HOMOGENEOUS LOCALLY NILPOTENT DERIVATIONS OF $\mathbf{k}[X,Y,Z]$

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Abstract. Let $\mathbf{k}$ be a field of characteristic zero and let $B = \mathbf{k}[X,Y,Z]$ be a polynomial ring in three variables. A derivation $D : B \to B$ is said to be *locally nilpotent* if for each $b \in B$ we have $D^n(b) = 0$ for $n \gg 0$. This paper describes the class of homogeneous locally nilpotent derivations of B, where homogeneity is relative to any $\mathbf{N}$-grading of B which is obtained by assigning weights to the variables, $w(X) = a$, $w(Y) = b$ and $w(Z) = c$, with $a, b, c \in \mathbf{N}$ and $\gcd(a,b,c) = 1$.

It is known that the kernel of a derivation from this class is a subalgebra $\mathbf{k}[f,g]$ of $B = \mathbf{k}[X,Y,Z]$, where f and g are w-homogeneous and algebraically independent, and it is also known that such a derivation is essentially determined by its kernel. The main results of this paper give explicit conditions on $f, g \in \mathbf{k}[X,Y,Z]$ which are equivalent to $\mathbf{k}[f,g]$ being the kernel of a homogeneous locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$.

Let B be a ring and let D be a derivation of B (by this, we mean a $\mathbf{Z}$-derivation going from B to B). We say that D is *locally nilpotent* if for each $b \in B$ we have $D^n(b) = 0$ for $n \gg 0$.

Several researchers have investigated locally nilpotent derivations of $\mathbf{k}[X_1, \dots, X_n]$, where $\mathbf{k}$ is a field of characteristic zero. These derivations are fully understood for $n < 3$ (see [14]), partially understood when $n = 3$, and very little is known when $n > 3$. Instead of counting the number n of variables, one may consider the "rank" of derivations (Definition 1.9); then one can say that locally nilpotent derivations D of $\mathbf{k}[X_1, \dots, X_n]$ are well understood when $\operatorname{rank} D < 3$ (see [4]), and very mysterious when $\operatorname{rank} D \geq 3$. In view of Freudenburg's recent discovery [6] that there exist rank 3 locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$, it is now clear that our understanding of the derivations of $\mathbf{k}[X,Y,Z]$ is very incomplete.

It would be very nice to classify all locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$, but that problem seems to be quite difficult. This paper addresses the less ambitious problem of describing the class of *homogeneous* locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$; to be precise, we consider homogeneity with respect to any $\mathbf{N}$-grading of $\mathbf{k}[X,Y,Z]$ which is obtained by assigning weights to the variables, say $w(X) = a$, $w(Y) = b$ and $w(Z) = c$, where $a, b, c \in \mathbf{N}$ and $\gcd(a,b,c) = 1$.

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It is known that the kernel of such a derivation is a subalgebra $\mathbf{k}[f,g]$ of $\mathbf{k}[X,Y,Z]$, where f and g are w-homogeneous and algebraically independent, and it is also known that such a derivation is essentially determined by its kernel. Hence our task is to understand which $f,g \in \mathbf{k}[X,Y,Z]$ are such that $\mathbf{k}[f,g]$ is a kernel. One should note that some ring-theoretical characterizations of the kernels are relatively easy to obtain (1.3 for instance), but these are unsatisfactory—they don't say anything explicit about f and g.

We now state the solution to our problem; for the purpose of this introduction, we restrict ourselves to the special case where $\mathbf{k}$ is algebraically closed and $a=b=c=1$, i.e., the grading is the standard one (the statement we give here is a corollary of Theorem 3.5 and Corollary 3.1):

Theorem . *Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and let $f,g \in B=\mathbf{k}[X,Y,Z]$ be homogeneous polynomials. Then the following conditions are equivalent:*

(1) *$\mathbf{k}[f,g]$ is the kernel of some homogeneous locally nilpotent derivation of B;*
(2) *f,g are irreducible and the algebraic surface $\mathbf{P}^2 \setminus V(fg)$ is isomorphic to $\mathbf{P}^2$ minus two lines.*

In this statement, $\mathbf{P}^2$ is the projective plane over $\mathbf{k}$ and $V(fg)$ denotes the union of the two curves "$f=0$" and "$g=0$" in $\mathbf{P}^2$.

The theorem enables us to construct infinite families of rank 3 homogeneous locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$. Indeed, if f,g satisfy the equivalent conditions of the theorem, then $\Delta_{(f,g)}$ (see 1.6 and 3.4) is a homogeneous locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$ with kernel $\mathbf{k}[f,g]$; moreover, $\Delta_{(f,g)}$ has rank 3 if and only if $\deg f > 1$ and $\deg g > 1$. Obviously, this raises the following:

Question . Which pairs of irreducible curves C_1, C_2 in $\mathbf{P}^2$ have the property that $\mathbf{P}^2 \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbf{P}^2$ minus two lines?

There is no doubt that this question is interesting in itself, and that there exists relevant literature about it ([9], [13], [16], [18], ...), but it is not entirely clear (to this author) to what extent it has been satisfactorily answered. In any case, we stress that we do not address the above question in this paper, except for the following:

Example . We answer the above question in the special case $\deg C_1 = 2 \le \deg C_2$. (In view of the above discussion, this gives rise to an infinite family of examples of rank three homogeneous locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$.)

(1) Given a finite sequence $n=(n_1,\dots,n_\ell)$ of positive integers (where $\ell \ge 1$), define an infinite sequence μ_n as follows:
  - Let $i_1,\dots,i_\ell$ be the integers given by $i_\ell = 2$ and $i_{\nu-1} = (4n_\nu+1)i_\nu$;
  - let $S_1,\dots,S_\ell$ be the sequences given by

$$S_\nu = (n_\nu i_\nu, n_\nu i_\nu, n_\nu i_\nu, n_\nu i_\nu, \underbrace{i_\nu,\dots,i_\nu}_{2n_\nu});$$

- let μ_n be the concatenation of $S_1, \dots, S_\ell$ and of the infinite constant sequence $(1, 1, \dots)$, i.e., $\mu_n = (n_1 i_1, \dots, i_\ell, 1, 1, \dots)$.

Observe that n can be recovered from μ_n, so that $n \mapsto \mu_n$ is an injective map.

(2) Fix an irreducible curve C_1 in $\mathbf{P}^2$, of degree 2, and let $P \in C_1$. One can show that, for each choice of a sequence n, there exists at least one irreducible curve $C_2^{(n)}$ (in $\mathbf{P}^2$) which satisfies:
- $C_1 \cap C_2^{(n)} = \{P\}$ and $C_2^{(n)}$ has a cusp at P;
- $\deg C_2^{(n)} = \displaystyle\prod_{\nu=1}^{\ell} (4n_\nu + 1)$;
- the multiplicity sequence[1] of $C_2^{(n)}$ at P is μ_n.

Moreover, one can see that any such $C_2^{(n)}$ satisfies: $\mathbf{P}^2 \setminus (C_1 \cup C_2^{(n)}) \cong \mathbf{P}^2$ minus two lines.

(3) Conversely, if C_1 and C_2 are irreducible curves in $\mathbf{P}^2$ such that $\deg C_1 = 2 \leq \deg C_2$ and $\mathbf{P}^2 \setminus (C_1 \cup C_2) \cong \mathbf{P}^2$ minus two lines, then one can prove that C_1 and C_2 meet in one point P, C_2 has a cusp at P and there exists a unique sequence $n = (n_1, \dots, n_\ell)$ such that μ_n is the multiplicity sequence of C_2 at P and $\deg C_2 = \prod_{\nu=1}^{\ell} (4n_\nu + 1)$.

Remark . We think that the above family of examples should be mentioned in this paper but, unfortunately, we have to omit the proof. A satisfactory proof would be rather long and would use techniques completely different from those of this paper.

Remark . In the notation of the above example, Freudenburg's example [6] corresponds to the sequence $n = (1)$ (i.e., $\ell = 1$ and $n_1 = 1$).

1. Preliminaries

Throughout this paper, all rings are commutative and have an identity element. If B is a ring then B^* denotes the group of units of B; if B is an integral domain then $\operatorname{qt} B$ denotes the field of fractions of B.

If B is an A-algebra and $n > 0$ is an integer then the notation $B = A^{[n]}$ means that B is A-isomorphic to the polynomial algebra in n variables over A. Suppose that $\mathbf{k}$ is a field and $B = \mathbf{k}^{[n]}$; by a *variable* of B we mean an element $b \in B$ which satisfies $B = \mathbf{k}[b, b_2, \dots, b_n]$ for some $b_2, \dots, b_n \in B$; by a *coordinate system* of B we mean an ordered n-tuple $(b_1, \dots, b_n)$ of elements of B such that $B = \mathbf{k}[b_1, \dots, b_n]$.

A subring A of a domain B is said to be *inert in* B (or *factorially closed in* B) if for all $x, y \in B$ we have $xy \in A \setminus \{0\} \Rightarrow x, y \in A$.

A subring A of a domain B is said to be *algebraically closed in* B if, for every $b \in B$ and every nonzero polynomial $f(T) \in A[T]$, we have $f(b) = 0 \Rightarrow b \in A$.

[1]We mean the sequence $(\mu(C_2^{(n)}, P_i))_{i=0}^{\infty}$, where $P_0 = P$, P_i is the i-th infinitely near point over P and $\mu(C_2^{(n)}, P_i)$ is the multiplicity of P_i on the suitable strict transform of $C_2^{(n)}$.

The following fact will be used in the proof of Theorem 3.5. It can be derived from the proof of Theorem 2.3.1 of [15].

1.1. *Let $A \subset B$ be integral domains, where B is finitely generated as an A-algebra. Suppose that there exists a multiplicatively closed subset S of A such that $S^{-1}B = (S^{-1}A)^{[1]}$, and that every element of $S \setminus A^*$ is a finite product of elements π of A satisfying:*

(1) *π is a prime element of B*
(2) *$\pi B \cap A = \pi A$*
(3) *$A/\pi A$ is algebraically closed in $B/\pi B$.*

Then $B = A^{[1]}$.

1.2. Basic properties of locally nilpotent derivations. Let B be an integral domain of characteristic zero, let $D : B \to B$ be a nonzero derivation of B, and let $A = \ker D$. Then the following hold:

(1) If D is locally nilpotent then A is an inert subring of B. In particular, if D is locally nilpotent and B is a UFD then A is a UFD.
(2) Let S be a multiplicatively closed subset of $B \setminus \{0\}$, and consider the derivation $S^{-1}D$ of $S^{-1}B$. Then
   (a) $S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subseteq A$.
   (b) If $S \subseteq A$ then $\ker S^{-1}D = S^{-1}A$ and $S^{-1}A \cap B = A$.
(3) A is algebraically closed in B; in particular, $B \cap \operatorname{qt} A = A$.
(4) Assume that $\mathbf{Q} \subseteq B$ and that D is locally nilpotent.
   (a) If $D(b) = 1$ for some $b \in B$, then $B = A[b] = A^{[1]}$.
   (b) More generally, let b be any element of B such that $Db \neq 0$ and $D^2b = 0$, and write $a = Db \in A \setminus \{0\}$. Then $B_a = A_a[b] = A_a^{[1]}$.
(5) Let $b \in B \setminus \{0\}$. The derivation bD (of B) is locally nilpotent if and only if D is locally nilpotent and $b \in A$.

References. (1) is given in 2^{o} of [14]; for (2), see the proposition in [8]; (3) is well-known; (4 a) is Proposition 2.1 of [17]; (4 b) is an easy consequence of (4 a), and is pointed out in [5]; the "if" part of (5) is trivial, and the "only if" part can be derived from the proof of the Claim in [8].

1.3. ([3], Proposition 1.4) *Let B be an integral domain of characteristic zero and let $A \neq B$ be a subring of B such that B is finitely generated as an A-algebra. Then the following are equivalent:*

(1) *A is the kernel of some locally nilpotent derivation $D : B \to B$;*
(2) *there exists a multiplicatively closed subset S of $A \setminus \{0\}$ such that $S^{-1}B = (S^{-1}A)^{[1]}$ and $B \cap (S^{-1}A) = A$;*
(3) *$S^{-1}B = (S^{-1}A)^{[1]}$ and $B \cap (S^{-1}A) = A$, where $S = A \setminus \{0\}$.*

The next result will be used in the proof of 3.10. Recall that a locally nilpotent derivation $D : B \to B$ is said to be *irreducible* if the only principal ideal of B which contains $D(B)$ is B itself.

Lemma 1.4. *Let $D : B \to B$ be an irreducible locally nilpotent derivation, where B is a UFD of characteristic zero, let $A = \ker D$ and let $S \subseteq A \setminus \{0\}$ be a multiplicatively closed set such that $S^{-1}B = (S^{-1}A)^{[1]}$. Then each $v \in B$ satisfying $S^{-1}B = (S^{-1}A)[v]$ also satisfies: $Dv \in A \setminus \{0\}$ and, for some $a \in A \setminus \{0\}$, $a\,Dv \in S$.*

Proof. Let $v \in B$ be such that $S^{-1}B = (S^{-1}A)[v]$ and let $\partial : (S^{-1}A)[v] \to (S^{-1}A)[v]$ denote the derivative with respect to v. Being an $(S^{-1}A)$-derivation of $(S^{-1}A)[v]$, $S^{-1}D$ is a multiple of ∂ ; more precisely, $S^{-1}D = \beta\partial$ with $\beta = (S^{-1}D)(v) = Dv \in B$.

Since D is assumed to be irreducible, it is in particular nonzero, so $S^{-1}D \neq 0$ and consequently $\beta \neq 0$. Let $p \in B$ be a prime factor of β. Since D is irreducible, there exists $b \in B$ such that $Db \notin pB$; on the other hand, $Db = \beta\partial b$ (where $\partial b \in S^{-1}B$) so there exists $s \in S$ such that $\beta \mid sDb$ (in B). Thus $p \mid sDb$ and, since $p \nmid Db$, we have $p \mid s$. Hence, each prime factor of $\beta = Dv$ divides some element of S and, consequently, Dv divides some element of S. Then the conclusion follows from the fact that A is an inert subring of B (see 1.2). □

The following theorem was proved by Miyanishi in [12], in the special case where $\mathbf{k}$ is algebraically closed; the general case is then a straightforward consequence of [10].

1.5. *Let $\mathbf{k}$ be a field of characteristic zero and let D be a nonzero locally nilpotent derivation of $\mathbf{k}^{[3]}$. Then $\ker D = \mathbf{k}^{[2]}$.*

It should be noted that, in the homogeneous case, V.D. Zurkowski [19] gave a different proof of 1.5.

1.6. ([3], Corollary 2.6) *Let $\mathbf{k}$ be a field of characteristic zero, $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$, let $D : B \to B$ be a nonzero locally nilpotent derivation and let $f, g \in B$ be such that $\ker D = \mathbf{k}[f, g]$. If $\Delta_{(f,g)} : B \to B$ denotes the $\mathbf{k}$-derivation defined by the jacobian determinant*

$$\Delta_{(f,g)}(b) = \left| \frac{\partial(f, g, b)}{\partial(X, Y, Z)} \right| \qquad (b \in B),$$

then $D = a\Delta_{(f,g)}$ for some $a \in \ker D \setminus \{0\}$. Moreover, $\Delta_{(f,g)}$ is locally nilpotent and irreducible.

We prove two facts which follow easily from 1.6:

Corollary 1.7. *Let $\mathbf{k}$ be a field of characteristic zero and let $f, g \in \mathbf{k}[X, Y, Z]$. Then the following are equivalent:*

1. *$\mathbf{k}[f, g]$ is the kernel of some locally nilpotent derivation of $\mathbf{k}[X, Y, Z]$;*
2. *for all field extensions $K/\mathbf{k}$, $K[f, g]$ is the kernel of some locally nilpotent derivation of $K[X, Y, Z]$;*

(3) *for some field extension $K/\mathbf{k}$, $K[f,g]$ is the kernel of a locally nilpotent derivation of $K[X,Y,Z]$.*

Remark . In the above statement, by $K[X,Y,Z]$ and $K[f,g]$ we mean $K\otimes_{\mathbf{k}}\mathbf{k}[X,Y,Z]$ and $K\otimes_{\mathbf{k}}\mathbf{k}[f,g]$ respectively.

Proof. If D is a locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$ with kernel $\mathbf{k}[f,g]$, and if $K/\mathbf{k}$ is any field extension, then $K\otimes_{\mathbf{k}} D$ is a locally nilpotent derivation of $K[X,Y,Z]$ with kernel $K[f,g]$. Hence (1) implies (2).

Obviously, (2) implies (3). Now assume that (3) holds and let $\Delta_{(f,g)}$ (resp. $\overline{\Delta}_{(f,g)}$) be the derivation of $\mathbf{k}[X,Y,Z]$ (resp. of $K[X,Y,Z]$) defined by means of a jacobian determinant, as in 1.6. Since (3) holds, 1.6 implies that $\overline{\Delta}_{(f,g)}$ is locally nilpotent and has kernel $K[f,g]$. It follows immediately that $\Delta_{(f,g)}$ is locally nilpotent and has kernel $K[f,g]\cap\mathbf{k}[X,Y,Z]=\mathbf{k}[f,g]$. Hence (3) implies (1). $\square$

Corollary 1.8. *Let $\mathbf{k}$ be a field of characteristic zero and let $f,g\in\mathbf{k}[X,Y,Z]$ be such that $\mathbf{k}[f,g]$ is the kernel of some locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$. Then f and g are irreducible elements of $\overline{\mathbf{k}}[X,Y,Z]$, where $\overline{\mathbf{k}}$ denotes the algebraic closure of $\mathbf{k}$.*

Proof. By 1.2 and 1.7, $\overline{\mathbf{k}}[f,g]$ is an inert subring of $\overline{\mathbf{k}}[X,Y,Z]$; since f and g are irreducible elements of $\overline{\mathbf{k}}[f,g]$, it follows that they are irreducible in $\overline{\mathbf{k}}[X,Y,Z]$. $\square$

Definition 1.9. ([7], section 3) Let $\mathbf{k}$ be a field of characteristic zero and D a $\mathbf{k}$-derivation of $B=\mathbf{k}^{[n]}$. The *rank* of D is the least integer $r\geq 0$ for which there exists a coordinate system $(X_1,\dots,X_n)$ of B satisfying $DX_i=0$ for all $i>r$.

Let the notation be as in the above definition. Obviously, $\operatorname{rank} D=0\Leftrightarrow D=0$, and it is easy to see that $\operatorname{rank} D=1$ is equivalent to the two conditions $B=(\ker D)^{[1]}$ and $\ker D=\mathbf{k}^{[n-1]}$. Locally nilpotent derivations of rank at most two are studied in [4]. About derivations of high rank, observe that $\operatorname{rank} D<n$ if and only if $\ker D$ contains some variable of B. Note that, for every integer $n>0$ other than 2, there exists a locally nilpotent derivation of $\mathbf{k}^{[n]}$ of rank n (see [6]).

Definition 1.10. ([2], Definition 2.3) Let B be a domain of characteristic zero, $D: B\to B$ a locally nilpotent derivation of B, $A=\ker D$, and let $\pi:\operatorname{Spec} B\to\operatorname{Spec} A$ be the morphism determined by the inclusion map $A\hookrightarrow B$. Given a prime ideal $\alpha\in\operatorname{Spec} A$, let $\kappa(\alpha)$ denote the field $A_\alpha/\alpha A_\alpha$. Then the fibre $\pi^{-1}(\alpha)$ is, in a natural way, a scheme over $\kappa(\alpha)$. We define

$$\begin{aligned}\mathcal{R}(D)&=\{\alpha\in\operatorname{Spec} A\mid \pi^{-1}(\alpha)\neq\mathbf{A}^1_{\kappa(\alpha)}\}\\&=\{\alpha\in\operatorname{Spec} A\mid \kappa(\alpha)\otimes_A B\neq\kappa(\alpha)^{[1]}\}.\end{aligned}$$

We will also consider the closure $\overline{\mathcal{R}}(D)$ of $\mathcal{R}(D)$ in $\operatorname{Spec} A$.

1.11. ([2], Lemma 2.4) *If D is a locally nilpotent derivation of $B=\mathbf{k}^{[3]}$ of rank 2, then*

(1) $\dim \overline{\mathcal{R}}(D) = 1$.
(2) *Let ξ be an irreducible element of $A = \ker D$ such that $V(\xi)$ is an irreducible component of $\overline{\mathcal{R}}(D)$, and let (X, Y, Z) be a coordinate system of B which satisfies $DX = 0$. Then $\mathbf{k}[X]$ is the integral closure of $\mathbf{k}[\xi]$ in B.*

1.12. Conventions on graded rings. Every graded ring considered in this paper is either $\mathbf{Z}$-graded or $\mathbf{N}$-graded. Let $B = \bigoplus_i B_i$ be a graded domain.

(1) The following notation is convenient: $\mathrm{d}(B) = \gcd\{i \mid B_i \neq 0\}$.
(2) If $b \in B_i \setminus \{0\}$ then we write $\deg b = i$, or sometimes $w(b) = i$.
(3) If S is a multiplicatively closed subset of $\bigcup_i (B_i \setminus \{0\})$ then the localized ring $S^{-1}B$ inherits a $\mathbf{Z}$-grading in a natural way: If $b \in B_i \setminus \{0\}$ and $s \in S$ then b/s is declared to be homogeneous of degree $\deg b - \deg s$. Note that $\mathrm{d}(S^{-1}B) = \mathrm{d}(B)$.
(4) A subring A of B is said to be *homogeneous* if $A = \bigoplus_i (A \cap B_i)$. If this is the case then we set $A_i = A \cap B_i$ and regard $A = \bigoplus_i A_i$ as a graded ring.
(5) If $D : B \to B$ is a derivation, we say that D is *homogeneous* if there exists an integer n such that $D(B_i) \subseteq B_{i+n}$ holds for all i. Note that if D is homogeneous then the subring $\ker D$ of B is homogeneous.
(6) Let $D : B \to B$ be a homogeneous derivation and let $A = \ker D$. Then $\mathrm{d}(B)$ divides $\mathrm{d}(A)$ and we define $\mathrm{d}(D) \in \mathbf{N}$ by

$$\mathrm{d}(D) = \begin{cases} \mathrm{d}(A)/\mathrm{d}(B), & \text{if } \mathrm{d}(B) \neq 0, \\ 1, & \text{if } \mathrm{d}(B) = 0. \end{cases}$$

(Note that $\mathrm{d}(D) = 1 \iff \mathrm{d}(A) = \mathrm{d}(B)$.)

1.13. In this paper, we are particularly interested in the following situation. Let $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is a field of characteristic zero, and let $w = (a, b, c) \in \mathbf{N}^3$ be such that $\gcd(a, b, c) = 1$. *By the symbol (B, w) we mean B regarded as an $\mathbf{N}$-graded ring*, where the grading is determined (in the usual way) by the weights $w(X) = a$, $w(Y) = b$ and $w(Z) = c$. Then we want to study locally nilpotent derivations of B which are w-homogeneous (i.e., homogeneous with respect to the grading determined by w).

If $w = (1, 1, 1)$ then we obtain the standard grading of B. In this case, w-homogeneous derivations are simply called homogeneous derivations.

We close this section by recalling that every algebraic action of $G_m(\mathbf{k})$ on the affine plane $\mathbf{A}^2_{\mathbf{k}}$ is linearizable ([1], [11]). This can be stated as follows:[2]

1.14. *Let $\mathbf{k}$ be a field, $A = \mathbf{k}^{[2]}$, and let $A = \bigoplus_i A_i$ be a $\mathbf{Z}$-grading. Then there exist homogeneous elements f, g of A such that $A = \mathbf{k}[f, g]$.*

[2]The given references assume that $\mathbf{k}$ is algebraically closed, but the result is known to be true for arbitrary $\mathbf{k}$.

2. Derivations of graded rings

If B is a graded ring and $D : B \to B$ is a homogeneous locally nilpotent derivation, then the integer $\mathrm{d}(D)$ (see 1.12) may or may not be equal to 1. The first result of this section gives some information about the case where $\mathrm{d}(D) \neq 1$; its two corollaries are concerned with the special case where $B = \mathbf{k}^{[n]}$.

Proposition 2.1. *Let $B = \bigoplus_i B_i$ be an $\mathbf{N}$-graded UFD containing $\mathbf{Q}$ and $D : B \to B$ a homogeneous locally nilpotent derivation such that $\mathrm{d}(D) \neq 1$. Let $A = \ker D$, $d = \mathrm{d}(A)$ and*

$$\mathcal{H} = \{H \in B \setminus \{0\} \mid H \text{ is homogeneous and prime, and } \deg H \not\equiv 0 \pmod{d}\}.$$

Also, let R be the homogeneous subring of B defined by $R_n = B_n$ if $n \equiv 0 \pmod{d}$, $R_n = 0$ if $n \not\equiv 0 \pmod{d}$. Then:

1. *$\mathcal{H} \neq \emptyset$ and any two elements of $\mathcal{H}$ are associates in B.*
2. *For each $H \in \mathcal{H}$, and for each homogeneous element $v \in B$ such that $Dv \neq 0$ and $D^2v = 0$, we have $v = aH$ for some $a \in A \setminus \{0\}$. In particular, each $H \in \mathcal{H}$ satisfies $DH \neq 0$ and $D^2H = 0$.*
3. *For some integer $n > 0$, $B_0 \cong B_n$ as B_0-modules.*
4. *$B = R[H]$, for every $H \in \mathcal{H}$.*
5. *Let $\delta = \mathrm{d}(D) = d/\mathrm{d}(B)$, assume that $\delta \neq 0$ and let $H \in \mathcal{H}$. Then the restriction of $H^{1-\delta}D$ to R is a homogeneous locally nilpotent derivation $\Delta : R \to R$ satisfying $\ker \Delta = A$ and consequently $\mathrm{d}(\Delta) = 1$. Moreover, R is a UFD, H^δ is a prime element of R, $\Delta(H^\delta) \neq 0$ and $\Delta^2(H^\delta) = 0$.*

Remark . In a sense, part (5) of Proposition 2.1 is a reduction to the case where $\mathrm{d}(D) = 1$. More precisely, if $\mathrm{d}(D) \neq 1$ then we may study the derivation $\Delta : R \to R$, which is closely related to D and which satisfies $\mathrm{d}(\Delta) = 1$.

Proof. Observe that the assumption $\mathrm{d}(D) \neq 1$ (or $d \neq \mathrm{d}(B)$) implies, in particular, that $\mathrm{d}(B) \neq 0$ and $D \neq 0$.

Since B is a UFD and $d \neq \mathrm{d}(B)$, $\mathcal{H}$ is nonempty. Consider $H \in \mathcal{H}$ and let $n = \deg H$. Note that $H \notin A$, because otherwise we would have $H \in A_n$, so $A_n \neq 0$, and this is impossible because $n \not\equiv 0 \pmod{d}$. By 1.2 we have $\mathrm{qt}\, A \cap B = A$, so we obtain

$$H \notin \mathrm{qt}\, A. \tag{1}$$

Since D is homogeneous, locally nilpotent and nonzero, we can find a homogeneous element $v \in B$ such that $Dv \neq 0$ and $D^2v = 0$. Let $\alpha = Dv \in A \setminus \{0\}$, then by part (4) of 1.2 we have

$$B_\alpha = A_\alpha[v] = A_\alpha^{[1]}\ .$$

In particular there exists an integer $m \geq 0$ such that $\alpha^m H \in A[v]$. So we may write $\alpha^m H = \sum_{i \in I} a_i v^i$, where I is a nonempty subset of $\mathbf{N}$ and, for each $i \in I$, $a_i \in A \setminus \{0\}$ is homogeneous and $\deg(a_i v^i) = \deg(\alpha^m H)$. Since $\deg a_i \equiv 0$ and

$\deg \alpha \equiv 0 \pmod{d}$, we obtain $i \deg v \equiv \deg H \not\equiv 0 \pmod{d}$, so in particular $i > 0$ for each $i \in I$. It follows that

$$\alpha^m H = bv, \tag{2}$$

for some $b \in B \setminus \{0\}$. We claim that $H \mid v$ (in B). Indeed, if $H \nmid v$ then $H \mid b$ (because H is a prime element of B), so equation (2) gives $v \mid \alpha^m$, so $v \in A$ by inertness of A in B. This is absurd, because $Dv \neq 0$. Hence $H \mid v$ and, if we write $v = aH$ (with $a \in B$), then equation (2) gives $ab = \alpha^m$. Since A is inert in B it follows that $a, b \in A$, so

$$v = aH, \quad \text{for some } a \in A \setminus \{0\}. \tag{3}$$

Now if $H' \in \mathcal{H}$ we have $v = a'H'$ ($a' \in A \setminus \{0\}$), so

$$aH = a'H'.$$

If $H \mid a'$ then $H \in A$ (because A is inert in B) and this contradicts (1); so $H \nmid a'$ and consequently $H \mid H'$. Similarly we obtain that $H' \mid H$, i.e., we proved part (1) of the proposition. Part (2) follows immediately from equation (3).

Consider the ring R and, again, let $H \in \mathcal{H}$. If β is a nonzero homogeneous element of B, we may write $\beta = \beta_0 H^s$, where $\beta_0 \in B$, $s \in \mathbf{N}$ and $H \nmid \beta_0$. Since $H \nmid \beta_0$, no prime factor of β_0 belongs to $\mathcal{H}$ (by part (1) of the proposition), so every prime factor of β_0 has degree divisible by d; consequently, $\beta_0 \in R$. Hence every nonzero homogeneous element β of B has the form

$$\beta = \beta_0 H^s, \quad \text{where } \beta_0 \in R,\ s \in \mathbf{N} \text{ and } H \nmid \beta_0. \tag{4}$$

In particular, $\beta \in R[H]$. This shows that $B = R[H]$, which is part (4) of the proposition.

From equation (4), we also deduce that the submodule $B_0 H$ of the B_0-module B_n is actually equal to B_n, where $n = \deg H$. Indeed, if $\deg \beta = n$ then $\deg \beta \not\equiv 0 \pmod{d}$, so $\beta \notin R$ and consequently $s > 0$ in (4). Since

$$\deg H = \deg \beta = \deg \beta_0 + s \deg H$$

and $\deg \beta_0 \geq 0$, we must have $s = 1$ and $\deg \beta_0 = 0$, i.e., $\beta \in B_0 H$. This shows that $B_n = B_0 H \cong B_0$, which proves part (3) of the proposition.

At this point, we assume that $\delta = \mathrm{d}(D)$ is nonzero (i.e., $d \neq 0$). Note the following property of δ:

$$\text{For all } i \in \mathbf{Z}, \text{ if } i \deg H \equiv 0 \pmod{d} \text{ then } \delta \mid i. \tag{5}$$

Indeed, the fact that $B = R[H]$ implies that $\gcd(d, \deg H) = \mathrm{d}(B)$, and this easily implies that (5) holds. Let us now check that

$$D(R) \subseteq RH^{\delta - 1}. \tag{6}$$

Since $DH \neq 0$ and $D^2 H = 0$, we have

$$B_\alpha = A_\alpha[H] = A_\alpha^{[1]},$$

where $\alpha = DH \in A\backslash\{0\}$. Thus, if $r \in R\backslash\{0\}$ is a homogeneous element, $\alpha^m r \in A[H]$ for some integer $m \geq 0$. So we may write $\alpha^m r = \sum_{i \in I} a_i H^i$, where I is a nonempty subset of $\mathbf{N}$ and, for each $i \in I$, $a_i \in A \setminus \{0\}$ is homogeneous and $\deg(a_i H^i) = \deg(\alpha^m r)$. Since $\deg a_i \equiv \deg \alpha \equiv \deg r \equiv 0 \pmod d$, we obtain $i \deg H \equiv 0 \pmod d$ and, by observation (5), this implies that $\delta \mid i$ (for each $i \in I$). Hence $\alpha^m r = f(H^\delta)$ for some $f(T) \in A[T]$, so

$$\alpha^m Dr = Df(H^\delta) = f'(H^\delta)\, \delta\, H^{\delta-1}\, \alpha,$$

and since $H^\delta \in R$ we have $\alpha^m Dr \in RH^{\delta-1}$. Since $H \nmid \alpha$ (by (1) and inertness of A in B), we have $H^{\delta-1} \mid Dr$. If we write $\rho = (Dr)/H^{\delta-1}$ then $\rho \in B$ and $\alpha^m \rho \in R$; thus $\deg \rho \equiv 0 \pmod d$, i.e., $\rho \in R$, which proves (6).

It follows that $\Delta : R \to R$ is a well-defined derivation with kernel A, and it is clear that Δ is homogeneous. To see that Δ is locally nilpotent, observe that the equation $\alpha^m r = f(H^\delta)$, which was obtained in the preceding paragraph, actually implies that $R_\alpha = A_\alpha[H^\delta] = A_\alpha^{[1]}$. Since $\Delta(H^\delta) = \delta\alpha$, the localization $\Delta_\alpha : R_\alpha \to R_\alpha$ of Δ is simply the partial derivative $\partial/\partial H^\delta$ multiplied by the element $\delta\alpha$ of A_α. Hence Δ_α is locally nilpotent and so is Δ. Also, we see that $\Delta(H^\delta) \neq 0$ and $\Delta^2(H^\delta) = 0$.

Since A is a UFD and $R_\alpha = A_\alpha^{[1]}$, it is clear that R_α is a UFD; so, in order to prove that R is a UFD, it suffices to show that α is a product of prime elements of R. Since α is homogeneous, it is enough to verify that every homogeneous prime element p of A is a prime element of R. So, let r_1, r_2 be homogeneous elements of R such that $p \mid r_1 r_2$ (in R). Since p is irreducible in A and A is inert in B, it follows that p is irreducible in B, hence p is a prime element of B and consequently we may assume that $r_1 = bp$, for some $b \in B$. Then b is homogeneous and $0 \equiv \deg r_1 = \deg(bp) \equiv \deg b \pmod d$, so $b \in R$. This shows that p is a prime element of R and hence that R is a UFD.

Finally, we show that H^δ is a prime element of R. Let Q be a prime factor of H^δ in R; factoring Q in B gives $Q = uH^i$, where $u \in B^*$ and $0 < i \leq \delta$. Then u is homogeneous of degree zero, so $i \deg H = \deg Q \equiv 0 \pmod d$. By observation (5), this implies that $\delta \mid i$, so $i = \delta$ and H^δ is a prime element of R. $\square$

Observe that if $D \neq 0$ is a locally nilpotent derivation of $\mathbf{k}[X] = \mathbf{k}^{[1]}$ which is homogeneous with respect to the standard grading, then $\mathrm{d}(D) = 0$. In the case of a polynomial ring in several variables, we have:

Corollary 2.2. *Let D be a locally nilpotent derivation of $B = \mathbf{k}[X_1, \ldots, X_n] = \mathbf{k}^{[n]}$, where $n > 1$ and where $\mathbf{k}$ is a field of characteristic zero. If D is homogeneous with respect to the standard grading, then* $\mathrm{d}(D) = \mathrm{d}(\ker D) = 1$.

Proof. Observe that $B_0 \not\supseteq B_r$ for each $r > 0$; by part (3) of Proposition 2.1, it follows that $\mathrm{d}(D) = 1$, so $\mathrm{d}(\ker D) = \mathrm{d}(B) = 1$. $\square$

In the following statement, observe that $\mathrm{d}(B) = 1$, and consequently $\mathrm{d}(D) = \mathrm{d}(\ker D)$.

Corollary 2.3. *Let* $\mathbf{k}$ *be a field of characteristic zero and consider the graded ring* (B, w)*, where* $B = \mathbf{k}[X_1, \dots, X_n] = \mathbf{k}^{[n]}$*,* $w \in \mathbf{N}^n$ *and* $\gcd(w) = 1$*. Let* $D : B \to B$ *be a* w*-homogeneous locally nilpotent derivation, let* $d = \mathrm{d}(D) = \mathrm{d}(\ker D)$ *and assume that* $d \neq 1$*. Then* $d \nmid w(X_j)$ *holds for exactly one value of* j*, and this* j *also satisfies:*

1. $DX_j \neq 0$*,* $D^2 X_j = 0$*;*
2. *assume that* $d \neq 0$ *and define* R *as in Proposition 2.1, then*

$$R = \mathbf{k}[X_1, \dots, X_{j-1}, X_j^d, X_{j+1}, \dots, X_n].$$

Proof. Define $\mathcal{H}$ as in Proposition 2.1. Since $d \neq 1 = \gcd(w(X_1), \dots, w(X_n))$, $d \nmid w(X_j)$ holds for at least one value of j; since $X_j \in \mathcal{H}$ for any such j, the uniqueness of j and the assertion (1) follow respectively from parts (1) and (2) of Proposition 2.1.

Write $R' = \mathbf{k}[X_1, \dots, X_{j-1}, X_j^d, X_{j+1}, \dots, X_n]$, then $R' \subseteq R$ is obvious. For the other inclusion, let $r \in R \setminus \{0\}$ be a homogeneous element and consider a monomial $X_1^{s_1} \cdots X_n^{s_n}$ which occurs in r. Observe that $B = R[X_j]$ and $\mathrm{d}(B) = 1$ imply that $\gcd(d, w(X_j)) = 1$. Thus,

$$0 \equiv w(r) = \sum_{i=1}^{n} s_i w(X_i) \equiv s_j w(X_j) \implies s_j \equiv 0 \pmod{d},$$

i.e., $X_1^{s_1} \cdots X_n^{s_n} \in R'$. This shows that $r \in R'$ and consequently that $R = R'$. $\square$

The second part of this section, which consists of the results or paragraphs labeled from 2.4 to 2.7, is concerned with the case $\mathrm{d}(D) = 1$, and in particular with the following type of question: *Let* B *be a graded ring and* A *a homogeneous subring of* B *such that* $\mathrm{d}(A) = \mathrm{d}(B)$*. Find conditions on* A *which are equivalent to the existence of a homogeneous locally nilpotent derivation* $D : B \to B$ *with kernel* A*.*

We begin with two lemmas:

Lemma 2.4. *Let* $R = \bigoplus R_n$ *be a* $\mathbf{Z}$*-graded UFD satisfying:*

For all $n \in \mathbf{Z}$*, if* $R_n \neq 0$ *then* $R_{-n} \cap R^* \neq \emptyset$*.*

Then R_0 *is a UFD.*

Proof. If p is a homogeneous prime of R, say $p \in R_n$, then choose $u \in R_{-n} \cap R^*$ and define $p_0 = up$. Then $p_0 \in R_0$ and p_0 is a prime element of R; moreover, it is easy to see that p_0 is a prime element of R_0.

These remarks imply that every element of R_0 (which is neither zero nor a unit) is a product of prime elements of R_0. $\square$

Lemma 2.5. *Let* $R = \bigoplus R_n$ *be a* $\mathbf{Z}$*-graded domain and* Q *a homogeneous subring of* R *satisfying:*

For all $n \in \mathbf{Z}$*, if* $R_n \neq 0$ *then* $Q_{-n} \cap Q^* \neq \emptyset$*.*

Then the following are equivalent:

1. *There exists a homogeneous element* v *of* R *such that* $R = Q[v] = Q^{[1]}$*;*

(2) $R_0 = (Q_0)^{[1]}$.

Proof. Suppose that (1) holds, with $v \in R_n \setminus \{0\}$. Since $R_n \neq 0$, we can pick $\theta \in Q_{-n} \cap Q^*$ and define $v_0 = \theta v \in R_0$. Then $Q[v_0] = R = Q^{[1]}$, so v_0 is algebraically independent over Q and in particular it is algebraically independent over Q_0. It is clear that $R_0 = Q_0[v_0]$, so (2) holds.

Conversely, assume that (2) holds and consider $v \in R_0$ such that $R_0 = Q_0[v] = (Q_0)^{[1]}$. If $r \neq 0$ is any homogeneous element of R (say $r \in R_n$) then, since $R_n \neq 0$, we may pick $\theta \in Q_{-n} \cap Q^*$ and define $r_0 = \theta r \in R_0$; then $r_0 \in Q_0[v]$, whence $r \in Q[v]$. This shows that $R = Q[v]$. In order to prove that v is algebraically independent over Q, let us assume the contrary and consider a polynomial $0 \neq f(T) \in Q[T]$ such that $f(v) = 0$. Write $f(T) = \sum_i f_i(T)$ with $f_i(T) \in Q_i[T]$, choose m such that $f_m(T) \neq 0$ and define $g(T) = \theta' f_m(T) \in Q_0[T]$, where $\theta' \in Q_{-m} \cap Q^*$. Then $g(v) = 0$, which contradicts the fact that v is algebraically independent over Q_0. Hence (1) holds. □

2.6. We consider pairs (A, B) of $\mathbf{N}$-graded rings satisfying:

(1) B is an integral domain containing the field $\mathbf{Q}$ and A is a homogeneous subring of B;
(2) $A \neq B$ and B is finitely generated as an A-algebra;
(3) $\mathrm{d}(A) = \mathrm{d}(B)$.

Given such a pair (A, B), consider the multiplicatively closed subset $S = \bigcup_i (A_i \setminus \{0\})$ of A and denote the localized rings $S^{-1}A$ and $S^{-1}B$ by $\mathcal{A} = \bigoplus_{i\in\mathbf{Z}} \mathcal{A}_i$ and $\mathcal{B} = \bigoplus_{i\in\mathbf{Z}} \mathcal{B}_i$ respectively. Note that $\mathcal{A}$ is a homogeneous subring of $\mathcal{B}$, that $\mathcal{A}_0$ is a field and that $\mathcal{B}_0$ is a finitely generated $\mathcal{A}_0$-domain. Moreover, observe that if we write $d = \mathrm{d}(A) = \mathrm{d}(B)$ then for all $n \in \mathbf{Z}$ we have: $\mathcal{A}_n \neq 0 \Leftrightarrow n \in d\mathbf{Z} \Leftrightarrow \mathcal{B}_n \neq 0$, and $\mathcal{A}_n \setminus \{0\} \subseteq \mathcal{A}^*$.

The following will be used in the proof of 3.5 :

Lemma 2.7. *Let A, B, $\mathcal{A}$ and $\mathcal{B}$ be as in 2.6. Then the following conditions are equivalent:*

(1) *There exists a nonzero homogeneous locally nilpotent derivation D of B such that $A \subseteq \ker D \subseteq \operatorname{qt} A$;*
(2) $\mathcal{B}_0 = \mathcal{A}_0^{[1]}$.

Proof. Suppose (1) holds. We claim that

$$\ker D \subseteq \mathcal{A}. \tag{7}$$

Since $\ker D$ is a homogeneous subring of B, the proof of (7) reduces to that of:

If $b \in B \setminus \{0\}$ is homogeneous and $Db = 0$, then $b \in \mathcal{A}$.

Let $b \in B \setminus \{0\}$, homogeneous and such that $Db = 0$; then $b \in \operatorname{qt} A$, so we may write $ab = a'$ for some $a, a' \in A \setminus \{0\}$. Moreover, we may arrange that a and a' be homogeneous, and this shows that $b \in \mathcal{A}$. Hence (7) holds.

If we define S as in 2.6 then (7) implies

$$S^{-1} \ker D = \mathcal{A}. \tag{8}$$

On the other hand, we may consider a homogeneous element v of B such that $Dv \neq 0$ and $D^2 v = 0$. Write $\alpha = Dv$, then $B_\alpha = (\ker D)_\alpha[v] = (\ker D)_\alpha^{[1]}$ by 1.2. This, together with (8) and the fact that $\alpha \in S$, implies that $\mathcal{B} = \mathcal{A}[v] = \mathcal{A}^{[1]}$. Then (2) follows from Lemma 2.5.

Conversely, assume that (2) holds; then Lemma 2.5 implies that

$$\mathcal{B} = \mathcal{A}[v] = \mathcal{A}^{[1]},$$

for some homogeneous element v of $\mathcal{B}$. Thus we may consider the locally nilpotent derivation $\partial : \mathcal{B} \to \mathcal{B}$, where ∂ denotes the derivative with respect to v. Observe that if $b \in B$ then $\partial b \in \mathcal{B} = S^{-1}B$; since B is assumed to be finitely generated as an A-algebra we see that, for some $s \in S$, the derivation $s\partial$ maps B into itself. Let $D : B \to B$ be the restriction of $s\partial$; then D is locally nilpotent (by part (5) of 1.2) and $\ker D = B \cap \ker \partial = B \cap \mathcal{A}$. Thus $A \subseteq \ker D \subseteq \operatorname{qt} A$ and (1) holds. □

We conclude this section with a fact concerning subrings of a graded $\mathbf{k}^{[2]}$; it will be used in the proof of Proposition 3.4.

Lemma 2.8. *Let u and v be indeterminates over a field $\mathbf{k}$ and let $A = \mathbf{k}[u, v]$. Let $m, n \in \mathbf{N}$ be such that $\gcd(m, n) = 1$ and regard A as an $\mathbf{N}$-graded ring, the grading being given by the weights $w(u) = m$ and $w(v) = n$. Let f and g be elements of A satisfying:*

- (i) *f and g are w-homogeneous and $\gcd(w(f), w(g)) = 1$;*
- (ii) *f and g are irreducible in $\overline{\mathbf{k}}[u, v]$, where $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$;*
- (iii) *f, g are not associates.*

Then $\mathbf{k}[f, g] = \mathbf{k}[u, v]$.

Proof. Consider the inclusion map $\nu : \mathbf{k}[f, g] \to \mathbf{k}[u, v]$ and the corresponding

$$\overline{\mathbf{k}} \otimes_{\mathbf{k}} \nu : \overline{\mathbf{k}}[f, g] \to \overline{\mathbf{k}}[u, v],$$

and recall that if $\overline{\mathbf{k}} \otimes_{\mathbf{k}} \nu$ is surjective then so is ν. Hence we may assume that $\mathbf{k}$ is algebraically closed.

Let "$x \sim y$" denote that x and y are associates in A. It is an easy exercise (left to the reader) to show that if P is an irreducible homogeneous element of $\mathbf{k}[u, v]$ then

$$P \sim u \quad \text{or} \quad P \sim v \quad \text{or} \quad P \sim u^n + \lambda v^m \ (\text{some } \lambda \in \mathbf{k}^*).$$

To verify that $\mathbf{k}[f, g] = \mathbf{k}[u, v]$, we consider four cases.
Case 1: $f \sim u$ and $g \sim v$. The claim is obvious in this case.
Case 2: $f \sim u$ and $g \sim u^n + \lambda v^m$, with $\lambda \in \mathbf{k}^*$. We have

$$1 = \gcd(w(f), w(g)) = \gcd(m, mn) = m,$$

so $m = 1$, so $g \sim u^n + \lambda v$. Hence $\mathbf{k}[f, g] = \mathbf{k}[u, u^n + \lambda v] = \mathbf{k}[u, v]$.

Case 3: $f \sim v$ and $g \sim u^n + \lambda v^m$, with $\lambda \in \mathbf{k}^*$. Similar to case 2.
Case 4: $f \sim u^n + \lambda v^m$ and $g \sim u^n + \mu v^m$, with $\lambda, \mu \in \mathbf{k}^*$. We have

$$1 = \gcd(w(f), w(g)) = \gcd(mn, mn) = mn,$$

so $m = 1 = n$, so $f \sim u + \lambda v$ and $g \sim u + \mu v$. By assumption (iii), $\lambda \neq \mu$, so $\mathbf{k}[f, g] = \mathbf{k}[u + \lambda v, u + \mu v] = \mathbf{k}[u, v]$.

This proves the lemma. □

3. Homogeneous derivations of $\mathbf{k}[X, Y, Z]$

Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$. We consider the problem of describing the set of nonzero, w-homogeneous locally nilpotent derivations of B (see 1.13 for definition). By 1.6, such a derivation is determined by its kernel, up to multiplication by a w-homogeneous element of that kernel; so our basic goal is to characterize the kernels of these derivations. Note that 1.5, 1.14 and corollaries 1.8 and 2.2 imply the following:

Corollary 3.1. *Let $\mathbf{k}$ be a field of characteristic zero and consider the $\mathbf{N}$-graded ring (B, w) (see 1.13), where $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ and where $w \in \mathbf{N}^3$ satisfies $\gcd(w) = 1$. Let $D \neq 0$ be a w-homogeneous locally nilpotent derivation of B. Then $\ker D = \mathbf{k}[f, g] = \mathbf{k}^{[2]}$, for some w-homogeneous polynomials f, g, and for any such f, g we have:*

1. *f and g are irreducible in $\overline{\mathbf{k}}[X, Y, Z]$, where $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$;*
2. *$\mathrm{d}(D) = \gcd(w(f), w(g))$;*
3. *if $w = (1, 1, 1)$ then $\mathrm{d}(D) = \gcd(\deg f, \deg g) = 1$.*

In view of this fact, we can phrase our question as follows:

Problem 3.2. Let f and g be w-homogeneous polynomials in $B = \mathbf{k}[X, Y, Z]$. Give conditions, on the pair (f, g), which are necessary and sufficient for the existence of a w-homogeneous locally nilpotent derivation of B with kernel $\mathbf{k}[f, g]$.

This investigation splits naturally into two cases, according to the value of the natural number $\mathrm{d}(D)$ defined in 1.12. Perhaps surprisingly, it turns out that "$\mathrm{d}(D) = 1$" is the generic case and that "$\mathrm{d}(D) \neq 1$" is very special. The reader should note that Theorem 3.7 implies, in particular, that all derivations satisfying $\mathrm{d}(D) \neq 1$ have rank strictly less than 3.

The case $\mathrm{d}(D) = 1$

3.3. Let $\mathbf{k}$ be a field of characteristic zero and consider the $\mathbf{N}$-graded ring (B, w), where $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ and where $w \in \mathbf{N}^3$ satisfies $\gcd(w) = 1$. Let $f, g \in B$ be w-homogeneous polynomials and consider the following conditions on f and g:

(i) $\gcd(w(f), w(g)) = 1$;
(ii) f and g are irreducible in $\overline{\mathbf{k}}[X, Y, Z]$, where $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$;
(iii) f, g are not associates in $\mathbf{k}[X, Y, Z]$.

Note that any f, g satisfying these conditions are algebraically independent over $\mathbf{k}$. Note, also, that these conditions are *necessary* for the existence of a w-homogeneous locally nilpotent derivation $D : B \to B$ with kernel $\mathbf{k}[f, g]$ and such that $\mathrm{d}(D) = 1$.

The following result clarifies the meaning of Problem 3.2:

Proposition 3.4. *Let $\mathbf{k}$ be a field of characteristic zero and consider the $\mathbf{N}$-graded ring (B, w), where $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ and where $w \in \mathbf{N}^3$ satisfies $\gcd(w) = 1$. Suppose that $f, g \in B$ are w-homogeneous and satisfy the conditions of 3.3. Then the following hold:*

1. *If $D \neq 0$ is any locally nilpotent derivation of B which satisfies $Df = 0 = Dg$, then $\ker D = \mathbf{k}[f, g]$.*
2. *For the pair (f, g), the following conditions are equivalent:*
   (a) *There exists a w-homogeneous locally nilpotent derivation D of B such that $\ker D = \mathbf{k}[f, g]$;*
   (b) *the derivation $\Delta_{(f,g)}$ of B (see 1.6) is locally nilpotent;*
   (c) *there exists a nonzero locally nilpotent derivation D of B such that $\ker D \supseteq \mathbf{k}[f, g]$.*

Proof. To prove (1), let $D \neq 0$ be a locally nilpotent derivation of B such that $Df = 0 = Dg$, and write $A = \ker D$. By 1.5, $A = \mathbf{k}[u, v]$ for some polynomials $u, v \in B$. By 1.6, we have $D = \alpha\Delta_{(u,v)}$ for some $\alpha \in A \setminus \{0\}$; note that $\Delta_{(u,v)}$ is locally nilpotent and has kernel A.

On the other hand, we may consider the derivation $\Delta_{(f,g)}$ of B. Since f and g are polynomials in u and v, the chain rule implies that $\Delta_{(f,g)} = a\Delta_{(u,v)}$ for some $a \in \mathbf{k}[u, v]$. Since f and g are algebraically independent over $\mathbf{k}$, we have $\Delta_{(f,g)} \neq 0$ and consequently $a \neq 0$. Now $\Delta_{(f,g)}$ is a w-homogeneous derivation; thus $a\Delta_{(u,v)}$ is w-homogeneous, and this clearly implies that $\Delta_{(u,v)}$ is w-homogeneous. By Corollary 3.1, $A = \mathbf{k}[u', v']$ for some *w-homogeneous* polynomials u' and v', and these polynomials satisfy $\gcd(w(u'), w(v')) = 1$ (because $f, g \in \mathbf{k}[u', v']$). Also, by condition (ii) of 3.3, f and g are irreducible in $\overline{\mathbf{k}}[u', v']$. Thus statement (1) follows from Lemma 2.8.

Assertion (2) is now easy: (a)$\Rightarrow$(b) follows from 1.6; (b)$\Rightarrow$(c) is obvious, since $\Delta_{(f,g)} \neq 0$ was noted in the proof of (1). To prove (c)$\Rightarrow$(a), assume that (c) holds; then, by assertion (1), $\ker D = \mathbf{k}[f, g]$ and consequently (by 1.6) $D = \alpha\Delta_{(f,g)}$ (some $\alpha \in \ker D \setminus \{0\}$) and $\Delta_{(f,g)}$ is locally nilpotent. Since $\ker \Delta_{(f,g)} = \ker D = \mathbf{k}[f, g]$, (a) holds. $\square$

The next result is the solution of Problem 3.2 in the case $\mathrm{d}(D) = 1$. In the statement of the theorem, $V(fg)$ is the closed subset "$fg = 0$" of the weighted projective space $\operatorname{Proj} \overline{B}$. In most cases, it is correct to say that the surface $\operatorname{Proj} \overline{B} \setminus V(fg)$ is obtained by removing the *two* curves $V(f)$, $V(g)$ from $\operatorname{Proj} \overline{B}$, but one should be aware that, in some cases, one of those two "curves" is empty! For example, let

$w = (0, 0, 1)$, $f = X$ and $g = Z$; then $V(f)$ is a line in $\operatorname{Proj} \overline{B} \cong \mathbf{A}^2_{\overline{\mathbf{k}}}$ and $V(g)$ is empty, so $\operatorname{Proj} \overline{B} \setminus V(fg)$ is the affine plane minus a line.

Theorem 3.5. *Let $\mathbf{k}$ be a field of characteristic zero, with algebraic closure $\overline{\mathbf{k}}$, and consider the $\mathbf{N}$-graded ring (B, w), where $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$ and where $w \in \mathbf{N}^3$ satisfies $\gcd(w) = 1$. Suppose that $f, g \in B$ are w-homogeneous and satisfy $\gcd(w(f), w(g)) = 1$. Then the following are equivalent:*

1. *There exists a w-homogeneous locally nilpotent derivation D of $\mathbf{k}[X, Y, Z]$ such that $\ker D = \mathbf{k}[f, g]$;*
2. *f and g are irreducible elements of $\overline{B} = \overline{\mathbf{k}}[X, Y, Z]$ and the surface $\operatorname{Proj} \overline{B} \setminus V(fg)$ is isomorphic to $\mathbf{P}^2_{\overline{\mathbf{k}}}$ minus two lines.*

Remark . The conditions (1) and (2) of the theorem are also equivalent to:

($1\frac{1}{2}$) $B_{(fg)} = (A_{(fg)})^{[1]}$.

Indeed, (1) $\Rightarrow$ ($1\frac{1}{2}$) follows from Corollary 3.10 and we shall now argue that ($1\frac{1}{2}$) $\Rightarrow$ (2). Assume that ($1\frac{1}{2}$) holds; then tensoring by $\overline{\mathbf{k}}$ yields $\overline{B}_{(fg)} = (\overline{A}_{(fg)})^{[1]}$, where $\overline{B} = \overline{\mathbf{k}}[X, Y, Z]$ and $\overline{A} = \overline{\mathbf{k}}[f, g]$, so we may assume that $\mathbf{k} = \overline{\mathbf{k}}$. In this case, the proof of the theorem shows that ($1\frac{1}{2}$) does imply (2) (see statement (15) in the proof, below).

Proof of Theorem 3.5. In view of Corollary 1.7, it is clear that we may assume that $\mathbf{k}$ is algebraically closed. Hence $\overline{\mathbf{k}} = \mathbf{k}$ and $\overline{B} = B$. Also, it is convenient to assume, throughout the proof, that

$$\text{at least one of the conditions (3.5.1), (3.5.2) is satisfied.}$$

Let $A = \mathbf{k}[f, g]$, $B = \mathbf{k}[X, Y, Z]$, $p = w(f)$ and $q = w(g)$. Since it is assumed that $\gcd(p, q) = 1$, we clearly have:

$$(A, B) \text{ satisfies the conditions of 2.6}$$

and we may consider the graded rings $\mathcal{A}$ and $\mathcal{B}$ defined in 2.6. On the other hand, let $A_{(fg)}$ and $B_{(fg)}$ denote the homogeneous localizations at fg, i.e., the degree zero components of the graded rings A_{fg} and B_{fg} respectively.

Observe that, since B is a UFD, so is the $\mathbf{Z}$-graded ring B_{fg}. Since $\gcd(p, q) = 1$, if $n \in \mathbf{Z}$ then there exist $i, j \in \mathbf{Z}$ such that $pi + qj = -n$, so $f^i g^j \in (B_{fg})_{-n} \cap (B_{fg})^*$. Thus 2.4 implies that

$$B_{(fg)} \text{ is a UFD.} \tag{9}$$

We will now verify that

$$(f, g) \text{ satisfies the conditions of 3.3.} \tag{10}$$

Condition (i) of 3.3 holds, since it is one of the assumptions of this theorem. Condition (ii) of 3.3 follows from Corollary 3.1 if (3.5.1) holds, and holds by assumption if (3.5.2) holds. Condition (iii) is obvious if (3.5.1) holds because, say, $\ker D = \mathbf{k}^{[2]}$ by 1.5;

there remains to check that (3.5.2) implies that f and g are not associates in B. To see this, observe that $\operatorname{Proj} B \setminus V(fg) \cong \operatorname{Spec} B_{(fg)}$ and that $\mathbf{P}^2_{\mathbf{k}}$ minus two lines is isomorphic to $\operatorname{Spec} \mathbf{k}[W, W^{-1}]^{[1]}$, so (3.5.2) implies that $B_{(fg)} \cong \mathbf{k}[W, W^{-1}]^{[1]}$ (W is an indeterminate); in particular, $\mathbf{k}^*$ is a proper subset of $B^*_{(fg)}$. On the other hand, pick $h \in \{f, g\}$ such that $w(h) > 0$; using only the fact that h is a homogeneous prime element of B and $w(h) > 0$, it is easy to see that $B^*_{(h)} = \mathbf{k}^*$, hence $B_{(h)}$ is not isomorphic to $B_{(fg)}$. Hence f, g cannot be associates, and (10) is true.

Let $\xi = f^q/g^p$. It is easy to see that every element of $A_n \setminus \{0\}$ (where $n \in \mathbf{N}$) has the form $f^i g^s H(f^q, g^p)$, for some $i, s \in \mathbf{N}$ and some homogeneous polynomial in two variables $H \in \mathbf{k}[T_1, T_2]$ (that is, homogeneous with respect to the standard grading of $\mathbf{k}[T_1, T_2]$). This can be rewritten as $f^i g^j H(\xi, 1)$ for some $j \in \mathbf{N}$, so we conclude that every $a \in A_n \setminus \{0\}$ has the form

$$a = f^i g^j \theta, \quad \text{for some } i, j \in \mathbf{N} \text{ and } \theta \in \mathbf{k}[\xi]. \tag{11}$$

From this, one easily deduces

$$A_{(fg)} = \mathbf{k}[\xi, \xi^{-1}], \tag{12}$$

$$B^*_{(fg)} = \bigcup_{n \in \mathbf{Z}} \mathbf{k}^* \xi^n, \tag{13}$$

$$\mathcal{A}_0 = T^{-1} A_{(fg)} \quad \text{and} \quad \mathcal{B}_0 = T^{-1} B_{(fg)}, \tag{14}$$

where $T = A_{(fg)} \setminus \{0\}$.

We now consider the equivalence of conditions (3.5.1) and (3.5.2). First, we prove:

$$\text{The condition (3.5.2) is equivalent to } B_{(fg)} = (A_{(fg)})^{[1]}. \tag{15}$$

Recall that $\operatorname{Spec} B_{(fg)} \cong \operatorname{Proj} B \setminus V(fg)$ and that, by (12), $\operatorname{Spec} ({A_{(fg)}}^{[1]})$ is isomorphic to $\mathbf{P}^2_{\mathbf{k}}$ minus two lines. So it is obvious that $B_{(fg)} = (A_{(fg)})^{[1]}$ implies (3.5.2). Conversely, suppose that (3.5.2) holds. Then, for some element ζ of $B_{(fg)}$, we have $B_{(fg)} = \mathbf{k}[\zeta, \zeta^{-1}]^{[1]}$. That element ζ must then satisfy

$$B^*_{(fg)} = \bigcup_{n \in \mathbf{Z}} \mathbf{k}^* \zeta^n,$$

and this, together with (13), implies that $\zeta = \lambda \xi^{\pm 1}$ (for some $\lambda \in \mathbf{k}^*$) and consequently that $\mathbf{k}[\zeta, \zeta^{-1}] = \mathbf{k}[\xi, \xi^{-1}] = A_{(fg)}$. Hence (3.5.2) implies that $B_{(fg)} = (A_{(fg)})^{[1]}$, and (15) is proved.

There remains to prove:

$$\text{The condition (3.5.1) is equivalent to } B_{(fg)} = (A_{(fg)})^{[1]}. \tag{16}$$

One way is easy: If $B_{(fg)} = (A_{(fg)})^{[1]}$ then (by (14)) $\mathcal{B}_0 = \mathcal{A}_0^{[1]}$, so (3.5.1) holds by Lemma 2.7 and Proposition 3.4.

Conversely, assume that (3.5.1) holds. We begin by observing that $A_{(fg)}$ is an inert subring of $B_{(fg)}$. To see this, consider a locally nilpotent derivation $D : B \to B$

with kernel A. Then, by 1.2, localization at fg yields a locally nilpotent derivation, $D_{fg} : B_{fg} \to B_{fg}$, with kernel A_{fg}. Thus (again by 1.2) A_{fg} is an inert subring of B_{fg}, and it follows immediately that $A_{(fg)}$ is an inert subring of $B_{(fg)}$.

We also point out that Lemma 2.7 and (14) imply that

$$T^{-1}B_{(fg)} = (T^{-1}A_{(fg)})^{[1]}. \tag{17}$$

We will now verify that, for each $a \in \mathbf{k}^*$, the following hold (where we write $\pi = \xi - a$):

(i) π is a prime element of $B_{(fg)}$,

(ii) $A_{(fg)} \cap \pi B_{(fg)} = \pi A_{(fg)}$,

(iii) $A_{(fg)}/\pi A_{(fg)}$ is algebraically closed in $B_{(fg)}/\pi B_{(fg)}$.

Since π is an irreducible element of $A_{(fg)}$ and $A_{(fg)}$ is an inert subring of $B_{(fg)}$, π is an irreducible element of $B_{(fg)}$. Since $B_{(fg)}$ is a UFD by (9), π is prime in $B_{(fg)}$ and (i) holds.

If $\beta \in B_{(fg)} \setminus \{0\}$ satisfies $\pi\beta \in A_{(fg)}$ then, again by inertness of $A_{(fg)}$, we have $\beta \in A_{(fg)}$; so (ii) holds.

Condition (iii) is obvious, since $\mathbf{k}$ is assumed to be algebraically closed.

In view of 1.1, conditions (17), (i), (ii) and (iii) imply that $B_{(fg)} = (A_{(fg)})^{[1]}$, which means that (16) is proved. This completes the proof of the theorem. $\square$

The case $\mathrm{d}(D) \neq 1$

Our first comment is that this case does occur. To obtain examples, it suffices to make sure that $\gcd(w(X), w(Y)) \neq 1$ in the following:

Example 3.6. Let $\mathbf{k}$ be a field of characteristic zero and consider (B, w), where $B = \mathbf{k}[X, Y, Z]$ and $w \in \mathbf{N}^3$ satisfies $\gcd w = 1$. Let $\varphi \in \mathbf{k}[X]$ and $\psi \in \mathbf{k}[X, Z]$ be w-homogeneous polynomials such that

(1) φ is not the zero polynomial,

(2) φ and $\partial\psi/\partial Z$ are relatively prime in B,

(3) P is w-homogeneous, where $P = \varphi(X)Y + \psi(X, Z)$.

Then $\Delta_{(X,P)} : B \to B$ is a w-homogeneous locally nilpotent derivation, $\ker \Delta_{(X,P)} = \mathbf{k}[X, P]$ and $\mathrm{d}(\Delta_{(X,P)}) = \gcd(w(X), w(Y))$.

(The verification of these claims is left to the reader. This is very easy if one uses the third section of [4].)

The aim of this sub-section is to prove that Example 3.6 is in fact the only way that the case $\mathrm{d}(D) \neq 1$ can occur. More precisely,

Theorem 3.7. *Let $\mathbf{k}$ be a field of characteristic zero and consider (B, w), where $B = \mathbf{k}[X', Y', Z']$ and $w \in \mathbf{N}^3$ satisfies $\gcd w = 1$. Suppose that $D : B \to B$ is a w-homogeneous locally nilpotent derivation such that $\mathrm{d}(D) \neq 1$. Then there exists a w-homogeneous coordinate system[3] (X, Y, Z) of B, and polynomials $\varphi \in \mathbf{k}[X]$ and*

[3]We mean that X, Y and Z are w-homogeneous elements of B satisfying $B = \mathbf{k}[X, Y, Z]$.

$\psi \in \mathbf{k}[X,Z]$ satisfying the three conditions of Example 3.6, such that $\ker D = \mathbf{k}[X,P]$. *Consequently, $D = a\Delta_{(X,P)}$ for some nonzero element a of* $\mathbf{k}[X,P]$.

In order to prove Theorem 3.7, we will establish two lemmas.

Lemma 3.8. *Let $D : B \to B$ be a locally nilpotent derivation, where $B = \mathbf{k}[X,Y,Z]$. Assume that $DZ \in \mathbf{k}[f] \setminus \{0\}$ for some variable f of* $\ker D$. *Then:*

1. *$B = \mathbf{k}[f,Z]^{[1]}$.*
2. *If D and f are w-homogeneous (for some $w \in \mathbf{N}^3$), then $B = \mathbf{k}[f,h,Z]$ for some w-homogeneous element h of B.*

Proof. We leave it to the reader to verify the following statement:

> *Let B be an $\mathbf{N}$-graded domain and A a homogeneous subring of B such that $B = A^{[1]}$. Then $B = A[h]$, for some* homogeneous *element h of B.*

Hence the second part of the lemma follows from the first. We prove part (1) of the lemma.

Since $DZ \neq 0$ we have $D \neq 0$ so, by 1.5, $\ker D = \mathbf{k}[f,g]$ for some $g \in B$. Write $DZ = \varphi \in \mathbf{k}[f] \setminus \{0\}$, then $B_\varphi = \mathbf{k}[f,g]_\varphi[Z] = \mathbf{k}[f,Z]_\varphi[g]$, so

$$B_\varphi = (A_\varphi)^{[1]}, \tag{18}$$

where $A = \mathbf{k}[f,Z]$. For future use, we also note that

$$f \notin \mathbf{k}[Z]. \tag{19}$$

This follows from equation (18) or, alternatively, from the fact that $\ker D$ is algebraically closed in B, together with $Df = 0$ and $DZ \neq 0$.

The first part of the proof consists in proving

$$A \text{ is the kernel of some locally nilpotent derivation of } B. \tag{20}$$

We first prove this claim under the assumption that $\mathbf{k}$ is algebraically closed. In view of 1.3 and equation (18), it suffices to prove that $A_\varphi \cap B = A$.

First we show that, for each $\lambda \in \mathbf{k}$, if we write $\pi = f - \lambda \in \mathbf{k}[f]$ then

$$A \cap \pi B = \pi A. \tag{21}$$

Since π is prime in $\mathbf{k}[f]$, it is prime in $\ker D = \mathbf{k}[f]^{[1]}$; since $\ker D$ is an inert subring of B, π is prime in B. Hence $\boldsymbol{p} = A \cap \pi B$ is a prime ideal of A and $\pi A \subseteq \boldsymbol{p}$. Suppose that $\boldsymbol{p} \neq \pi A$; then $\boldsymbol{p}$ is a maximal ideal of A, so $\boldsymbol{p} = (f-\lambda, Z-\mu)$ (where $\mu \in \mathbf{k}$) and consequently $Z - \mu \in \pi B$, which contradicts (19). Hence $\boldsymbol{p} = \pi A$ and equality (21) holds.

To see that $A_\varphi \cap B = A$, consider $b \in A_\varphi \cap B$. Then there exist prime factors $\pi_1, \dots, \pi_n$ of φ in $\mathbf{k}[f]$ such that $\pi_1 \cdots \pi_n b \in A$. If $n > 0$ then $\pi_1 \cdots \pi_n b \in A \cap \pi_n B = \pi_n A$ by equality (21), so $\pi_1 \cdots \pi_{n-1} b \in A$. Clearly, this argument shows that $b \in A$ and consequently that $A_\varphi \cap B = A$. It follows that the claim (20) holds (if $\mathbf{k}$ is algebraically closed).

Actually, (20) holds without assuming that $\mathbf{k}$ is algebraically closed. Indeed, consider the algebraic closure $\overline{\mathbf{k}}$ of $\mathbf{k}$, the rings $\overline{B} = \overline{\mathbf{k}}[X,Y,Z]$ and $\overline{A} = \overline{\mathbf{k}}[f,Z]$ and the locally nilpotent derivation $\overline{D} = \overline{\mathbf{k}} \otimes_{\mathbf{k}} D : \overline{B} \to \overline{B}$. Since $\ker \overline{D} = \overline{\mathbf{k}}[f,g]$, $\overline{D}$ satisfies all hypotheses of this lemma ($\overline{D}Z \in \overline{\mathbf{k}}[f] \setminus \{0\}$, where f is a variable of $\ker \overline{D}$). Since (20) holds in the special case where $\mathbf{k}$ is algebraically closed, $\overline{A}$ is the kernel of some locally nilpotent derivation of $\overline{B}$ and, by Corollary 1.7, it follows that (20) holds.

Consider a locally nilpotent derivation $\Delta : B \to B$ with kernel A. Note that $\operatorname{rank} \Delta < 3$, because A contains the variable Z of B. In order to prove this lemma, there remains to show that $\operatorname{rank} \Delta \neq 2$ (then $\operatorname{rank} \Delta = 1$ and consequently $B = A^{[1]}$).

Assume that $\operatorname{rank} \Delta = 2$. We have $\dim \overline{\mathcal{R}}(D) = 1$ by 1.11, so we may consider a prime element ξ of A such that $V(\xi) \subseteq \overline{\mathcal{R}}(D)$; then, again by 1.11, $\mathbf{k}[Z]$ is the integral closure of $\mathbf{k}[\xi]$ in B. On the other hand, equality (18) implies that $\overline{\mathcal{R}}(D)$ is disjoint from the open subset $\operatorname{Spec} A_\varphi$ of $\operatorname{Spec} A$. Thus $V(\xi) \subseteq V(\varphi)$, and consequently $\xi \mid \varphi$ in A, whence $\xi \in \mathbf{k}[f]$. We obtain that f is integral over $\mathbf{k}[\xi]$, so $f \in \mathbf{k}[Z]$, which contradicts statement (19). We conclude that Δ cannot have rank two, which completes the proof. $\square$

Lemma 3.9. *Let $\mathbf{k}$ be a field of characteristic zero and consider (B, w), where $B = \mathbf{k}[X,Y,Z]$ and $w \in \mathbf{N}^3$. Let $D : B \to B$ be a w-homogeneous locally nilpotent derivation. If $w(Z) > 0$, $DZ \neq 0$ and $D^2 Z = 0$, then $B = \mathbf{k}[u,v,Z]$ for some w-homogeneous variable u of $\ker D$ and some w-homogeneous element v of B.*

Proof. Let $A = \ker D$. Since $DZ \neq 0$, $D \neq 0$ and consequently $\ker D = \mathbf{k}[f,g]$ for some w-homogeneous elements $f, g \in B$ (by 3.1); note that

$$f, g \notin \mathbf{k}[Z], \tag{22}$$

because $Df = 0 = Dg$, $DZ \neq 0$ and A is algebraically closed in B. Note, also, that we may assume that $D = \Delta_{(f,g)}$ (see 1.6). Write $p = w(f)$, $q = w(g)$ and $w = (a,b,c)$.

If $p = 0 = q$ then $A \subseteq B_0$, so B_0 has transcendence degree 2 over $\mathbf{k}$. Consequently, $B_0 = \mathbf{k}[X,Y]$. Since A is algebraically closed in B, $A \subseteq \mathbf{k}[X,Y]$ implies $A = \mathbf{k}[X,Y]$, thus the assertion is satisfied with $u = X$, $v = Y$.

We may therefore assume that $q \neq 0$. Then we claim that $a + b > 0$. Indeed, if it is not the case then $a = 0 = b$, so $B_0 = \mathbf{k}[X,Y]$, and since $q > 0$ we have $g = \alpha Z^n$ for some $n > 0$ and $\alpha \in \mathbf{k}[X,Y]$. Thus $Z \mid g$ and, since g is irreducible by 3.1, we obtain that g and Z are associates, which contradicts statement (22).

Since we assumed that $D = \Delta_{(f,g)}$, we have

$$w(DZ) = w\left(\begin{vmatrix} f_X & f_Y \\ g_X & g_Y \end{vmatrix} \right) = p + q - a - b < p + q.$$

If the monomial $f^i g^j$ occurs in the polynomial $DZ \in \mathbf{k}[f,g]$ then

$$p + q > w(DZ) = w(f^i g^j) = pi + qj,$$

so $i = 0$ or $j = 0$. In other words, $DZ \in \mathbf{k}[f] + \mathbf{k}[g]$; since DZ is w-homogeneous and $q > 0$, we have

$$DZ = \varphi(f) + \mu\, g^j \qquad \text{(for some } \varphi(T) \in \mathbf{k}[T],\ \mu \in \mathbf{k} \text{ and } j \in \mathbf{N}). \tag{23}$$

In view of Lemma 3.8, it suffices to show that

$$DZ \in \mathbf{k}[u], \text{ for some } w\text{-homogeneous variable } u \text{ of } A. \tag{24}$$

If $p = 0$ then $w(DZ) < p + q = q$, so $j = 0$ in equation (23), thus $DZ \in \mathbf{k}[f]$ and (24) holds.

Assume that $p > 0$ then, since DZ is w-homogeneous, (23) reads

$$DZ = \lambda f^i + \mu\, g^j \qquad \text{(for some } \lambda, \mu \in \mathbf{k}).$$

It is clear that (24) holds if one of the following conditions is satisfied:

$$\lambda = 0, \quad \mu = 0, \quad i < 2, \quad \text{or} \ \ j < 2.$$

Now we claim that if none of these conditions is satisfied then we have a contradiction. Indeed, if $p \le q$ then

$$p + q > w(DZ) = qj \ge 2q \ge p + q$$

is absurd, and the argument is similar if $p > q$. This completes the proof of the lemma. $\square$

Proof of Theorem 3.7. Let $d = \mathrm{d}(D)$. Since $d \neq 1$, Corollary 2.3 implies that a unique element Z of $\{X', Y', Z'\}$ satisfies $d \nmid w(Z)$, and that Z also satisfies $DZ \neq 0$ and $D^2 Z = 0$. Thus, by Lemma 3.9, there exist a w-homogeneous variable X of $\ker D$ and a w-homogeneous element Y of B such that $B = \mathbf{k}[X, Y, Z]$. Since $\ker D = \mathbf{k}^{[2]}$ and X is a w-homogeneous variable of $\ker D$, we have $\ker D = \mathbf{k}[X, P]$ for some w-homogeneous polynomial $P \in \mathbf{k}[X, Y, Z]$. By result 1.6, $D = a\Delta_{(X,P)}$ for some $a \in \mathbf{k}[X, P] \setminus \{0\}$.

Write $\Delta = \Delta_{(X,P)}$, then $\Delta Z \neq 0$ and $\Delta^2 Z = 0$; since $\Delta Z = P_Y$ (partial derivative with respect to Y), we obtain $P_Y \in \mathbf{k}[X, P] \setminus \{0\}$. Write

$$P_Y = H(X, P), \tag{25}$$

where $H(T_1, T_2) \in \mathbf{k}[T_1, T_2]$ is a nonzero polynomial in two variables, and let $n = \deg_{T_2} H$ (the degree of H as a polynomial in T_2 with coefficients in $\mathbf{k}[T_1]$). Then equation (25) gives

$$\deg_Y P > \deg_Y(P_Y) = n \deg_Y P,$$

so $n = 0$ and $P_Y \in \mathbf{k}[X]$. Thus $P = \varphi Y + \psi$, where $\varphi \in \mathbf{k}[X] \setminus \{0\}$ and $\psi \in \mathbf{k}[X, Z]$.

There remains to verify that φ and $\partial\psi/\partial Z$ are relatively prime. Let h be the gcd of these two polynomials; then $\psi - \psi(X, 0)$ is divisible by h, and consequently $P = hQ + \psi(X, 0)$ for some $Q \in B$. It follows that $\mathbf{k}[X, hQ] = \mathbf{k}[X, P] = \ker D$; then Corollary 3.1 implies that hQ is an irreducible element of B, so $h \in B^*$. $\square$

Some consequences

Our last result gathers some consequences of Theorem 3.5 and of the second section.

Corollary 3.10. *Let* $\mathbf{k}$ *be a field of characteristic zero and consider the* $\mathbf{N}$*-graded ring* (B, w)*, where* $B = \mathbf{k}[X, Y, Z]$*,* $w \in \mathbf{N}^3$ *and* $\gcd w = 1$*. Let* $0 \neq D : B \to B$ *be a* w*-homogeneous locally nilpotent derivation and let* $A = \mathbf{k}[f, g] = \ker D$*, where* f *and* g *are* w*-homogeneous. Then the following hold.*

If $\mathrm{d}(D) \geq 1$ *then:*
1. $B_{(fg)} = (A_{(fg)})^{[1]}$*;*
2. $\mathrm{Proj}\, B \setminus V(fg)$ *is isomorphic to* $\mathbf{P}^2_{\mathbf{k}}$ *minus two lines.*

If $\mathrm{d}(D) \leq 1$ *then:*

3. $B_{fg} = A_{fg}[v] = (A_{fg})^{[1]}$*, for some* w*-homogeneous element* v *of* B*;*
4. $\mathrm{rank}\, D < 3 \Leftrightarrow f$ *or* g *is a variable of* B*;*
5. $\Delta_{(f,g)}(v) = f^i g^j$*, for some* $i, j \in \mathbf{N}$ *and some* w*-homogeneous* $v \in B$*.*

Remark . If $\mathrm{d}(D) < 1$ (i.e., $\mathrm{d}(D) = 0$) then assertions (1) and (2) do not hold, because we have $A = B_0$ and consequently $B_{(fg)} = A_{(fg)}$.

Assertions (3–5) do not always hold when $\mathrm{d}(D) > 1$. For instance, consider the derivation $D = \Delta_{(X,P)}$ as in Example 3.6, with $w = (n, 0, 1)$ (where $n > 1$), $\varphi = X$ and $\psi = Z^n$. Let $f = X + P = X + XY + Z^n$ and $g = P = XY + Z^n$. Then (3–5) do not hold.

Proof. The first part of the proof consists in showing that condition (5) holds if $\mathrm{d}(D) = 1$. By assumption, the condition (1) of Theorem 3.5 is satisfied; if $\mathbf{k}$ is algebraically closed then the proof of that result implies that assertion (1) (of this corollary) holds, so assertion (3) holds by Lemma 2.5 and assertion (5) follows from Lemma 1.4. Hence, if we drop the assumption that $\mathbf{k}$ is algebraically closed, we obtain that there exists a w-homogeneous element $\overline{v} \in \overline{B} = \overline{\mathbf{k}}[X, Y, Z]$ which satisfies $\overline{\Delta}(\overline{v}) = f^i g^j$ for some $i, j \in \mathbf{N}$ (here, $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$ and $\overline{\Delta} : \overline{B} \to \overline{B}$ is the extension of $\Delta_{(f,g)} : B \to B$). It follows from linear algebra that $\Delta_{(f,g)}(v) = f^i g^j$ for some w-homogeneous element v of B. Indeed, let $\omega = w(\overline{v})$ and let E be a finite set of monomials $X^\alpha Y^\beta Z^\gamma$ of weight ω such that $\overline{v} \in \mathrm{Span}_{\overline{\mathbf{k}}}(E)$; let $V = \mathrm{Span}_{\mathbf{k}}(E)$, $W = \Delta_{(f,g)}(V)$ and restrict $\Delta_{(f,g)}$ to a linear map of finite dimensional vector spaces over $\mathbf{k}$, $\delta : V \to W$. Then $f^i g^j$ is in the image of the $\overline{\mathbf{k}}$-linear map $\overline{\mathbf{k}} \otimes_{\mathbf{k}} \delta : \overline{\mathbf{k}} \otimes_{\mathbf{k}} V \to \overline{\mathbf{k}} \otimes_{\mathbf{k}} W$, and it follows that $f^i g^j$ is in the image of δ. In other words, (5) holds.

Next, we show that if $\mathrm{d}(D) = 1$ then assertions (1–5) hold. Since we have shown that (5) holds, part (4) of 1.2 implies that assertion (3) holds, which implies that (1) holds (by Lemma 2.5); clearly, (2) follows from (1). As to (4), the only nontrivial part is "$\Rightarrow$" when $\mathrm{rank}\, D = 2$. Suppose that $\mathrm{rank}\, D = 2$; then $\dim \overline{\mathcal{R}}(D) = 1$ by 1.11, so we may consider an irreducible element ξ of A such that $V(\xi)$ is an irreducible component of $\overline{\mathcal{R}}(D)$. Since $B_{fg} = (A_{fg})^{[1]}$, we have $\overline{\mathcal{R}}(D) \subseteq V(f) \cup V(g)$ in $\mathrm{Spec}\, A$, so $V(\xi) \subseteq V(f)$ or $V(\xi) \subseteq V(g)$. Suppose, for instance, that $V(\xi) \subseteq V(f)$; then ξ and f are associates, so $\mathbf{k}[\xi] = \mathbf{k}[f]$, so 1.11 implies that the integral closure of $\mathbf{k}[f]$

in B is $\mathbf{k}[T]$ for some variable T of B. Since $A = \mathbf{k}[f]^{[1]}$ is algebraically closed in B, $\mathbf{k}[f]$ is integrally closed in B; we therefore obtain that f is a variable of B and that (4) holds. Hence, (1–5) hold if $\mathrm{d}(D) = 1$.

If $\mathrm{d}(D) > 1$ then consider the homogeneous subring R of B and the homogeneous locally nilpotent derivation $\Delta : R \to R$ which are defined in Proposition 2.1. Because of the obvious equality $B_{(fg)} = R_{(fg)}$, in order to show that (1) holds it suffices to show that

$$R_{(fg)} = A_{(fg)}. \tag{26}$$

By Corollary 2.3, we have $R = \mathbf{k}[X', Y', Z'] = \mathbf{k}^{[3]}$ and the grading of R is determined by some $w' \in \mathbf{N}^3$. Since $\ker \Delta = A$ and $\mathrm{d}(\Delta) = 1$, the preceding paragraph implies that assertions (1–5) hold for Δ; in particular assertion (1) holds for Δ, i.e., (26) holds. (To be precise, consider the graded ring (R, w'') where $w'' = w'/\gcd(w')$, then $\gcd(w'') = 1$ so the preceding paragraph implies that assertions (1–5) hold for the w''-homogeneous derivation Δ. Note that the rings $R_{(fg)}$ and $A_{(fg)}$ are not affected by that change of grading.) Hence (1) holds—and (2) follows from (1).

If $\mathrm{d}(D) < 1$ then $\mathrm{d}(A) = \mathrm{d}(D) = 0$, so $A \subseteq B_0$ and it follows that A is one of the rings $\mathbf{k}[X, Y]$, $\mathbf{k}[X, Z]$, or $\mathbf{k}[Y, Z]$. Thus it is clear that assertions (3–5) hold in this case. This completes the proof. $\square$

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