alpha' \in A^*$ and the lemma is proved. $\square$

We now list the facts needed for the proof of Theorem 2.2. We begin with an "exercise" left to the reader:

1.6. *Let $\mathbf{k}$ be a field, $A = \mathbf{k}^{[r]}$ ($r \geq 1$) and let $A = \oplus_{i \in \mathbb{N}} A_i$ be a grading such that $A_0 = \mathbf{k}$. If $f_1, \dots, f_n$ are homogeneous elements of A satisfying $\mathbf{k}[f_1, \dots, f_n] = A$, then there is a subset $\{g_1, \dots, g_r\}$ of $\{f_1, \dots, f_n\}$ satisfying $A = \mathbf{k}[g_1, \dots, g_r]$.*

Part 1 of 1.7 is due to Miyanishi [7] when $\mathbf{k}$ is algebraically closed; then one uses [6] to deduce the general case. For part 2 of 1.7 (in particular for irreducibility of $\Delta_{(f,g)}$), see Corollary 2.6 of [2].

1.7. *Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}[X_0, X_1, X_2] = \mathbf{k}^{[3]}$.*

1. *If $0 \neq D : B \to B$ is a locally nilpotent derivation, then $\ker D = \mathbf{k}^{[2]}$.*
2. *If $f, g \in B$ are such that $\mathbf{k}[f, g]$ is the kernel of some locally nilpotent derivation of B, then the derivation $\Delta_{(f,g)} : B \to B$ defined by the jacobian determinant*

$$\Delta_{(f,g)}(b) = \left| \frac{\partial(f, g, b)}{\partial(X_0, X_1, X_2)} \right| \qquad (b \in B)$$

is locally nilpotent, irreducible and has kernel $\mathbf{k}[f, g]$.

For the next two facts, let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}[X_0, X_1, X_2] = \mathbf{k}^{[3]}$, let ω_0, ω_1, ω_2 be positive integers satisfying $\gcd(\omega_0, \omega_1, \omega_2) = 1$, and let $B = \oplus_{i \in \mathbb{N}} B_i$ be the grading determined by $B_0 = \mathbf{k}$ and $X_i \in B_{\omega_i}$.

1.8 ([1], Theorem 3.5)**.** *Let $f, g \in B$ be homogeneous and geometrically irreducible. If $\gcd(\deg f, \deg g) = 1$, then the following are equivalent:*

(1) *$B_{(fg)}$ is a polynomial ring in one variable over $\mathbf{k}[f,g]_{(fg)}$.*
(2) *$\mathbf{k}[f,g]$ is the kernel of a homogeneous locally nilpotent derivation of B.*

1.9. *Assume that $\omega_0, \omega_1, \omega_2$ are pairwise relatively prime. If $\mathbf{k}[f,g]$ is the kernel of some locally nilpotent derivation $D : B \to B$, where $f, g \in B$ are homogeneous, then $\gcd(\deg f, \deg g) = 1$.*

Proof. Suppose that $\mathbf{k}[f,g] = \ker D$. Theorem 3.7 of [1] implies, in particular, that if $\gcd(\deg f, \deg g) > 1$ then there exists a homogeneous coordinate system[1] (X, Y, Z) of B satisfying $\gcd(\deg X, \deg Y) > 1$. However, it is easy to see that if some homogeneous coordinate system of B has pairwise relatively prime degrees (which is the case here), then all homogeneous coordinate systems have that property. So we must have $\gcd(\deg f, \deg g) = 1$. $\square$

In 1.10, we gather some facts which can be found in [3];[2] then we deduce 1.11 from 1.10. The proof of Theorem 2.2 requires 1.11.

1.10. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and X a projective algebraic surface over $\mathbf{k}$; assume that X is normal, rational and affine-ruled, and that $\text{Pic}(X_s)$ is a group of rank one, where X_s is the smooth locus of X; moreover, assume that all singularities of X are cyclic quotient (in [3], surfaces satisfying these conditions are said to "satisfy the condition (‡)"). Suppose that $U \neq \emptyset$ is an open subset of X isomorphic to $\mathbb{A}^1 \times \Gamma$, for some curve Γ. Since X is normal and rational, Γ must be an open subset of $\mathbb{P}^1$, so the projection $U \to \Gamma$ determines a rational map $X \to \mathbb{P}^1$; let us consider the linear system[3] Λ on X, without fixed components, determined by that rational map. The following facts are proved in [3]:

(i) *Every member F of Λ has irreducible support, i.e., $F = \nu C$ where $\nu \geq 1$ is an integer and C is an irreducible curve on X.* If $\nu = 1$ (resp. $\nu > 1$) we call F a "reduced" (resp. "multiple") member of Λ.
(ii) *At most two members of Λ are multiple.*
(iii) *$U = X \setminus \text{supp}(F_1 + \cdots + F_n)$ for some distinct members $F_1, \ldots, F_n$ of Λ (then define positive integers $\nu_1, \ldots, \nu_n$ by $F_i = \nu_i C_i$, where C_i is an irreducible curve).*

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[1] We mean: X, Y, Z are homogeneous elements of B such that $B = \mathbf{k}[X, Y, Z]$.
[2] At the time of writing, the numbering of the results, in [3], is not available.
[3] We view Λ as a set of effective divisors; so a "member" of Λ is a divisor of X.

(iv) *All multiple members of* Λ *belong to* $\{F_1, \dots, F_n\}$.

(v) *For a subset* $\{F_i, F_j\}$ *of* $\{F_1, \dots, F_n\}$ *(with* $i \neq j$*), the following are equivalent:*
- $\{F_i, F_j\}$ *contains all multiple members of* Λ*;*
- *the isomorphism* $U \cong \mathbb{A}^1 \times \Gamma$ *extends to an isomorphism* $X \setminus \operatorname{supp}(F_i + F_j) \cong \mathbb{A}^1 \times (\mathbb{P}^1 - \text{two points})$.

Moreover, if these conditions hold then $\operatorname{Pic}(X_s) = \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$*, with* $d = \gcd(\nu_i, \nu_j)$.

In 1.11, given a homogeneous polynomial $h \in B$ with prime factorization $h = p_1^{e_1} \cdots p_r^{e_r}$, let $\operatorname{div}_0(h)$ denote the effective divisor $\sum_i e_i V(p_i)$ of X, where $V(p_i) \subset X$ is the zero set of p_i.

Corollary 1.11. *Let* $B = \mathbf{k}[X_0, X_1, X_2]$ *be the graded polynomial ring defined in the statement of Theorem* 2.2*, assume that* $\mathbf{k}$ *is algebraically closed and consider the weighted projective plane* $X = \operatorname{Proj} B$. *Suppose that* $U \neq \emptyset$ *is an open subset of* X *isomorphic to* $\mathbb{A}^1 \times \Gamma$*, for some curve* Γ*, and consider the linear system* Λ *on* X *determined by the projection* $U \to \Gamma$*, as in* 1.10*. Then:*

(1) $U = X \setminus (V(f_1) \cup \cdots \cup V(f_n))$*, for some homogeneous irreducible elements* $f_1, \dots, f_n$ *of* B *(no two of which are associates).*

(2) *For each* $i = 1, \dots, n$*, there exists an integer* $\nu_i \geq 1$ *such that* $\operatorname{div}_0(f_i^{\nu_i}) \in \Lambda$.

(3) *If* $n \geq 2$ *then there exist distinct elements* $i, j \in \{1, \dots, n\}$ *satisfying:*
  (a) $X \setminus (V(f_i) \cup V(f_j)) \cong \mathbb{A}^1 \times (\mathbb{P}^1$ *minus two points*$)$
  (b) $\Lambda = \left\{ \operatorname{div}_0(\lambda f_i^{\nu_i} + \mu f_j^{\nu_j}) \mid (\lambda : \mu) \in \mathbb{P}^1 \right\}$
  (c) *For every* $(\lambda : \mu) \in \mathbb{P}^1 \setminus \{(0:1), (1:0)\}$*,* $\lambda f_i^{\nu_i} + \mu f_j^{\nu_j}$ *is irreducible in* B*.*
  (d) *For every* $k \in \{1, \dots, n\} \setminus \{i, j\}$*,* $\nu_k = 1$ *and* $f_k = \lambda f_i^{\nu_i} + \mu f_j^{\nu_j}$ *for some* $(\lambda : \mu) \in \mathbb{P}^1 \setminus \{(0:1), (1:0)\}$*.*
  (e) $\gcd(\nu_i, \nu_j) = 1$.

Proof. The weighted projective plane X is normal and rational, the Picard group of its smooth locus is $\mathbb{Z}$, and all its singularities are cyclic quotient, so we may apply 1.10. Assertions (1) and (2) follow immediately from parts (i) and (iii) of 1.10. Assume that $n \geq 2$. By (ii) and (iv), there exists a subset $\{i, j\}$ of $\{1, \dots, n\}$ (with $i \neq j$) satisfying

(2) $\{\operatorname{div}_0(f_i^{\nu_i}), \operatorname{div}_0(f_j^{\nu_j})\}$ contains all multiple members of Λ.

Then (v) gives (3a) and (3e); for (3b), simply note that Λ has (projective) dimension 1 and that $\operatorname{div}_0(f_i^{\nu_i})$ and $\operatorname{div}_0(f_j^{\nu_j})$ are distinct members of Λ. If $(\lambda : \mu) \notin \{(0:1), (1:0)\}$ then, since $\{i, j\}$ satisfies (2), $\operatorname{div}_0(\lambda f_i^{\nu_i} + \mu f_j^{\nu_j})$ is a reduced member of Λ; this gives (3c). For (3d), note that $\operatorname{div}_0(f_k^{\nu_k}) \in \Lambda$ implies $f_k^{\nu_k} = \lambda f_i^{\nu_i} + \mu f_j^{\nu_j}$, and $k \notin \{i, j\}$ implies that $\operatorname{div}_0(f_k^{\nu_k})$ is reduced, so $\nu_k = 1$. □

Remark. One consequence of this paper is that $(\nu_i, \nu_j) = (\deg f_j, \deg f_i)$, in part (3) of 1.11. Indeed, we have $(\nu_i, \nu_j) = \frac{1}{d}(\deg f_j, \deg f_i)$, where $d = \gcd(\deg f_i, \deg f_j)$, and the Corollary stated in the introduction implies that $d = 1$, because $B_{(f_i f_j)}$ is a polynomial ring in one variable over a subring.

2. The results

Theorem 2.1. *Let B be an affine UFD over a field $\mathbf{k}$ of characteristic zero and let $x_1, \dots, x_n$ ($n \geq 2$) be prime elements of B no two of which are associates. Suppose that $B = \mathbf{k}[x_1, \dots, x_n]$ and that $B = \oplus_{i \in \mathbb{Z}} B_i$ is a $\mathbb{Z}$-grading such that $\mathbf{k} \subseteq B_0$, each x_i is homogeneous and*

(i) $\gcd(\deg(x_1), \dots, \deg(x_{i-1}), \deg(x_{i+1}), \dots, \deg(x_n)) = 1$, *for all $i = 1, \dots, n$.*

Suppose that A is a homogeneous subalgebra of B satisfying $A \not\subseteq B_0$ and the following conditions:

(ii) *$A^* = B^*$, A is a UFD and every homogeneous prime element of A is a prime element of B.*

(iii) *$A = \mathbf{k}[S]$ and $B_{(S)} = A_{(S)}{}^{[1]}$, for some homogeneous multiplicatively closed subset S of A.*

Then $\gcd\{i \mid A_i \neq 0\} = 1$ and A is the kernel of a homogeneous locally nilpotent derivation $D : B \to B$.

Proof. Let $d = \gcd\{i \mid A_i \neq 0\}$ and let $R = \oplus_{i \in \mathbb{Z}} R_i$ be the homogeneous subring of B defined by $R_i = B_i$ for all $i \in d\mathbb{Z}$ and $R_i = 0$ otherwise. Note that $A \subseteq R$ and that R is finitely generated as a $\mathbf{k}$-algebra. Since $A \not\subseteq B_0$, we have $d \geq 1$; in particular, B is integral over R. Also, observe that

(3) If $r \in R \setminus \{0\}$ is homogeneous, then $\deg r = \deg s_1 - \deg s_2$ for some $s_1, s_2 \in S$.

To see this, note that the assumptions $\mathbf{k} \subseteq B_0$ and $A = \mathbf{k}[S]$ imply that the set $E = \{\deg s \mid s \in S\}$ is equal to $\{i \mid A_i \neq 0\}$, so $\deg r$ belongs to the ideal (of $\mathbb{Z}$) generated by E; since E is closed under addition, $\deg r = e_1 - e_2$ for some $e_1, e_2 \in E$.

We have $B_{(S)} = A_{(S)}[\frac{h}{\sigma}]$, for some $\frac{h}{\sigma} \in B_{(S)}$, where h is a homogeneous element of B, $\sigma \in S$ and $\deg h = \deg \sigma$ (so $h \in R$). We claim that

$$S^{-1}R = (S^{-1}A)[h] = (S^{-1}A)^{[1]}, \tag{4}$$

where $S^{-1}R \supseteq (S^{-1}A)[h]$ is obvious. If r is any nonzero homogeneous element of R then, by (3), $rs_2/s_1 \in B_{(S)}$ for some $s_1, s_2 \in S$. Thus $rs_2/s_1 \in A_{(S)}[\frac{h}{\sigma}]$ and it follows that $r \in (S^{-1}A)[h]$. This shows that $R \subseteq (S^{-1}A)[h]$, so the equality $S^{-1}R = (S^{-1}A)[h]$ holds. It remains to show that h is transcendental over $S^{-1}A$. If not, then h/σ is algebraic over $S^{-1}A$, hence algebraic over A, so there is a nonzero $f(T) = \sum a_i T^i \in A[T]$ satisfying $f(h/\sigma) = 0$. We may arrange that all nonzero a_i are homogeneous and of the same degree; then, by (3), we can find $s_1, s_2 \in S$ such that $\frac{s_2}{s_1} f(T) \in A_{(S)}[T]$, which is absurd because h/σ is transcendental over $A_{(S)}$. So, (4) holds.

Next, we show:

(5) If at least one of $b, b' \in B$ is homogeneous and $bb' \in A \setminus \{0\}$, then $b, b' \in A$.

For this, it's enough to prove the case where both b and b' are homogeneous. Consider a factorization $bb' = \mu \prod_{i\in I} p_i$ where $\mu \in A^*$ and each p_i is a prime (and homogeneous) element of A. By assumption (ii), each p_i is then a prime element of B so $b = \lambda \prod_{j\in J} p_j$ where $\lambda \in B^* \subset A$ and $J \subseteq I$. So $b \in A$ and, similarly, $b' \in A$.

From (5), we easily deduce that

$$R \cap S^{-1}A = A. \tag{6}$$

In fact, if $r \in R \cap S^{-1}A$ then $r = a/s$ ($a \in A$, $s \in S$), so $rs \in A$; since $s \neq 0$ is homogeneous, (5) implies that $r \in A$.

By (4), (6) and the fact that R is **k**-affine, we obtain

(7) $A = \ker \Delta$, for some irreducible, homogeneous locally nipotent derivation $\Delta : R \to R$.

In fact, the "h-derivative" $\frac{d}{dh} : (S^{-1}A)[h] \to (S^{-1}A)[h]$ is a homogeneous locally nilpotent derivation with kernel $S^{-1}A$. Since R is finitely generated as an A-algebra, there exists $s \in S$ such that the derivation $s\frac{d}{dh}$ maps R into itself; the restriction $\Delta' : R \to R$ of $s\frac{d}{dh}$ is a homogeneous derivation with kernel $R \cap S^{-1}A = A$, and is locally nilpotent because $s \in \ker(\frac{d}{dh})$ (see 1.4). By 1.3, we have $\Delta' = \rho'\Delta$, where $\rho' \in R$ and $\Delta : R \to R$ is an irreducible derivation; since Δ is homogeneous and locally nilpotent (1.4) and has the same kernel as Δ', we proved (7).

Extend Δ to a derivation $D' : \operatorname{Frac} B \to \operatorname{Frac} B$ and let $m = \left(\prod_{i=1}^n x_i\right)^{d-1}$; then mD' maps B into itself. Indeed, for each i we have $dx_i^{d-1}D'(x_i) = D'(x_i^d) = \Delta(x_i^d) \in R$, so $mD'x_i \in B$. Hence, the restriction $D'' : B \to B$ of mD' is a derivation and satisfies

$$D''(r) = m\Delta(r), \quad \text{for all } r \in R.$$

Note that D'' must be homogeneous, because its restriction to R is.

Using 1.3, write $D'' = \beta D$ where β is a homogeneous element of B and $D : B \to B$ is an irreducible, homogeneous derivation. Then

$$D(r) = \frac{m}{\beta}\Delta(r), \quad \text{for all } r \in R.$$

We claim that β divides m in B. To see this, consider the set $\mathcal{M}$ of all monomials $M = x_1^{i_1} \cdots x_n^{i_n}$ ($i_1, \dots, i_n \in \mathbb{N}$) satisfying $\deg(M) + \deg(D) \in d\mathbb{Z}$. Given any $M \in \mathcal{M}$, the derivation $MD : B \to B$ maps R into itself, so we may consider the restriction $\Delta_M : R \to R$ of MD.

Observe that $\Delta : R \to R$ satisfies the hypothesis of Lemma 1.5 (if p is a homogeneous prime element of A then, by assumption (ii), p is a prime element of B, and it follows immediately that p is a prime element of R). Since $\ker \Delta_M = \ker \Delta$, Lemma 1.5 implies that $\Delta_M = \rho_M \Delta$ for some $\rho_M \in R$. Note that $\Delta \neq 0$, choose $r \in R$ such that $\Delta r \neq 0$ and write

$$\rho_M \Delta r = M\, Dr = \frac{Mm}{\beta}\Delta r,$$

which implies that $Mm/\beta = \rho_M \in R$. In particular, $\beta \mid Mm$ in B, and this holds for all $M \in \mathcal{M}$. By assumption (i) we have $\gcd(\mathcal{M}) = 1$ in B, so $\beta \mid m$ in B. Thus,

$$Dr = \gamma\, \Delta r, \quad \text{for all } r \in R,$$

where $\gamma = m/\beta = \lambda \prod_{i=1}^{n} x_i^{e_i}$, $\lambda \in B^*$, $e_i \in \mathbb{N}$.

Suppose that $e_1 > 0$. By assumption (i), we may choose $q_2, \dots, q_n \in \mathbb{N}$ such that $\deg(x_1) + q_2 \deg(x_2) + \cdots + q_n \deg(x_n) \in d\mathbb{Z}$. Let $N = x_2^{q_2} \cdots x_n^{q_n}$, then $\deg(x_1 N) \in d\mathbb{Z}$, so $x_1 N \in R$ and consequently

$$\gamma\, \Delta(x_1 N) = D(x_1 N) = (Dx_1)N + x_1 DN \implies x_1 \mid Dx_1.$$

Moreover, for each $j \neq 1$ we have $x_j^d \in R$, so

$$\gamma\, \Delta(x_j^d) = D(x_j^d) = d x_j^{d-1} Dx_j \implies x_1 \mid Dx_j,$$

which is absurd because D is irreducible. Hence, $e_1 = 0$ and, by symmetry, $e_j = 0$ for all j. So $\gamma \in B^*$ and we proved:

(8) Δ extends to a homogeneous derivation $D : B \to B$.

Since B is integral over R, 1.1 gives

(9) $D : B \to B$ is locally nilpotent.

Note that if α is a homogeneous element of $\ker D$ then $\alpha^d \in R \cap \ker D = \ker \Delta = A$, so $\alpha \in A$ by (5). This implies that $\ker D \subseteq A$, because $\ker D$ is a homogeneous subring of B. So

$$\ker D = A.$$

Let $a = \Delta h = Dh$, where $h \in R$ is as in (4). Then $a \in A \setminus \{0\}$; since $D : B \to B$ (resp. $\Delta : R \to R$) is locally nilpotent and has kernel A, 1.2 implies that

$$B_a = A_a[h] \quad (\text{resp. } R_a = A_a[h])$$

so $B_a = R_a$. It follows that $R = B$, so $d = 1$. $\square$

Theorem 2.2. *Let $B = \mathbf{k}[X_0, X_1, X_2] = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is a field of characteristic zero, let ω_0, ω_1, ω_2 be pairwise relatively prime positive integers and let $B = \oplus_{i \in \mathbb{N}} B_i$ be the grading determined by $B_0 = \mathbf{k}$ and $X_i \in B_{\omega_i}$. Consider elements $f_1, \dots, f_n$ of B ($n \geq 2$) which are homogeneous, geometrically irreducible and no two of which are associates. Then the following are equivalent:*

1. *$B_{(f_1 \cdots f_n)}$ is a polynomial ring in one variable over a subring.*
2. *$\mathbf{k}[f_1, \dots, f_n]$ is the kernel of a nonzero homogeneous locally nilpotent derivation $D : B \to B$.*

Moreover, if these equivalent conditions are satisfied then

3. *$\mathbf{k}[f_1, \dots, f_n] = \mathbf{k}[f_i, f_j]$, for some distinct $i, j \in \{1, \dots, n\}$, and any such i, j satisfy* $\gcd(\deg f_i, \deg f_j) = 1$.
4. $B_{(f_1 \cdots f_n)} = (\mathbf{k}[f_1, \dots, f_n]_{(f_1 \cdots f_n)})^{[1]}$.

Proof.

Step 1. We show that, under the assumption that $\mathbf{k}$ is algebraically closed, (1) implies (2) and (3).

Assume that (1) holds and let $A = \mathbf{k}[f_1, \dots, f_n]$. Consider the weighted projective plane $X = \operatorname{Proj} B$; by (1), the open set $U = X \setminus (V(f_1) \cup \dots \cup V(f_n))$ is isomorphic to the product of $\mathbb{A}^1$ with a curve. Consider distinct $i, j \in \{1, \dots, n\}$ satisfying (3a–e) of 1.11. Then part (3d) gives that $A = \mathbf{k}[f_i, f_j]$, so $A = \mathbf{k}^{[2]}$ is a UFD; and it follows from part (3c) that every homogeneous prime element of A is prime in B. Now we claim:

(10) $$B_{(f_i f_j)} = A_{(f_i f_j)}{}^{[1]}.$$

If this is the case then (2) and (3) follow immediately from Theorem 2.1, using $S = \{f_i^k f_j^\ell \mid k, \ell \in \mathbb{N}\}$.

By part (3a) of 1.11, we have $B_{(f_i f_j)} = R^{[1]}$ for a subring R of $B_{(f_i f_j)}$ satisfying $R = \mathbf{k}[\zeta, \zeta^{-1}]$ with ζ transcendental over $\mathbf{k}$. Thus

$$(B_{(f_i f_j)})^* = R^* = \bigcup_{n \in \mathbb{Z}} \mathbf{k}^* \zeta^n.$$

On the other hand, if we define $p' = \deg(f_i)$, $q' = \deg(f_j)$, $(p, q) = \frac{1}{\gcd(p', q')}(p', q')$ and $\xi = f_i^q / f_j^p$ then it is easy to see that

$$(B_{(f_i f_j)})^* = \bigcup_{n \in \mathbb{Z}} \mathbf{k}^* \xi^n,$$

from which we obtain $\zeta = \lambda \xi^{\pm 1}$ ($\lambda \in \mathbf{k}^*$). So $R = \mathbf{k}[\xi, \xi^{-1}] = A_{(f_i f_j)}$, (10) holds and Step 1 is complete.

Step 2. We show that (1) implies (2) and (3) (without assuming that $\mathbf{k}$ is algebraically closed).

Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$ and $\bar{B} = \bar{\mathbf{k}}[X_0, X_1, X_2] = \bar{\mathbf{k}}^{[3]}$. If (1) holds, it follows that $\bar{B}_{(f_1 \cdots f_n)}$ is a polynomial ring in one variable over a subring. Since the f_i are irreducible in $\bar{B}$ by assumption, Step 1 implies that $\bar{\mathbf{k}}[f_1, \dots, f_n] = \ker \bar{D}$ for some homogeneous locally nilpotent derivation $0 \neq \bar{D} : \bar{B} \to \bar{B}$, and that, for some i, j, $\bar{\mathbf{k}}[f_i, f_j] = \bar{\mathbf{k}}[f_1, \dots, f_n]$ and $\gcd(\deg f_i, \deg f_j) = 1$. By 1.7, the derivation $\bar{D} = \Delta_{(f_i, f_j)} : \bar{B} \to \bar{B}$ satisfies the requirements. Since this $\bar{D}$ maps the X_i to elements of B, it restricts to a derivation $D : B \to B$ (locally nilpotent and homogeneous). Since $\ker D = \bar{\mathbf{k}}[f_i, f_j] \cap B = \mathbf{k}[f_i, f_j]$ and $\mathbf{k}[f_1, \dots, f_n] \subseteq \bar{\mathbf{k}}[f_i, f_j] \cap B = \mathbf{k}[f_i, f_j]$, (2) and (3) hold and Step 2 is complete.

Step 3. We show that (2) implies (4).

Assume that (2) holds. Then 1.7 implies that $\mathbf{k}[f_1, \dots, f_n] = \mathbf{k}^{[2]}$, so, by 1.6, $\mathbf{k}[f_1, \dots, f_n] = \mathbf{k}[f_i, f_j]$ for some i, j. Since (by 1.9) $\gcd(\deg f_i, \deg f_j) = 1$, we may apply 1.8 and conclude that

(11) $$B_{(f_i f_j)} = (A_{(f_i f_j)})^{[1]},$$

where $A = \mathbf{k}[f_i, f_j] = \mathbf{k}[f_1, \ldots, f_n]$. Now (11) implies that $B_{(f_1 \cdots f_n)} = (A_{(f_1 \cdots f_n)})^{[1]}$, so (4) holds and the proof is complete. □

Remark. The Corollary stated in the introduction (hence, also Theorem 2.2) is no longer true if we replace the assumption "geometrically irreducible" by the weaker "irreducible". Indeed, consider $B = \mathbb{Q}[X_0, X_1, X_2]$ with the standard total degree grading ($\deg(X_i) = 1$), and let $f = X_0$ and $g = X_0^2 + X_1^2$. Then $B_{(fg)} = (\mathbf{k}[X_0, X_1]_{(fg)})[\frac{X_2}{X_0}] = (\mathbf{k}[X_0, X_1]_{(fg)})^{[1]}$ but $\mathbf{k}[f, g] = \mathbf{k}[X_0, X_1^2]$ is not the kernel of a derivation of B.

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