# ON POLYNOMIALS IN THREE VARIABLES ANNIHILATED BY TWO LOCALLY NILPOTENT DERIVATIONS

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Abstract. Let $B$ be a polynomial ring in three variables over an algebraically closed field $\mathbf{k}$ of characteristic zero. We are interested in irreducible polynomials $f \in B$ satisfying the following condition: there exist nonzero locally nilpotent derivations $D_1, D_2 : B \to B$ such that $\ker(D_1) \neq \ker(D_2)$ and $D_1(f) = 0 = D_2(f)$. The main result asserts that a nonconstant polynomial $f \in B$ satisfies the above requirement if and only if its "generic fiber" $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ is isomorphic, as an algebra over the field $K = \mathbf{k}(f)$, to $K[X,Y,Z]/(XY - \varphi(Z))$ for some nonconstant $\varphi(Z) \in K[Z]$.

## 1. Introduction

The main objective of this paper is to prove 1.10, below. Before stating that result, we introduce some notations and concepts.

**Conventions.** All rings and algebras are commutative, associative and unital. If $R \subseteq A$ are rings, "$A = R^{[n]}$" means that $A$ is $R$-isomorphic to the polynomial algebra in $n$ variables over $R$. We write $A^*$ for the units of $A$. If $a \in A$ then $A_a$ denotes the localized ring $S^{-1}A$ with respect to the multiplicative set $S = \{1, a, a^2, \dots\}$. If $E$ is any subset of a ring $R$ then $V_R(E) = V(E) = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq E \}$. All curves and surfaces are irreducible and reduced, unless the contrary is stated explicitely.

1.1. If $R$ is a ring of characteristic zero, a derivation $D : R \to R$ is said to be *locally nilpotent* if for each $r \in R$ there exists $n \in \mathbb{N}$ (depending on $r$) such that $D^n(r) = 0$. We use the following notations:

$$\begin{aligned} \text{LND}(R) &= \text{set of locally nilpotent derivations } D : R \to R \text{ such that } D \neq 0 \\ \text{KLND}(R) &= \{ \ker D \mid D \in \text{LND}(R) \} \quad (\text{observe that } R \notin \text{KLND}(R)) \\ \text{ML}(R) &= \text{intersection of all elements of KLND}(R), \end{aligned}$$

where we understand that $\text{ML}(R) = R$ when $\text{KLND}(R) = \varnothing$.

1.2. If $K$ is a field of characteristic zero, let $\mathfrak{D}(K)$ be the set of $K$-algebras isomorphic to

$$K[X,Y,Z]/(XY - \varphi(Z))$$

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for some nonconstant polynomial $\varphi(Z) \in K[Z] \setminus K$, where $X, Y, Z$ are indeterminates over $K$. This class of algebras was studied in [18], [6] and [7], in particular. It is well-known (see e.g. [7, 2.3]) that if $K$ is a field of characteristic zero and $R \in \mathfrak{D}(K)$ then $R$ is a normal domain and $\mathrm{ML}(R) = K$.

In [7], one defines a "Danielewski surface" to be a pair $(R, K)$ such that $R \in \mathfrak{D}(K)$, but we will not use that terminology here. In this paper we adopt the following definition, which is equivalent to the one given in [12]:

1.3. **Definition.** Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. A *Danielewski surface over* $\mathbf{k}$ is an integral affine surface $S$ over $\mathbf{k}$ together with a surjective morphism $\pi : S \to \mathbb{A}^1_{\mathbf{k}}$ with reduced fibers such that there exists a closed point $x \in \mathbb{A}^1_{\mathbf{k}}$ satisfying:

(i) $\pi^{-1}(\mathbb{A}^1_{\mathbf{k}} \setminus \{x\})$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}} \times (\mathbb{A}^1_{\mathbf{k}} \setminus \{x\})$ over $\mathbb{A}^1_{\mathbf{k}} \setminus \{x\}$

(ii) $\pi^{-1}(x)$ is a disjoint union of affine lines $\mathbb{A}^1_{\mathbf{k}}$.

**Basic elements.** Until the end of this section, let $\mathbf{k}$ be an algebraically closed field of characteristic zero and let $B = \mathbf{k}^{[3]}$. We propose the idea that the following elements of $B$ are interesting:

1.4. **Definition.** An element of $B$ is said to be *basic* if it is irreducible and belongs to at least two elements of $\mathrm{KLND}(B)$.

It is easy to see that all variables of $B$ are basic (recall that an element $f \in B$ is called a *variable* if there exist $g, h$ such that $B = \mathbf{k}[f, g, h]$). Moreover, the converse is true if we think fieldwise, i.e., each basic element is a good field generator:

1.5. *If $f$ is a basic element of $B$ then there exist $g, h \in B$ such that $\mathbf{k}(f, g, h)$ is the field of fractions of $B$.*

*Proof.* Choose $A \in \mathrm{KLND}(B)$ such that $f \in A$, then (4.1) $A = \mathbf{k}[f, g]$ for some $g$. Let $S = A \setminus \{0\}$ then by the well-known 2.3 we have $S^{-1}B = \mathbf{k}(f, g)^{[1]}$, so for some $h \in B$ we have $\mathbf{k}[f, g, h] \subseteq B \subset \mathbf{k}(f, g)[h]$. □

As the set of basic elements includes all variables and is included in the set of good field generators, it seems legitimate to state that *the concept of basic element is a natural generalization of that of variable.* Consequently, basic elements should be appealing even to those who are not primarily interested in locally nilpotent derivations.

Let us now look at concrete examples.

1.6. **Example.** Choose $X, Y, Z$ such that $B = \mathbf{k}[X, Y, Z]$, choose a nonconstant $\varphi(Z) \in \mathbf{k}[Z]$ and let $f = XY - \varphi(Z)$; then $f$ is an irreducible element of $B$ and $\mathbf{k}[X, f]$, $\mathbf{k}[Y, f]$ are two elements of $\mathrm{KLND}(B)$; thus $f$ is a basic element of $B$.

1.7. **Definition.** A basic element $f$ of $B$ is *trivial* if there exist $X, Y, Z$ satisfying $B = \mathbf{k}[X, Y, Z]$ and $f = XY - \varphi(Z)$ for some nonconstant $\varphi(Z) \in \mathbf{k}[Z]$.

Note that all variables of $B$ are trivial basic elements. To prove the existence of nontrivial basic elements, we make the following obvious observation:

1.8. *If $f$ is a trivial basic element of $B$ then $B/(f-\lambda) \in \mathfrak{D}(\mathbf{k})$ for every $\lambda \in \mathbf{k}$.*

1.9. **Example.** Consider the following homogeneous polynomial of degree 5:[1]

$$f = X^5 + 2X^3YZ - 2X^2Y^3 + X^2Z^3 - 2XY^2Z^2 + Y^4Z \in B = \mathbf{k}[X,Y,Z].$$

In the notation of [14] we have $f = H_3$, and $\mathbf{k}[H_2, H_3]$, $\mathbf{k}[H_3, H_4]$ are distinct elements of $\text{KLND}(B)$; so $f$ is a basic element of $B$. As the domain $B/fB$ is not normal, $B/fB \notin \mathfrak{D}(\mathbf{k})$. By 1.8, $f$ is a nontrivial basic element of $B$.

The main result 1.10 characterizes basic elements in terms of their generic fiber; in a sense, this result asserts that all basic elements have properties in common with the trivial ones:

1.10. **Main theorem.** *Let $B$ be a polynomial ring in three variables over an algebraically closed field $\mathbf{k}$ of characteristic zero. For a nonconstant $f \in B$, the following conditions are equivalent:*

(a) *$f$ is irreducible and $f \in A \cap A'$ for some distinct elements $A, A' \in \text{KLND}(B)$;*
(b) *$\text{ML}(\mathcal{B}) = \mathbf{k}(f)$, where we define $\mathcal{B} = \mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$;*
(c) *$\mathcal{B} \in \mathfrak{D}(\mathbf{k}(f))$, where $\mathcal{B}$ is defined as in* (b).

Note that implications $(a) \Leftrightarrow (b) \Leftarrow (c)$ are straightforward; we shall give a complete proof of 1.10 but the main point is to prove that (a) implies (c). The fact that (b) implies (c) is noteworthy because it is well-known that, for a two-dimensional $K$-algegra $R$, $\text{ML}(R) = K$ is striclty weaker than $R \in \mathfrak{D}(K)$. Also note the following consequence of our results (compare with 1.8):

1.11. *If $f$ is a basic element of $B$ then $B/(f-\lambda) \in \mathfrak{D}(\mathbf{k})$ for general $\lambda \in \mathbf{k}$. Consequently, for general $\lambda$ the hypersurface "$f = \lambda$" of $\mathbb{A}^3_{\mathbf{k}}$ is isomorphic to a hypersurface with equation "$xy = \varphi(z)$".* (Note, however, that there does not necessarely exist an automorphism of $\mathbb{A}^3_{\mathbf{k}}$ which maps one hypersurface onto the other, as the existence of such an automorphism is equivalent to $f$ being a trivial basic element.)

Result 1.11 is part of 4.4, but one can also derive it from condition (c) of 1.10. Indeed, it is not difficult to show that if the *generic fiber* of $f$ belongs to the class $\mathfrak{D}$, then its *general fibers* also belong to that class.

1.12. **Example.** Consider again the nontrivial basic element $f = X^5 + 2X^3YZ - 2X^2Y^3 + X^2Z^3 - 2XY^2Z^2 + Y^4Z$ of $B = \mathbf{k}[X,Y,Z]$ (cf. 1.9). Then one can show that $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ is $\mathbf{k}(f)$-isomorphic to $\mathbf{k}(f)[X_1, X_2, X_3]/(X_1X_2 - X_3^5 - f^3)$, where $X_1, X_2, X_3$ are indeterminates over $\mathbf{k}(f)$. Moreover, if $\lambda \in \mathbf{k}^*$ then the hypersurface $f = \lambda$ in $\mathbb{A}^3_{\mathbf{k}}$ is isomorphic to the hypersurface $xy = z^5 + 1$, but no automorphism of $\mathbb{A}^3_{\mathbf{k}}$ maps one hypersurface onto the other.

We think that the study of basic elements may lead to significant progress in the classification of locally nilpotent derivations of $B$. To support this claim we give the following result, which is a consequence of 1.10 and results 5.3 and 4.8 of [7].

[1] The zero-set of $f$ in $\mathbb{P}^2$ is "Yoshihara's rational quintic" [25].

1.13. **Corollary.** *Let $f$ be a basic element of $B$ and let $A$ and $A'$ be elements of* $\text{KLND}(B)$ *which contain $f$. Then there exists a finite sequence of local slice constructions which transforms $A$ into $A'$.*

*Proof.* In the notation of [7], the Corollary is equivalent to the following claim: *Let $f$ be a basic element of $B$ and let $R = \mathbf{k}[f]$; then the graph $\underline{\text{KLND}}_R(B)$ is connected.* So let us prove this claim.

If $A$ is any element of $\text{KLND}(B)$ which contains $f$ then by 4.1 we have $A = R^{[1]}$, so $R$ is factorially closed in $B$; so we have $R \in \mathcal{R}^{\text{in}}(B)$ (cf. [7, 5.2]) and consequently (cf. [7, 5.3]) we have an isomorphism of graphs $\underline{\text{KLND}}_R(B) \cong \underline{\text{KLND}}_K(\mathcal{B})$ where we define $\mathcal{B} = \mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ and $K = \mathbf{k}(f)$. As 1.10 asserts that $\mathcal{B} \in \mathfrak{D}(K)$, we may apply [7, 4.8] and conclude that $\underline{\text{KLND}}_K(\mathcal{B})$ (and hence $\underline{\text{KLND}}_R(B)$) is connected. $\square$

Let us also recall that it was once hoped that every element of $\text{KLND}(B)$ contained a variable of $B$, because that would have enabled one to solve the classification problem; however Freudenburg [15] exhibited a kernel which did not contain a variable. In this regard, we propose:

1.14. **Conjecture.** *Each element of* $\text{KLND}(B)$ *contains a basic element of $B$.*

Note that this conjecture is true in the homogeneous case (a proof will appear elsewhere). We think that if 1.14 is true in general, we are in a good position for developing a classification.

## 2. Preliminaries on locally nilpotent derivations

2.1. **Definition.** A subring $A$ of an integral domain $B$ is said to be *factorially closed* in $B$ if the conditions $x, y \in B$ and $xy \in A \setminus \{0\}$ imply $x, y \in A$. Note that if $A$ is factorially closed in $B$ then the following hold: $B^* = A^*$; $A$ is algebraically closed in $B$; an element of $A$ is irreducible in $A \Leftrightarrow$ it is irreducible in $B$; for $x, y \in A$, $x \mid y$ in $A$ $\Leftrightarrow x \mid y$ in $B$; if $B$ is a UFD then so is $A$.

2.2. *Let $B$ be an integral domain of characteristic zero. If $A \in \text{KLND}(B)$, then $A$ is factorially closed in $B$.*

2.3. *Let $B$ be a $\mathbb{Q}$-domain, $D \in \text{LND}(B)$ and $A = \ker D$. There exists $s \in B$ satisfying $D(s) \in A \setminus \{0\}$ and, given any such $s$, we have $B_a = A_a[s] = A_a^{[1]}$ where $a = D(s)$. In particular,* $\text{trdeg}_A(B) = 1$ *for all $A \in \text{KLND}(B)$.*

Proofs of the above two facts can be found in [23].

2.4. **Definition.** A derivation $D : B \to B$ (where $B$ is any ring) is said to be *irreducible* if the only principal ideal of $B$ which contains $D(B)$ is $B$ itself. Now assume that $B$ is a UFD of characteristic zero. It is easy to see that if $A \in \text{KLND}(B)$ then there exists an irreducible $\Delta_A \in \text{LND}(B)$ such that $\ker(\Delta_A) = A$, and such a $\Delta_A$ is unique up to multiplication by a unit of $A$. For each $A \in \text{KLND}(B)$, define

$$I_A = A \cap \Delta_A(B).$$

Note that $I_A$ is a *nonzero* ideal of $A$ and is uniquely determined by $A$ and $B$.

2.5. Let $R$ be a $\mathbf{k}$-algebra. If $D : R \to R$ is a locally nilpotent derivation and $T$ is an indeterminate over $R$ then we may define a ring homomorphism $\xi : R \to R[T]$ by $\xi(r) = \sum_{i=0}^{\infty} \frac{1}{n!} D^n(r) T^n$ $(r \in R)$. Applying the "Spec" functor yields a morphism $\operatorname{Spec}(\xi) : \operatorname{Spec} \mathbf{k}[T] \times \operatorname{Spec} R \to \operatorname{Spec} R$ which is in fact a $G_a$-action on $\operatorname{Spec} R$. Let us denote it by $\alpha_D : G_a \times \operatorname{Spec} R \to \operatorname{Spec} R$, where $\alpha_D = \operatorname{Spec}(\xi)$. It is well-known that $D \mapsto \alpha_D$ is a bijection from the set of locally nilpotent derivations of $R$ to the set of $G_a$-actions on $\operatorname{Spec} R$. Also well-known is the fact that the set of fixed points of $\alpha_D$ is $\operatorname{Fix}(\alpha_D) = V(DR) \subseteq \operatorname{Spec} R$. This justifies the following:

2.6. **Definition.** If $D$ is a locally nilpotent derivation of a $\mathbf{k}$-algebra $R$, we define the *set of fixed points of $D$* to be:

$$\operatorname{Fix}(D) = V(DR) \subseteq \operatorname{Spec} R.$$

In other words, we define $\operatorname{Fix}(D) = \operatorname{Fix}(\alpha_D)$.

## Locally nilpotent derivations of $\mathbf{k}^{[3]}$

From now-on, let $B = \mathbf{k}^{[3]}$ where $\mathbf{k}$ is a field of characteristic zero.

2.7. *If $A \in \operatorname{KLND}(B)$ then:*

1. $A = \mathbf{k}^{[2]}$
2. *$B$ is faithfully flat as an $A$-module.*
3. *The ideal $I_A$ of $A$ is principal (cf. 2.4).*

In the case $\mathbf{k} = \mathbb{C}$, assertion (1) was proved by Miyanishi [20] and (2) and (3) by Bonnet [3]. See [9] for the generalizations to all fields of characteristic zero.

2.8. **Notation.** Given $A \in \operatorname{KLND}(B)$, let us write $\psi_A$ for a generator of the principal ideal $I_A$ of $A$. So $\psi_A$ is a nonzero element of $A$ determined up to multiplication by an element of $\mathbf{k}^*$.

2.9 ([4]-2.6). *Given $A \in \operatorname{KLND}(B)$, the derivation $\Delta_A$ (cf. 2.4) can be obtained as follows: choose $X, Y, Z$ such that $B = \mathbf{k}[X, Y, Z]$ and $f, g$ such that $A = \mathbf{k}[f, g]$, then*

$$\Delta_A(h) = \left| \frac{\partial(f, g, h)}{\partial(X, Y, Z)} \right| \quad (\textit{for all } h \in B).$$

2.10. Fix $A \in \operatorname{KLND}(B)$ and consider the morphism $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by the inclusion $A \hookrightarrow B$. As $B$ is faithfully flat over $A$ (cf. 2.7), $\mathcal{Q}$ is surjective and every fiber of $\mathcal{Q}$ has pure dimension 1. Moreover, 2.3 implies that the general fibers of $\mathcal{Q}$ are affine lines. Now consider the unique irreducible locally nilpotent derivation $D : B \to B$ with kernel $A$ (i.e., $D = \Delta_A$, cf. 2.4). We make two observations regarding the fixed points of $D$ (cf. 2.6) in relation with the morphism $\mathcal{Q}$:

(a) $\operatorname{Spec} B \setminus \operatorname{Fix}(D) = \operatorname{Smooth}(\mathcal{Q})$

(b) $\mathcal{Q}$ *maps* $\operatorname{Fix}(D)$ *to a finite set of closed points in* $\operatorname{Spec} A$.

*Proof.* To prove (a), choose $X, Y, Z$ such that $B = \mathbf{k}[X, Y, Z]$ and $f, g$ such that $A = \mathbf{k}[f, g]$. Then $\mathrm{Fix}(D) = V(J)$ where $J$ is the ideal of $B$ generated by $D(B)$, i.e., $J = (DX, DY, DZ)$. Using 2.9, we find that $DX, DY, DZ$ are the $2 \times 2$ minors of the $2 \times 3$ jacobian matrix $M = \frac{\partial(f,g)}{\partial(X,Y,Z)}$, so the complement of $\mathrm{Fix}(D)$ is the set of points in $\mathrm{Spec}\, B = \mathbb{A}^3$ at which $\mathrm{rank}\, M = 2$. As $\mathcal{Q}$ is a flat morphism by 2.7, the set of points at which $\mathrm{rank}\, M = 2$ is exactly the set of points at which the morphism $\mathcal{Q}$ is smooth.

To prove (b), recall that $\mathrm{Fix}(D) = \mathrm{Fix}(\alpha_D)$ where $\alpha_D$ is the $G_a$-action on $\mathrm{Spec}\, B$ determined by $D$. Consider a closed point $P \in \mathrm{Fix}(\alpha_D)$; then $P$ belongs to the fiber $F_P = \mathcal{Q}^{-1}(\mathcal{Q}(P))$, which is a finite union of curves since every fiber of $\mathcal{Q}$ has pure dimension 1. The $G_a$-action $\alpha_D$ restricts to a $G_a$-action on $F_P$, and hence to a $G_a$-action on each irreducible component of $F_P$; let $\Gamma_P$ be an irreducible component of $F_P$ such that $P \in \Gamma_P$, then the action on $\Gamma_P$ is trivial (any $G_a$-action on a curve and having a fixed point is trivial), so $\Gamma_P \subseteq \mathrm{Fix}(\alpha_D)$. This shows that $\mathrm{Fix}(\alpha_D)$ is a union of curves $\Gamma_P$, where each $\Gamma_P$ is mapped to a point by $\mathcal{Q}$. Since $D$ is irreducible, $\mathrm{Fix}(\alpha_D)$ has codimension at least 2 in $\mathrm{Spec}\, B = \mathbb{A}^3$ and consequently $\mathrm{Fix}(\alpha_D)$ is a finite union of curves $\Gamma_P$, which proves (b). $\square$

**2.11.** Let $A \in \mathrm{KLND}(B)$ and consider the fibers of $\mathcal{Q} : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$. If $\mathfrak{p} \in \mathrm{Spec}\, A$, consider the residue field $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and the "fiber ring" $B \otimes_A \kappa(\mathfrak{p})$. Define the following subset of $\mathrm{Spec}\, A$:

$$\mathcal{X}_A = \left\{ \mathfrak{p} \in \mathrm{Spec}\, A \mid B \otimes_A \kappa(\mathfrak{p}) \neq \kappa(\mathfrak{p})^{[1]} \right\}$$

By 2.3 the general fibers of $\mathcal{Q}$ are affine lines, i.e., $\mathcal{X}_A$ is not dense in $\mathrm{Spec}\, A$. The following fact is needed in the proof of assertion (b-iii) of 3.3.

2.12. **Lemma.** *If $A \in \mathrm{KLND}(B)$, then $\mathcal{X}_A$ is a constructible subset of* $\mathrm{Spec}\, A$.

**2.12.1.** Some comments are in order. Given a morphism of schemes $\varphi : X \to Y$ one may ask whether $\mathcal{A}(\varphi) = \left\{ y \in Y \mid X_y \cong \mathbb{A}^1_{\kappa(y)} \right\}$ is a constructible subset of $Y$, where $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$ is the residue field of $Y$ at $y$ and $X_y = \mathrm{Spec}\, \kappa(y) \times_Y X$ is the fiber of $\varphi$ over $y$. Apparently, the answer is affirmative whenever $\varphi$ is a morphism of integral schemes of finite type over a field of characteristic zero. We don't know a reference for this fact, so we will sketch the proof for the special case $\varphi = \mathcal{Q}$ (see the proof of 2.12). We shall need the following facts:

(i) (Verification left to the reader.) *Let $Q \subset R$ be integral domains and $K$ the field of fractions of $Q$. If $R$ is a finitely generated $Q$-algebra and $K \otimes_Q R = K^{[n]}$, then $R_s = Q_s^{[n]}$ for some $s \in Q \setminus \{0\}$.*

(ii) (Verification left to the reader.) *Consider a field extension $\mathbf{k}'/\mathbf{k}$, a scheme $Y$ of finite type over* $\mathrm{Spec}\, \mathbf{k}$ *and the canonical morphism $\varepsilon : Y' \to Y$ where $Y' = \mathrm{Spec}\, \mathbf{k}' \times_{\mathrm{Spec}\, \mathbf{k}} Y$. For a subset $E$ of $Y$, $E$ is constructible in $Y \iff \varepsilon^{-1}(E)$ is constructible in $Y'$.*

(iii) (Verification left to the reader.) *Let $\varphi : X \to Y$ and $\varepsilon : Y' \to Y$ be morphisms of schemes over a field of characteristic zero, and consider the morphism $\varphi' : Y' \times_Y X \to Y'$ obtained by base extension. Then $\mathcal{A}(\varphi') = \varepsilon^{-1}\big(\mathcal{A}(\varphi)\big)$.*

(iv) *Let $\varphi : X \to Y$ be a morphism of complex irreducible quasi-projective algebraic varieties. Then there exists a Zariski-dense open subset $Y_0$ of $Y$ so that the fibers of $\varphi$ over any two closed points of $Y_0$ are homeomorphic in the standard topology.* (Special case of the Varchenko equisingularity theorem [24, Thm 5.2].)

(v) *Let $S$ be a geometrically integral scheme of finite type over a field of characteristic zero and let $\varphi : X \to S$ be an affine, flat morphism of finite type. Assume that there is a nonempty open subset $S_0$ of $S$ such that every closed point of $S_0$ belongs to $\mathcal{A}(\varphi)$. Then the generic point of $S$ belongs to $\mathcal{A}(\varphi)$.* (Special case of [17, Lemma 2.2].)

*Proof of* 2.12. Let $\mathbf{k}$ be a field of characteristic zero and $B = \mathbf{k}^{[3]}$. We say that "2.12 is valid over $\mathbf{k}$" if for every $A \in \text{KLND}(B)$ the set $\mathfrak{X}_A$ is constructible in $\operatorname{Spec} A$, or equivalently, for every $A \in \text{KLND}(B)$ the set $\mathcal{A}(\mathfrak{Q})$ is constructible in $\operatorname{Spec} A$ (where $\mathfrak{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ is determined by $A \hookrightarrow B$). We have to show that 2.12 is valid over each field of characteristic zero. We first show that it is valid over $\mathbb{C}$.

So let $B = \mathbb{C}^{[3]}$, $A \in \text{KLND}(B)$ and $\mathfrak{Q} : X \to Y$ where $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We want to show that $\mathcal{A}(\mathfrak{Q})$ is constructible. Let $C$ be an irreducible closed subset of $Y$ such that $\mathcal{A}(\mathfrak{Q}) \cap C$ is dense in $C$; in view of (6.C) of [19], it suffices to show that $\mathcal{A}(\mathfrak{Q}) \cap C$ contains a nonempty Zariski-open subset of $C$. This is obvious if $C$ is a closed point; this is also easy if $C = Y$, because 2.3 implies that $\mathcal{A}(\mathfrak{Q})$ contains a nonempty open subset of $Y$; so we may assume that $C$ is a curve. Consider the irreducible element $p \in A$ such that $C = V(p)$. Give $C$ the reduced induced structure of a closed subscheme of $Y$ (i.e., $C \cong \operatorname{Spec}(A/pA)$) and consider the morphism $\mathfrak{Q}' : X \times_Y C \to C$ obtained by base extension. As $A$ is factorially closed in $B$, it follows that $p$ is irreducible in $B$, so $X \times_Y C = \operatorname{Spec}(B/pB)$ is an integral scheme, i.e., an integral surface. The fibers of $\mathfrak{Q}$ and $\mathfrak{Q}'$ over any point of $C$ are the same, so $\mathcal{A}(\mathfrak{Q}) \cap C = \mathcal{A}(\mathfrak{Q}')$ (this also follows from (iii)). This means that the proof in the complex case reduces to proving:

$(*)$ *Let $\varphi : S \to C$ be a morphism of affine complex algebraic varieties, where $S$ is a surface and $C$ is a curve. If $\mathcal{A}(\varphi)$ is dense in $C$, then it contains a nonempty Zariski-open subset of $C$.*

Note that if $U$ is a nonempty affine open subset of $C$ then the restriction $\varphi' : \varphi^{-1}(U) \to U$ of $\varphi$ satisfies the hypothesis of $(*)$, and it suffices to prove $(*)$ for $\varphi'$. Using this observation, we see that it suffices to prove $(*)$ in the special case where the following additional conditions are satisfied: (a) $\varphi$ is surjective; (b) $C$ is a smooth curve; (c) $\varphi$ is a smooth morphism; (d) (using (iv)) the fibers of $\varphi$ over any two closed points of $C$ are homeomorphic in the standard topology. Moreover, as $\mathcal{A}(\varphi)$ is dense in $C$, at least one of the following conditions is satisfied:

(e) $\mathcal{A}(\varphi)$ contains infinitely many closed points of $C$;

(f) the generic point of $C$ belongs to $\mathcal{A}(\varphi)$.

Suppose that (e) holds. If $P$ is any closed point of $C$ then the fiber $\varphi^{-1}(P)$ is smooth and homeomorphic to $\mathbb{C}$ in the standard topology, so it is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ as an algebraic variety. This shows that all closed points of $C$ belong to $\mathcal{A}(\varphi)$, so applying (v) to $\varphi$ implies that (f) holds. So (f) must hold in all cases and by (i) we obtain that some nonempty open subset of $C$ is included in $\mathcal{A}(\varphi)$, which proves $(*)$. This also proves that 2.12 is valid over $\mathbb{C}$.

Using (ii), (iii) and a straightforward "Lefschetz Principle" argument, we now extend validity to all fields of characteristic zero. We begin by proving:

$(**)$ *If* $\mathbf{k}'/\mathbf{k}$ *is an extension of fields of characteristic zero and if* 2.12 *is valid over* $\mathbf{k}'$*, then it is valid over* $\mathbf{k}$.

Indeed, consider $B = \mathbf{k}^{[3]}$, $A \in \text{KLND}(B)$ and $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$. Define $B' = \mathbf{k}' \otimes_{\mathbf{k}} B = \mathbf{k}'^{[3]}$ and $A' = \mathbf{k}' \otimes_{\mathbf{k}} A$; note that $A' \in \text{KLND}(B')$ and let $\mathcal{Q}' : \operatorname{Spec} B' \to \operatorname{Spec} A'$ be the morphism determined by the inclusion $A' \hookrightarrow B'$. Then in the diagram

$$\begin{array}{ccc} \operatorname{Spec} B' & \xrightarrow{\mathcal{Q}'} & \operatorname{Spec} A' \\ \big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ \operatorname{Spec} B & \xrightarrow{\mathcal{Q}} & \operatorname{Spec} A \end{array}$$

$\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by base extension via $\varepsilon : \operatorname{Spec} A' \to \operatorname{Spec} A$. Since we are assuming that 2.12 is valid over $\mathbf{k}'$, $\mathcal{A}(\mathcal{Q}')$ is constructible in $\operatorname{Spec} A'$. We have $\mathcal{A}(\mathcal{Q}') = \varepsilon^{-1}\big(\mathcal{A}(\mathcal{Q})\big)$ by (iii) and consequently $\mathcal{A}(\mathcal{Q})$ is constructible by (ii). This proves $(**)$.

Now let $\mathbf{k}$ be any field of characteristic zero and consider $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$, $A \in \text{KLND}(B)$ and $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$. We leave it to the reader to show that there exist a subfield $\mathbf{k}_0$ of $\mathbf{k}$ finitely generated over $\mathbb{Q}$ and an element $A_0 \in \text{KLND}(B_0)$ (where $B_0 = \mathbf{k}_0[X, Y, Z]$) such that $B = \mathbf{k} \otimes_{\mathbf{k}_0} B_0$ and $A = \mathbf{k} \otimes_{\mathbf{k}_0} A_0$; so if $\mathcal{Q}_0 : \operatorname{Spec} B_0 \to \operatorname{Spec} A_0$ is the morphism determined by $A_0 \hookrightarrow B_0$ then $\mathcal{Q}$ is obtained from $\mathcal{Q}_0$ by base extension via $\varepsilon : \operatorname{Spec} A \to \operatorname{Spec} A_0$. Since $\mathbf{k}_0$ is isomorphic to a subfield of $\mathbb{C}$, $(**)$ implies that 2.12 is valid over $\mathbf{k}_0$ and hence that $\mathcal{A}(\mathcal{Q}_0)$ is constructible. Since $\mathcal{A}(\mathcal{Q}) = \varepsilon^{-1}\big(\mathcal{A}(\mathcal{Q}_0)\big)$ by (iii), $\mathcal{A}(\mathcal{Q})$ is constructible by (ii). $\square$

2.13. **Lemma.** *Let* $A \in \text{KLND}(B)$*. For a multiplicative subset* $S$ *of* $A$*, the following are equivalent:*

(1) $S^{-1}B = (S^{-1}A)^{[1]}$
(2) $S \cap I_A \neq \varnothing$.

*Proof.* It follows from 2.3 that if $a \in I_A \setminus \{0\}$ then $B_a = (A_a)^{[1]}$, so (2) implies (1). The converse is Lemma 1.4 of [5]. $\square$

2.14. **Lemma.** *If* $A \in \text{KLND}(B)$ *then* $V(I_A)$ *is the closure of* $\mathcal{X}_A$ *in* $\operatorname{Spec} A$.

*Remark.* In fact, 2.13 and 2.14 are valid (with the same proofs) under weaker hypotheses: for 2.13, it suffices to assume that $B$ is a factorial $\mathbb{Q}$-algebra; for 2.14, assume that $B$ is a factorial $\mathbb{Q}$-algebra and that $A \in \text{KLND}(B)$ satisfies:

$(*)$ $A$ is noetherian and $B$ is flat as an $A$-module and finitely generated as an $A$-algebra.

*Proof of* 2.14. Let $\overline{\mathcal{X}_A}$ be the closure of $\mathcal{X}_A$ in $\operatorname{Spec} A$. Then $\overline{\mathcal{X}_A} \subseteq V(I_A)$ easily follows from 2.13.

For the reverse inclusion, consider a basic affine open set $\varnothing \neq \mathcal{D}(f) \subseteq \operatorname{Spec}(A)$ such that $\mathcal{D}(f) \cap \mathcal{X}_A = \varnothing$ (where $f \in A$ and $\mathcal{D}(f) = \operatorname{Spec}(A) \setminus V(f)$); it suffices to show that $\mathcal{D}(f) \cap V(I_A) = \varnothing$. Now $(*)$ implies that $A_f$ is noetherian and that $B_f$ is flat as an $A_f$-module and finitely generated as an $A_f$-algebra. Moreover, the condition $\mathcal{D}(f) \cap \mathcal{X}_A = \varnothing$ implies that $B_f \otimes_{A_f} \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})^{[1]}$ holds for every $\mathfrak{p} \in \operatorname{Spec}(A_f)$. Thus, by Theorem 3.4 of [1], $A_f \subset B_f \subset (A_f)^{[n]}$ for some $n > 0$. Together with the fact that $A_f$ and $B_f$ are UFD's such that $\operatorname{trdeg}_{A_f}(B_f) = 1$, the last condition implies that $B_f = (A_f)^{[1]}$. Then 2.13 gives $f^m \in I_A$ for some $m$, and consequently $\mathcal{D}(f) \cap V(I_A) = \varnothing$. ☐

2.15. **Corollary.** *Let $A \in \text{KLND}(B)$ and consider $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ determined by $A \hookrightarrow B$. Let $U = \operatorname{Spec} A \setminus \overline{\mathcal{X}}_A$, where $\overline{\mathcal{X}}_A$ is the closure of $\mathcal{X}_A$ in $\operatorname{Spec} A$. Then $\mathcal{Q}^{-1}(U)$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}} \times U$ over $U$.*

*Proof.* By 2.14, and since $I_A = (\psi_A)$, it suffices to show that $B_{\psi_A} = A_{\psi_A}^{[1]}$. This follows from 2.13, because $\psi_A \in I_A$. ☐

2.16. **Proposition.** *Let $B = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. Let $A \in \text{KLND}(B)$ and let $p$ be a prime factor of $\psi_A$. Then $A/pA$ is a subalgebra of $\mathbf{k}^{[1]}$.*

*Proof.* By a straightforward Lefschetz Principle argument which is left to the reader, it suffices to prove the case $\mathbf{k} = \mathbb{C}$. Assume that $\mathbf{k} = \mathbb{C}$, and let us obtain this result as a corollary to results 2.2 and 3.2 of Kaliman's [16]. The "$\Gamma$" which occurs in the statement of those results is a closed subset of $\operatorname{Spec} A$ and satisfies the following conditions: $\Gamma$ *is a finite union of curves in* $\operatorname{Spec} A$*, and if we define* $E = \mathcal{Q}^{-1}(\Gamma) \subset \operatorname{Spec}(B)$ *then the variety* $\operatorname{Spec}(B) \setminus E$ *is naturally isomorphic to* $(\operatorname{Spec}(A) \setminus \Gamma) \times \mathbb{C}$ *over* $\operatorname{Spec}(A) \setminus \Gamma$ (as before, $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ is the morphism determined by $A \hookrightarrow B$). It follows that $\Gamma \supseteq \mathcal{X}_A$, and consequently that $\Gamma \supseteq \overline{\mathcal{X}_A} = V_A(I_A) = V_A(\psi_A) \supseteq V_A(p)$ by 2.14 and 2.8; so if we define $\Gamma^1 = V_A(p)$ then

$$\Gamma^1 \subseteq \overline{\mathcal{X}_A} \tag{1}$$

(2) $\Gamma^1$ *is an irreducible component of the set* $\Gamma$ *of* [16].

By [16, 2.2, 2.3], there exist a (not necessarely irreducible) curve $Z$, a finite morphism $\theta : Z \to \Gamma$ and a surjective morphism $\kappa : E \to Z$ so that $\mathcal{Q}|_E = \theta \circ \kappa : E \to \Gamma$ and $Z$ contains a Zariski dense subset $Z^*$ for which

(3) $\kappa^{-1}(Z^*)$ *is isomorphic to* $Z^* \times \mathbb{C}$ *over* $Z^*$.

Define $Z^1 = \theta^{-1}(\Gamma^1)$ and $E^1 = \mathcal{Q}^{-1}(\Gamma^1)$. Note that $E^1$ is irreducible, since $E^1 = V_B(p)$ and $p$ is irreducible in $B$. As $\kappa$ is surjective, we have $Z^1 = \kappa\big(\kappa^{-1}(Z^1)\big) = \kappa(E^1)$, so $Z^1$ is irreducible. As $\theta$ is surjective, $\Gamma^1 = \theta\big(\theta^{-1}(\Gamma^1)\big) = \theta(Z^1)$. Hence,

(4) $Z^1$ *is an irreducible component of* $Z$ *and* $\theta(Z^1) = \Gamma^1 = V_A(p)$.

We claim:

$$\theta|_{Z^1} : Z^1 \to \Gamma^1 \textit{ is not bijective.} \tag{5}$$

Indeed, if (5) is false then $\theta|_{Z^1}$ is a birational morphism, so there is a Zariski open subset $U \subseteq \Gamma^1$ such that $(\theta|_{Z^1})^{-1}(U) \to U$ is an isomorphism. By (3), it follows that for a general point $P \in \Gamma^1$ we have $\mathcal{Q}^{-1}(P) = \mathbb{A}^1$ (scheme-theoretically), i.e., $P \notin \mathcal{X}_A$. This contradicts (1), so (5) is true.

Now (2), (4) and (5) allow us to make use of [16, 3.2] and conclude that $\Gamma^1$ is a "polynomial curve" (i.e., its normalization is $\mathbb{A}^1_{\mathbb{C}}$). This implies that $A/pA$ is a subalgebra of $\mathbb{C}^{[1]}$, which completes the proof. $\square$

## 3. Irreducible polynomials belonging to at least one kernel

In this section, $\mathbf{k}$ is an algebraically closed field of characteristic zero. We say that a condition is satisfied for general $\lambda \in \mathbf{k}$ if there are at most finitely many $\lambda \in \mathbf{k}$ for which that condition is false.

3.1. **Global Assumptions.** Throughout the section, the following assumptions and notations are in effect. Let $B = \mathbf{k}^{[3]}$ and fix a pair $(f, A)$ such that $A \in \text{KLND}(B)$, $f \in A$ and $f - \lambda_0$ is irreducible[2] for at least one $\lambda_0 \in \mathbf{k}$.

The following objects are determined by the pair $(f, A)$:

- Let $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ be the morphism determined by the inclusion $A \hookrightarrow B$. Let $D : B \to B$ be the essentially unique (cf. 2.4) irreducible locally nilpotent derivation such that $\ker D = A$.
- For each $\lambda \in \mathbf{k}$, define the $\mathbf{k}$-algebra $B_\lambda = B/(f - \lambda)$ and consider the locally nilpotent derivation $D_\lambda : B_\lambda \to B_\lambda$ determined by the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{D} & B \\ {\scriptstyle \pi_\lambda}\downarrow & & \downarrow{\scriptstyle \pi_\lambda} \\ B_\lambda & \xrightarrow[D_\lambda]{} & B_\lambda \end{array}$$

  where $\pi_\lambda$ is the canonical epimorphism. Let $\mathcal{Q}_\lambda : \operatorname{Spec} B_\lambda \to \operatorname{Spec}\big(\ker D_\lambda\big)$ be the morphism determined by the inclusion $\ker(D_\lambda) \hookrightarrow B_\lambda$.

*Warning.* Several results in this section begin with the phrase "Let the setup be as in 3.1." Each such result gives information about the pair $(f, A)$ or the objects ($B_\lambda$, $\mathcal{Q}_\lambda$, etc) which are determined by that pair, and this is the case even when $A$ does not appear in the statement of the result. For instance, 3.12 appears to be asserting that *if $f \in B$ is such that* $\mathrm{ML}(B/(f - \lambda)) = \mathbf{k}$ *for infinitely many $\lambda \in \mathbf{k}$, then* $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ *belongs to* $\mathfrak{D}(\mathbf{k}(f))$*;* however, one also requires as a hypothesis that $f$ belongs to some $A \in \text{KLND}(B)$, and that $f - \lambda_0$ is irreducible for some $\lambda_0$.

[2]Note that an element of $A$ is irreducible in $A$ if and only if it is irreducible in $B$, because $A$ is factorially closed in $B$ (2.1, 2.2).

3.2. **Notation.** Let $\varphi : X \to Y$ be a morphism of schemes such that $Y \setminus \mathcal{A}(\varphi)$ is a finite set (cf. 2.12.1). Then the cardinality of the set $Y \setminus \mathcal{A}(\varphi)$ is denoted $N(\varphi)$.

3.3. **Lemma.** *Let the setup be as in* 3.1. *For general* $\lambda \in \mathbf{k}$, *the following hold:*

(a) $B_\lambda$ *is an integral domain and* $\operatorname{Spec}(B_\lambda)$ *is isomorphic to a smooth hypersurface of* $\mathbb{A}^3$

(b) $\operatorname{Fix}(D_\lambda) = \varnothing$, $\ker(D_\lambda) = \pi_\lambda(A) \cong A/(f-\lambda)$ *and the morphism*

$$\mathcal{Q}_\lambda : \operatorname{Spec} B_\lambda \to \operatorname{Spec}\big(\ker D_\lambda\big)$$

*has the following properties:*

(i) $\mathcal{Q}_\lambda$ *is surjective and smooth; in particular every fiber of* $\mathcal{Q}_\lambda$ *is reduced*

(ii) *every fiber of* $\mathcal{Q}_\lambda$ *which is not an affine line is a disjoint union of (reduced) affine lines;*

(iii) $N(\mathcal{Q}_\lambda) = |V_A(\psi_A, f-\lambda)|$ *(cf.* 3.2, 2.8*).*

*Proof.* We have $A = \mathbf{k}^{[2]}$ by 2.7, and for polynomials in two variables it is well-known (cf. [22]) that the set $\Lambda = \big\{\, \lambda \in \mathbf{k} \mid f - \lambda \text{ is reducible in } A \,\big\}$ is either finite or equal to $\mathbf{k}$. As $\lambda_0 \notin \Lambda$, $\Lambda$ is finite and consequently $B_\lambda$ is an integral domain for general $\lambda$. So the general fibers of the morphism $\mathbb{A}^3 = \operatorname{Spec} B \to \operatorname{Spec} \mathbf{k}[f]$ are integral schemes. Since in characteristic zero the general fibers are smooth, (a) is proved.

The same argument applied to $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{k}[f]$ shows that

(6) *for general* $\lambda \in \mathbf{k}$, $A/(f-\lambda)$ *is an integral domain and is smooth over* $\mathbf{k}$.

Let $E = \mathcal{Q}\big(\operatorname{Fix}(D)\big) \subset \operatorname{Spec} A$ and recall from 2.10 that $E$ is a finite set of closed points of $\operatorname{Spec} A$. Consequently, $V_A(f-\lambda)$ is disjoint from $E$ for general $\lambda \in \mathbf{k}$. If $\operatorname{Fix}(D_\lambda) \neq \varnothing$ then there exists $\mathfrak{m} \in \operatorname{Spec} B_\lambda$ such that $D_\lambda(B_\lambda) \subseteq \mathfrak{m}$; let $\mathfrak{M} = \pi_\lambda^{-1}(\mathfrak{m})$, then $\mathfrak{M} \in \operatorname{Fix}(D)$ and $f - \lambda \in \mathfrak{M}$, so $\mathfrak{M} \cap A \in E \cap V_A(f-\lambda)$. As $E \cap V_A(f-\lambda) = \varnothing$ for general $\lambda$, $\operatorname{Fix}(D_\lambda) = \varnothing$ for general $\lambda$.

Define the $\mathbf{k}$-subalgebra $A_\lambda = \pi_\lambda(A)$ of $B_\lambda$ and note that $A_\lambda \subseteq \ker(D_\lambda)$. We claim that equality holds (for general $\lambda$). Before proving this, first note that $\pi_\lambda$ restricts to a surjective homomorphism $A \to A_\lambda$ with kernel $A \cap (f-\lambda)B$; as $A$ is factorially closed in $B$ we get $A \cap (f-\lambda)B = (f-\lambda)A$, so $A_\lambda \cong A/(f-\lambda)A$. By (6), it follows that $\operatorname{Spec}(A_\lambda)$ is a smooth integral curve.

As $B_\lambda$ is an integral domain, it follows that $\ker(D_\lambda)$ is factorially closed in $B_\lambda$; in particular, $\ker(D_\lambda) = K \cap B_\lambda$ where $K$ is the field of fractions of $\ker(D_\lambda)$. By (a), $B_\lambda$ is in particular normal; so it follows from a theorem of Zariski [26] that $\ker(D_\lambda)$ is finitely generated as a $\mathbf{k}$-algebra. By 2.3, $\ker(D_\lambda)$ has transcendence degree 1 over $\mathbf{k}$. Consequently, $C_\lambda = \operatorname{Spec}(\ker D_\lambda)$ is an algebraic curve. Since $B_\lambda$ is a normal domain and $\ker D_\lambda$ is integrally closed in $B_\lambda$, $\ker D_\lambda$ is a normal domain; hence $C_\lambda$ is a smooth integral curve. Consider the dominant morphisms

$$\operatorname{Spec}(B_\lambda) \xrightarrow{u} C_\lambda \xrightarrow{v} \operatorname{Spec} A_\lambda$$

determined by the inclusions $A_\lambda \subseteq \ker(D_\lambda) \subset B_\lambda$ (i.e., $u = \mathcal{Q}_\lambda$). To complete the proof that $\ker(D_\lambda) = A_\lambda$, it suffices to show that $v$ is an isomorphism. As $C_\lambda$ and $\operatorname{Spec} A_\lambda$ are smooth curves, it suffices to show that $v$ is surjective and birational.

We know from 2.10 that the map $\mathcal{Q} : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, so the restriction $\rho_\lambda : \mathcal{Q}^{-1}(L_\lambda) \to L_\lambda$ of $\mathcal{Q}$ is also surjective, where we define $L_\lambda$ to be the closed subscheme of $\operatorname{Spec} A$ determined by the principal ideal $(f - \lambda)A$ of $A$. Now it is clear that we have a commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec}(B_\lambda) & \xrightarrow{\cong} & \mathcal{Q}^{-1}(L_\lambda) \\ {\scriptstyle v\circ u}\big\downarrow & & \big\downarrow{\scriptstyle \rho_\lambda} \\ \operatorname{Spec}(A_\lambda) & \xrightarrow{\cong} & L_\lambda \end{array} \qquad \text{(for general } \lambda) \tag{7}$$

so the fact that $\rho_\lambda$ is surjective implies that $v$ is surjective.

As the general fibers of $\mathcal{Q}$ are $\mathbb{A}^1$, a general $\lambda \in \mathbf{k}$ satisfies the condition:

$$\mathcal{Q}^{-1}(P) = \mathbb{A}^1 \textit{ for general } P \in L_\lambda$$

or equivalently:

$$(v \circ u)^{-1}(P) = \mathbb{A}^1 \textit{ for general } P \in \operatorname{Spec} A_\lambda. \tag{8}$$

As $u$ is dominant, the complement of $\operatorname{im}(u)$ is a finite set; thus for a general point $P \in \operatorname{Spec} A_\lambda$ we have $v^{-1}(P) \subset \operatorname{im}(u)$, and consequently

$$|v^{-1}(P)| \leq \textit{number of connected components of } (v \circ u)^{-1}(P).$$

By (8) we obtain $|v^{-1}(P)| \leq 1$, so $v$ is birational and we have shown that

$$\ker(D_\lambda) = \pi_\lambda(A) \cong A/(f - \lambda)A$$

for general $\lambda \in \mathbf{k}$. In fact $v$ is the identity map and $u = \mathcal{Q}_\lambda$, so (7) is really:

$$\begin{array}{ccc} \operatorname{Spec}(B_\lambda) & \xrightarrow{\cong} & \mathcal{Q}^{-1}(L_\lambda) \\ {\scriptstyle \mathcal{Q}_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \rho_\lambda} \\ \operatorname{Spec}(\ker D_\lambda) & \xrightarrow{\cong} & L_\lambda \end{array} \qquad \text{(for general } \lambda). \tag{9}$$

As $\rho_\lambda$ is surjective, it follows that $\mathcal{Q}_\lambda$ is surjective.

Consider the open subset $W = \operatorname{Spec} A \setminus E$ of $\operatorname{Spec} A$ where, as before, $E$ is the finite set $\mathcal{Q}\big(\operatorname{Fix}(D)\big)$; using 2.10 again, we see that the restriction $\mathcal{Q}^{-1}(W) \to W$ of $\mathcal{Q}$ is a smooth morphism. As $L_\lambda \subset W$ (for general $\lambda$), it follows that $\mathcal{Q}^{-1}(L_\lambda) \xrightarrow{\rho_\lambda} L_\lambda$ is a smooth morphism and consequently (by (9)) that $\mathcal{Q}_\lambda : \operatorname{Spec} B_\lambda \to \operatorname{Spec}(\ker D_\lambda)$ is a smooth morphism. It follows that all fibers of $\mathcal{Q}_\lambda$ are reduced.

Consider $x \in \operatorname{Spec}(\ker D_\lambda)$ such that $\mathcal{Q}_\lambda^{-1}(x)$ is not an affine line (over the suitable residue field). Then $x$ is a closed point of $\operatorname{Spec}(\ker D_\lambda)$. Moreover, the fixed point free $G_a$-action on $\operatorname{Spec} B_\lambda$ determined by $D_\lambda$ restricts to a fixed point free $G_a$-action on each irreducible component of $\mathcal{Q}_\lambda^{-1}(x)$; thus $\mathcal{Q}_\lambda^{-1}(x)$ is a disjoint union of (reduced) affine lines.

Only (b–iii) remains to be proved. By (9) we have $N(\mathcal{Q}_\lambda) = N(\rho_\lambda)$, and since we have an isomorphism of schemes $\rho_\lambda^{-1}(P) \cong \mathcal{Q}^{-1}(P)$ for each $P \in L_\lambda$, we also have $N(\rho_\lambda) = |L_\lambda \cap \mathfrak{X}_A|$. Since $\mathfrak{X}_A$ is constructible (2.12) and not dense in $\operatorname{Spec} A$, $\bar{\mathfrak{X}}_A \setminus \mathfrak{X}_A$ is a finite set of closed points and consequently $L_\lambda \cap \mathfrak{X}_A = L_\lambda \cap \bar{\mathfrak{X}}_A$ for general $\lambda$. We have $\bar{\mathfrak{X}}_A = V_A(\psi_A)$ (2.14), so $N(\mathcal{Q}_\lambda) = |L_\lambda \cap V_A(\psi_A)|$ for general $\lambda$, from which we obtain the last assertion. □

3.4. **Lemma.** *Let the setup be as in* 3.1 *and consider an element* $\lambda \in \mathbf{k}$ *for which all conclusions of* 3.3 *are valid. The following conditions are equivalent:*

(a) $\ker(D_\lambda) = \mathbf{k}^{[1]}$ *and* $N(\mathcal{Q}_\lambda) \leq 1$

(b) $\mathcal{Q}_\lambda : \operatorname{Spec} B_\lambda \to \operatorname{Spec}\big(\ker D_\lambda\big)$ *defines a Danielewski surface.*

*Proof.* It is obvious that (b) implies (a). Assume that (a) holds then, as $\ker(D_\lambda) = \mathbf{k}^{[1]}$, the codomain of $\mathcal{Q}_\lambda$ is $\mathbb{A}^1_{\mathbf{k}}$; $\operatorname{Spec} B_\lambda$ is a smooth affine surface and $\mathcal{Q}_\lambda$ is a surjective morphism with reduced fibers, and such that each fiber is a disjoint union of affine lines; so there only remains to check condition (i) of 1.3. Consider $\rho_\lambda$ as in (9); $N(\mathcal{Q}_\lambda) \leq 1$ and $N(\mathcal{Q}_\lambda) = |V_A(\psi_A, f-\lambda)|$ imply that the curve $L_\lambda = V_A(f-\lambda) \subset \operatorname{Spec} A$ meets $\overline{\mathfrak{X}}_A$ in at most one point $x$, and 2.15 implies that $\rho_\lambda^{-1}(L_\lambda \setminus \{x\})$ is isomorphic to $\mathbb{A}^1_{\mathbf{k}} \times (L_\lambda \setminus \{x\})$ over $L_\lambda \setminus \{x\}$; via (9), this implies that $\mathcal{Q}_\lambda$ satisfies condition (i) of 1.3, so (b) holds. □

3.5. **Lemma.** *Let* $\operatorname{Spec} R$ *be a smooth affine surface such that* $\operatorname{ML}(R) = \mathbf{k}$*, let* $R_0 \in \textsc{klnd}(R)$ *and let* $\pi : \operatorname{Spec} R \to \operatorname{Spec} R_0$ *be the morphism determined by* $R_0 \hookrightarrow R$*. Then* $R_0 = \mathbf{k}^{[1]}$ *and* $N(\pi) \leq 1$.

*Proof.* This is well-known, see for instance [10, 2.3], [11, 2.15]. □

3.6. **Proposition.** *Let the setup be as in* 3.1*. If* $\operatorname{ML}(B_\lambda) = \mathbf{k}$ *for infinitely many* $\lambda \in \mathbf{k}$ *(resp. for general* $\lambda \in \mathbf{k}$*) then:*

(a) $A = \mathbf{k}[f]^{[1]}$

(b) $B_\lambda \in \mathfrak{D}(\mathbf{k})$ *for infinitely many* $\lambda \in \mathbf{k}$ *(resp. for general* $\lambda \in \mathbf{k}$*).*

*Proof.* There is a cofinite subset $\Lambda \subseteq \mathbf{k}$ such that all conclusions of 3.3 are valid for each $\lambda \in \Lambda$; define $\Lambda_\infty = \big\{ \lambda \in \Lambda \mid \operatorname{ML}(B_\lambda) = \mathbf{k} \big\}$ and note that $\Lambda_\infty$ is an infinite (resp. a cofinite) subset of $\mathbf{k}$ included in $\Lambda$.

Let $\lambda \in \Lambda_\infty$. As $\operatorname{Spec}(B_\lambda)$ is a smooth affine surface satisfying $\operatorname{ML}(B_\lambda) = \mathbf{k}$, 3.5 gives

$$\ker(D_\lambda) = \mathbf{k}^{[1]} \text{ and } N(\mathcal{Q}_\lambda) \leq 1 \tag{10}$$

so 3.4 implies that $\mathcal{Q}_\lambda$ defines a Danielewski surface. Since $\operatorname{Spec}(B_\lambda)$ is isomorphic to a smooth hypersurface of $\mathbb{A}^3$, its canonical sheaf is trivial and $\mathcal{Q}_\lambda$ therefore satisfies condition (a) of [12, 5.6]; by that result,[3] it follows that $\operatorname{Spec}(B_\lambda)$ is isomorphic to a hypersurface of $\mathbb{A}^3$ with equation $xy = p(z)$, for some nonconstant univariate polynomial $p(z)$ with distinct roots; so $B_\lambda \in \mathfrak{D}(\mathbf{k})$ (for each $\lambda \in \Lambda_\infty$), which proves (b).

[3]In the case $\mathbf{k} = \mathbb{C}$ we may use instead the result of Bandman and Makar-Limanov [2].

If $\lambda \in \Lambda_\infty$ then (10) and 3.3 give $A/(f-\lambda) \cong \ker(D_\lambda) = \mathbf{k}^{[1]}$, so the Abhyankar-Moh-Suzuki Theorem implies that $A = \mathbf{k}[f]^{[1]}$, which proves (a). $\square$

3.7. **Lemma.** *Let the setup be as in* 3.1 *and suppose that the condition*

$$\ker(D_\lambda) = \mathbf{k}^{[1]} \quad \textit{and} \quad N(\mathfrak{Q}_\lambda) \leq 1 \tag{11}$$

*holds for infinitely many $\lambda \in \mathbf{k}$. Then there exists $g$ such that $A = \mathbf{k}[f, g]$ and $\psi_A = a(f)g^n$ for some $a(f) \in \mathbf{k}[f] \setminus \{0\}$ and $n \in \mathbb{N}$.*

*Proof.* There is a cofinite subset $\Lambda \subseteq \mathbf{k}$ such that, for each $\lambda \in \Lambda$, all conclusions of 3.3 are valid, and an infinite subset $\Lambda_\infty \subseteq \Lambda$ such that (11) holds for every $\lambda \in \Lambda_\infty$.

Let $\lambda \in \Lambda_\infty$. Since $A/(f-\lambda) \cong \ker(D_\lambda) = \mathbf{k}^{[1]}$ by 3.3 and (11), the Abhyankar-Moh-Suzuki Theorem gives $A = \mathbf{k}[f, h]$ for some $h$. So $\psi_A$ is a polynomial in two variables $f, h$ and (by 3.3 and (11)) satisfies

$$|V_A(\psi_A, f-\lambda)| = N(\mathfrak{Q}_\lambda) \leq 1$$

for all $\lambda \in \Lambda_\infty$. This implies that $\psi_A = a(f)\big(u(f)h + v(f)\big)^n$, for some $n \in \mathbb{N}$ and $a, u, v \in \mathbf{k}[f]$ such that $a \neq 0$ and $\gcd(u, v) = 1$.

Note that $u \in \mathbf{k}$. Indeed, if $u \notin \mathbf{k}$ then $v \neq 0$, because $\gcd(u, v) = 1$; so $p = uh + v$ is a prime factor of $\psi_A$ which violates 2.16 ($A/pA$ has at least two places at infinity, so cannot be embedded in $\mathbf{k}[t]$). Moreover, if $u = 0$ then $\psi_A \in \mathbf{k}[f]$ and we are done. So we may assume from now-on that $u \in \mathbf{k}^*$. Define $g = uh + v$, then $A = \mathbf{k}[f, g]$ and $\psi_A = a(f)g^n$, where $a \in \mathbf{k}[f] \setminus \{0\}$ and $n \in \mathbb{N}$. $\square$

3.8. **Lemma.** *Suppose that $R \in \mathfrak{D}(\mathbf{k})$ and let $D : R \to R$ be an irreducible locally nilpotent derivation. Then there exists $s \in R$ such that* $\ker D = \mathbf{k}[Ds] = \mathbf{k}^{[1]}$.

*Proof.* Write $R = \mathbf{k}[X, Y, Z]/(XY - P(Z))$ where $P(Z) \in \mathbf{k}[Z] \setminus \mathbf{k}$ and let $x, y, z$ be the images in $R$ of $X, Y, Z$ respectively. The main result of [6] implies, in particular, that the group of $\mathbf{k}$-automorphisms of $R$ acts transitively on the set $\text{KLND}(R)$; so we may assume that $\ker D = \mathbf{k}[x]$. Then it is easy to see that $D(z) = ax$ for some $a \in \mathbf{k}^*$, which proves the claim. $\square$

3.9. **Lemma.** *Let $x, y$ be elements of an affine $\mathbf{k}$-domain $R$ such that*

(1) *$x$ is transcendental over $\mathbf{k}$*
(2) *for each $\lambda$ in an infinite subset of $\mathbf{k}$, there exists $c_\lambda \in \mathbf{k}$ such that $y - c_\lambda \in (x-\lambda)R$.*

*Then there exist $\alpha, \beta \in \mathbf{k}[x]$ such that $\alpha \neq 0$ and $\alpha y + \beta = 0$.*

*Proof.* Consider the subalgebras $\mathbf{k}[x]$ and $\mathbf{k}[y]$ of $R$. If $y \in \mathbf{k}$ then the conclusion of the lemma holds trivially, so we may assume throughout that $\mathbf{k}[y] = \mathbf{k}^{[1]}$ (and we have $\mathbf{k}[x] = \mathbf{k}^{[1]}$ by assumption (1)). Let $R_0 = \mathbf{k}[x, y] \subseteq R$ and consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}\mathbf{k}[x] & \operatorname{Spec} R \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "w"] & \operatorname{Spec}\mathbf{k}[y] \\
& \operatorname{Spec} R_0 \arrow[ul, "u_0"] \arrow[ur, "v_0"'] &
\end{tikzcd}$$

where the morphisms are determined by the corresponding inclusions. For a closed point $P \in \operatorname{Spec}\mathbf{k}[x]$, write $F_P = u_0^{-1}(P) \subseteq \operatorname{Spec} R_0$. The reader may verify that

$$w\big(u^{-1}(P)\big) \textit{ is a dense subset of } F_P$$

holds for a general closed point $P \in \operatorname{Spec}\mathbf{k}[x]$; consequently, $v_0\big(w(u^{-1}(P))\big)$ is dense in $v_0\big(F_P\big)$; as $v_0\big(w(u^{-1}(P))\big) = v\big(u^{-1}(P)\big)$, we conclude that

(12) *for a general closed point $P \in \operatorname{Spec}\mathbf{k}[x]$, $v\big(u^{-1}(P)\big)$ is dense in $v_0\big(F_P\big)$.*

On the other hand, the assumptions of the lemma imply:

(13) *For infinitely many closed points $P \in \operatorname{Spec}\mathbf{k}[x]$,*
*$v$ maps $u^{-1}(P)$ to a closed point of $\operatorname{Spec}\mathbf{k}[y]$.*

By (12) and (13), it follows that $v_0\big(F_P\big)$ is one closed point, i.e.,

(14) *For infinitely many closed points $P \in \operatorname{Spec}\mathbf{k}[x]$,*
*$v_0$ maps $u_0^{-1}(P)$ to a closed point of $\operatorname{Spec}\mathbf{k}[y]$.*

From (14) it is clear that $x, y$ must be algebraically dependent. Indeed, if they are not then $\operatorname{Spec} R_0 = \mathbb{A}^2$ and $u_0, v_0$ are the standard projections

$$\operatorname{Spec}\mathbf{k}[x] \xleftarrow{u_0} \mathbb{A}^2 \xrightarrow{v_0} \operatorname{Spec}\mathbf{k}[y],$$

but obviously these maps do not satisfy (14).

So there exists an irreducible $\Phi(X,Y) \in \mathbf{k}[X,Y] = \mathbf{k}^{[2]}$ such that $\Phi(x,y) = 0$. Moreover, assumption (1) implies that $\deg_Y \Phi(X,Y) > 0$. As $R_0 \cong \mathbf{k}[X,Y]/\Phi\mathbf{k}[X,Y]$, we may identify $\operatorname{Spec} R_0$ with the curve $V(\Phi) \subset \mathbb{A}^2$; then $u_0$, $v_0$ are the projections $\mathbb{A}^1 \xleftarrow{u_0} V(\Phi) \xrightarrow{v_0} \mathbb{A}^1$, i.e., $u_0(x,y) = x$ and $v_0(x,y) = y$. Then (14) states that, for infinitely many $\lambda \in \mathbf{k}$, the equation $\Phi(\lambda, Y) = 0$ has only one solution $Y$ in $\mathbf{k}$. This implies that $\deg_Y(\Phi) = 1$, which proves the lemma. $\square$

3.10. **Proposition.** *Let the setup be as in* 3.1 *and suppose that* $\mathrm{ML}(B_\lambda) = \mathbf{k}$ *for infinitely many $\lambda \in \mathbf{k}$. Then there exists $g$ such that $A = \mathbf{k}[f,g]$ and $\psi_A = a(f)g^n$ for some $a(f) \in \mathbf{k}[f] \setminus \{0\}$ and $n \in \{0,1\}$.*

*Proof.* There is a cofinite subset $\Lambda \subseteq \mathbf{k}$ such that, for each $\lambda \in \Lambda$, all conclusions of 3.3 are valid; by 3.6, there is an infinite subset $\Lambda_\infty \subseteq \Lambda$ such that $B_\lambda \in \mathfrak{D}(\mathbf{k})$ (and hence $\mathrm{ML}(B_\lambda) = \mathbf{k}$) holds for every $\lambda \in \Lambda_\infty$.

Let $\lambda \in \Lambda_\infty$. Then $\operatorname{Spec}(B_\lambda)$ is a smooth affine surface and $\mathrm{ML}(B_\lambda) = \mathbf{k}$, so 3.5 implies that $\ker(D_\lambda) = \mathbf{k}^{[1]}$ and $N(\mathfrak{Q}_\lambda) \leq 1$; since this holds for infinitely many $\lambda$, 3.7 implies that there exists $g$ such that $A = \mathbf{k}[f,g]$ and $\psi_A = a(f)g^n$ for some $a(f) \in$

$\mathbf{k}[f] \setminus \{0\}$ and $n \in \mathbb{N}$. There only remains to show that $n \leq 1$. By contradiction, we assume:

$$n > 1. \tag{15}$$

If $\lambda \in \Lambda_\infty$ then $B_\lambda \in \mathfrak{D}(\mathbf{k})$ and, by 3.3, $\mathrm{Fix}(D_\lambda) = \varnothing$; as the last condition implies that $D_\lambda$ is irreducible, we may apply 3.8 to $B_\lambda$ and $D_\lambda$ and obtain

(16) *for each $\lambda \in \Lambda_\infty$, there exists $\sigma_\lambda \in B_\lambda$ such that* $\ker D_\lambda = \mathbf{k}[D_\lambda(\sigma_\lambda)]$.

For $\lambda \in \Lambda_\infty$, define $g_\lambda = \pi_\lambda(g) \in B_\lambda$; then 3.3 gives $\ker(D_\lambda) = \pi_\lambda(A) = \mathbf{k}[g_\lambda] = \mathbf{k}^{[1]}$. Define the ideal $I_\lambda = D_\lambda(B_\lambda) \cap \mathbf{k}[g_\lambda]$ of $\mathbf{k}[g_\lambda]$.

Note that if $\lambda$ is such that $I_\lambda = \mathbf{k}[g_\lambda]$ then 2.3 implies that $B_\lambda = (\ker D_\lambda)^{[1]} = \mathbf{k}[g_\lambda]^{[1]}$, i.e., $B/(f-\lambda)B = \mathbf{k}^{[2]}$. If this is the case for infinitely many $\lambda$, then [9, Thm 3] implies that $f$ is a variable of $B$ and in this case it is well-known (see for instance part (5) of [8, 3.2]) that $\psi_A \in \mathbf{k}[f]$, which contradicts (15). So there exists an infinite subset $\Lambda'_\infty$ of $\Lambda_\infty$ for which:

$$I_\lambda \neq \mathbf{k}[g_\lambda] \textit{ for all } \lambda \in \Lambda'_\infty.$$

From now-on, let $\lambda \in \Lambda'_\infty$.

Choose a maximal ideal $\mathfrak{m}_\lambda$ of $\mathbf{k}[g_\lambda]$ such that $I_\lambda \subseteq \mathfrak{m}_\lambda$. By definition of $\psi_A$ there exists $s \in B$ such that $D(s) = \psi_A = a(f)g^n$, so $D_\lambda(\pi_\lambda(s)) = \pi_\lambda(Ds) = a(\lambda)g_\lambda^n$ where $a(\lambda) \in \mathbf{k}^*$. This shows that $g_\lambda^n \in I_\lambda \subseteq \mathfrak{m}_\lambda$ and consequently

$$\mathfrak{m}_\lambda = g_\lambda\, \mathbf{k}[g_\lambda].$$

The element $D_\lambda(\sigma_\lambda)$ of condition (16) must belong to $I_\lambda \subseteq \mathfrak{m}_\lambda$ and must generate $\mathbf{k}[g_\lambda]$ as a $\mathbf{k}$-algebra; so $D_\lambda(\sigma_\lambda) = \alpha_\lambda g_\lambda$ for some $\alpha_\lambda \in \mathbf{k}^*$, and we may therefore choose $\sigma_\lambda$ so that

$$D_\lambda(\sigma_\lambda) = g_\lambda.$$

Choose $t_\lambda \in B$ such that $\pi_\lambda(t_\lambda) = \sigma_\lambda$ and consider $s - ag^{n-1}t_\lambda \in B$. Then

$$\begin{aligned}\pi_\lambda(D(s - ag^{n-1}t_\lambda)) = \pi_\lambda(ag^n - ag^{n-1}D(t_\lambda)) &= a(\lambda)g_\lambda^n - a(\lambda)g_\lambda^{n-1}\pi_\lambda(D(t_\lambda)) \\ &= a(\lambda)g_\lambda^n - a(\lambda)g_\lambda^{n-1}D_\lambda(\sigma_\lambda) = 0.\end{aligned}$$

Since (for $x \in B$) the condition $\pi_\lambda(D(x)) = 0$ is equivalent to $x \in (f-\lambda)B + \mathbf{k}[g]$, there exists $P_\lambda(g) \in \mathbf{k}[g]$ such that $s - ag^{n-1}t_\lambda - P_\lambda(g) \in (f-\lambda)B$. If $c_\lambda \in \mathbf{k}$ is the constant coefficient of $P_\lambda$ then (using $n > 1$) we conclude that $s - c_\lambda \in (g, f-\lambda)B$. We record:

(17) *For each $\lambda \in \Lambda'_\infty$, $\exists\, c_\lambda \in \mathbf{k}$ such that $s - c_\lambda \in (g, f-\lambda)B$.*

Applying 3.9 to the elements $x = f + gB$ and $y = s + gB$ of $R = B/gB$, we obtain that there exist $\alpha(f), \beta(f) \in \mathbf{k}[f]$ such that $\alpha(f) \neq 0$ and $\alpha(f)s + \beta(f) \in gB$. Define $s' = (\alpha(f)s + \beta(f))/g \in B$; then $D(s') = \alpha(f)a(f)g^{n-1}$, so $\alpha(f)a(f)g^{n-1} \in I_A \setminus \{0\}$. This is absurd, because $I_A$ is the *principal* ideal of $\mathbf{k}[f,g]$ generated by $\psi_A = a(f)g^n$. So (15) is false and the proof is complete. $\square$

The following is a corollary to [7, 2.6.2]:

3.11. *Let $\mathcal{B}$ be a factorial domain containing a field $K$ of characteristic zero, and suppose that $\mathcal{D} \in \mathrm{LND}(\mathcal{B})$ and $s \in \mathcal{B}$ satisfy:*

$$\ker \mathcal{D} = K[\mathcal{D}s] = K^{[1]}.$$

*Then $\mathcal{B} \in \mathfrak{D}(K)$. Moreover, if $\mathcal{D}$ is irreducible then there exist $h \in \mathcal{B}$ and a nonconstant polynomial $P(T) \in K[T]$ such that $\mathcal{B} = K[\mathcal{D}(s), h, s]$ and $\mathcal{D}(s)h = P(s)$.*

3.12. **Proposition.** *Let the setup be as in* 3.1 *and suppose that* $\mathrm{ML}(B_\lambda) = \mathbf{k}$ *for infinitely many* $\lambda \in \mathbf{k}$. *Then* $\mathcal{B} \in \mathfrak{D}(K)$, *where we define* $\mathcal{B} = \mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ *and* $K = \mathbf{k}(f)$.

*Proof.* By definition of $\psi_A$, there exists $s_0 \in B$ such that $D(s_0) = \psi_A$. By 3.10, there exists $g$ such that $A = \mathbf{k}[f, g]$ and $\psi_A = ag^n$ for some $a \in \mathbf{k}[f] \setminus \{0\}$ and $n \in \{0, 1\}$. Define

$$s = \begin{cases} s_0, & \text{if } n = 1 \\ gs_0, & \text{if } n = 0, \end{cases}$$

so that $D(s) = ag$. Consider the locally nilpotent derivation $\mathcal{D} = S^{-1}D : \mathcal{B} \to \mathcal{B}$ obtained by localizing $D$ with respect to $S = \mathbf{k}[f] \setminus \{0\}$ and note that

$$\ker \mathcal{D} = S^{-1}\mathbf{k}[f, g] = K[g] = K[\mathcal{D}(s)] = K^{[1]}.$$

Then 3.11 asserts that $\mathcal{B} \in \mathfrak{D}(K)$. $\square$

## 4. Irreducible polynomials belonging to at least two kernels

In this section we set $B = \mathbf{k}^{[3]}$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero.

The following fact is due to Freudenburg and this author, and is in fact valid without assuming that $\mathbf{k}$ is algebraically closed; a proof should appear in the forthcoming [13].

4.1. *Suppose that $A, A'$ are distinct elements of $\mathrm{KLND}(B)$ such that $A \cap A' \neq \mathbf{k}$. Then $A \cap A' = \mathbf{k}^{[1]}$, $A = (A \cap A')^{[1]}$ and $A' = (A \cap A')^{[1]}$.*

We also need the following:

4.2. **Lemma.** *Suppose that $R$ is an integral domain and a finitely generated $\mathbf{k}$-algebra, and that $f, g, h \in R$ are algebraically independent over $\mathbf{k}$. For each $\lambda \in \mathbf{k}$, let $\pi_\lambda : R \to R/(f - \lambda)R$ be the canonical epimorphism. Then, for general $\lambda \in \mathbf{k}$, the elements $\pi_\lambda(g), \pi_\lambda(h) \in R/(f - \lambda)R$ are algebraically independent over $\mathbf{k}$.*

*Proof.* Let $\varphi : \mathbf{k}[X, Y, Z] \to R$ be the $\mathbf{k}$-homomorphism which maps $X, Y, Z$ to $f, g, h$ respectively, and let $\varphi^* : \operatorname{Spec} R \to \mathbb{A}^3$ be the morphism determined by $\varphi$. The hypothesis implies that $\varphi$ is injective, so $\varphi^*$ is dominant and consequently there exists an open subset $U \neq \varnothing$ of $\mathbb{A}^3$ such that $U \subseteq \mathrm{im}(\varphi^*)$.

Let $\lambda \in \mathbf{k}$ be such that $\pi_\lambda(g), \pi_\lambda(h)$ are algebraically dependent over $\mathbf{k}$. Then there exists $P(Y, Z) \in \mathbf{k}[Y, Z] \setminus \{0\}$ such that $P(g, h) \in (f - \lambda)R$. We claim that

$$V(X - \lambda) \cap \mathrm{im}(\varphi^*) \subseteq V(X - \lambda, P). \tag{18}$$

Indeed, if $\mathfrak{p} \in V(X-\lambda) \cap \operatorname{im}(\varphi^*)$ then $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$ for some $\mathfrak{q} \in \operatorname{Spec} R$; then $X - \lambda \in \mathfrak{p}$ implies that $f - \lambda = \varphi(X-\lambda) \in \mathfrak{q}$, so $\varphi(P) = P(g,h) \in (f-\lambda)R \subseteq \mathfrak{q}$ and consequently $P \in \mathfrak{p}$, proving (18). It follows that $V(X-\lambda) \cap U \subseteq V(X-\lambda, P)$; as $V(X-\lambda, P(Y,Z))$ has dimension at most 1, whereas $V(X-\lambda) \cap U$ is either empty or two-dimensional, we obtain that $V(X-\lambda) \cap U = \varnothing$, a condition which can hold for at most finitely many $\lambda$. $\square$

4.3. **Lemma.** *Suppose that $f$ is an irreducible element of $B$ which belongs to at least two elements of* $\text{KLND}(B)$. *Define $B_\lambda = B/(f-\lambda)B$ for each $\lambda \in \mathbf{k}$.*

(a) *For each $A \in \text{KLND}(B)$ satisfying $f \in A$, we have $A = \mathbf{k}[f]^{[1]}$.*
(b) *$B_\lambda$ is an integral domain, for every $\lambda \in \mathbf{k}$.*
(c) $\operatorname{ML}(B_\lambda) = \mathbf{k}$ *for general $\lambda \in \mathbf{k}$.*

*Proof.* Let $A, A'$ be distinct elements of $\text{KLND}(B)$ such that $f \in A \cap A'$. By 4.1 we have $A = \mathbf{k}[f]^{[1]}$ and $A' = \mathbf{k}[f]^{[1]}$, which proves (a). Note that $A = \mathbf{k}[f]^{[1]}$ implies that, for every $\lambda \in \mathbf{k}$, $f - \lambda$ is an irreducible element of $A$, hence an irreducible element of $B$; so (b) is proved. Choose $g, h \in B$ such that $A = \mathbf{k}[f,g]$ and $A' = \mathbf{k}[f,h]$. Then $h \notin A$; it follows that $h$ is not algebraic over $A$ and hence that $f, g, h$ are algebraically independent over $\mathbf{k}$; so, by 4.2, the condition

$$\pi_\lambda(g), \pi_\lambda(h) \textit{ are algebraically independent over } \mathbf{k}$$

is satisfied for general $\lambda \in \mathbf{k}$ (where $\pi_\lambda : B \to B_\lambda$ is the canonical epimorphism). Let $D$ (resp. $D'$) be an irreducible locally nilpotent derivation of $B$ such that $\ker D = A$ (resp. $\ker D' = A'$). Consider $D_\lambda, D'_\lambda : B_\lambda \to B_\lambda$, the locally nilpotent derivations determined by $D$ and $D'$ respectively. Then $D_\lambda, D'_\lambda$ are nonzero, because $D$ and $D'$ are irreducible, so each one of $R = \ker(D_\lambda)$, $R' = \ker(D'_\lambda)$ has transcendence degree 1 over $\mathbf{k}$. Clearly, $D_\lambda(\pi_\lambda(g)) = 0$ and $D'_\lambda(\pi_\lambda(h)) = 0$. If $D_\lambda(\pi_\lambda(h)) = 0$ then $\pi_\lambda(g), \pi_\lambda(h)$ must be algebraically dependent over $\mathbf{k}$; so for general $\lambda$ we have $D_\lambda(\pi_\lambda(h)) \neq 0$ and consequently $R, R'$ are distinct elements of $\text{KLND}(B_\lambda)$. Then $R \cap R'$ has transcendence degree 0 over $\mathbf{k}$; as $\mathbf{k}$ is algebraically closed, $R \cap R' = \mathbf{k}$ and consequently $\operatorname{ML}(B_\lambda) = \mathbf{k}$. $\square$

4.4. **Corollary.** *Let $f$ be an irreducible element of $B$ which belongs to at least two elements of* $\text{KLND}(B)$. *Then the following hold:*

(a) *$B/(f-\lambda) \in \mathfrak{D}(\mathbf{k})$ for general $\lambda \in \mathbf{k}$*
(b) *$\mathcal{B} \in \mathfrak{D}(K)$, where we define $\mathcal{B} = \mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ and $K = \mathbf{k}(f)$*
(c) *For any $A \in \text{KLND}(B)$ satisfying $f \in A$, there exists $g$ such that $A = \mathbf{k}[f,g]$ and $\psi_A = a(f)g^n$ for some $a(f) \in \mathbf{k}[f] \setminus \{0\}$ and $n \in \{0,1\}$.*

*Proof.* Choose $A \in \text{KLND}(B)$ which contains $f$. Then the pair $(f, A)$ satisfies the global assumptions 3.1 and we may apply the results of Section 3. By 4.3, the hypotheses of 3.6, 3.12 and 3.10 are satisfied; so we are done. $\square$

We may now obtain the Main Theorem 1.10 as an easy consequence of 4.4.

*Proof of* 1.10. Let $f \in B \setminus \mathbf{k}$, $S = \mathbf{k}[f] \setminus \{0\}$, $\mathcal{B} = S^{-1}B$ and $K = \mathbf{k}(f)$.

By 4.4, we obtain that 1.10(a) $\Rightarrow$ 1.10(c). Implication 1.10(c) $\Rightarrow$ 1.10(b) is well-known (see for instance [7, 2.3(d)]). Implication 1.10(b) $\Rightarrow$ 1.10(a) is also quite well-known, but let us recall the argument. Suppose that $f$ is such that $\mathrm{ML}(\mathcal{B}) = K$. Let us first show that if $\mathcal{A} \in \mathrm{KLND}(\mathcal{B})$ then $B \cap \mathcal{A} \in \mathrm{KLND}(B)$. Choose $\mathcal{D} \in \mathrm{LND}(\mathcal{B})$ such that $\ker \mathcal{D} = \mathcal{A}$. Let $X_1, X_2, X_3$ be such that $B = \mathbf{k}[X_1, X_2, X_3]$. For each $i \in \{1, 2, 3\}$ we have $\mathcal{D}(X_i) \in S^{-1}B$, so there exists $s \in S$ such that (for all $i = 1, 2, 3$) $s\mathcal{D}(X_i) \in B$. Then $s\mathcal{D}(B) \subseteq B$ and (since $S \subset \ker \mathcal{D}$) $s\mathcal{D} \in \mathrm{LND}(\mathcal{B})$; so if we let $D : B \to B$ be the restriction of $\mathcal{D}$ then $D \in \mathrm{LND}(B)$ and $\ker D = B \cap \mathcal{A}$. So we have a well-defined map

$$\mathrm{KLND}(\mathcal{B}) \longrightarrow \big\{ A \in \mathrm{KLND}(B) \mid f \in A \big\}, \qquad \mathcal{A} \mapsto B \cap \mathcal{A}$$

which is easily seen to be injective. As $\mathrm{trdeg}_K(\mathcal{B}) = 2$ and $\mathrm{ML}(\mathcal{B}) = K$, it follows that $\mathrm{KLND}(\mathcal{B})$ has at least 2 elements, so $f \in A \cap A'$ for some distinct elements $A, A' \in \mathrm{KLND}(B)$. By 4.1 we have $A \cap A' = \mathbf{k}[f_0] = \mathbf{k}^{[1]}$ for some $f_0$, and this implies that $\mathbf{k}[f] \subseteq \mathbf{k}[f_0]$. As $\mathbf{k}(f_0)$ is algebraic over $\mathbf{k}(f)$ and $\mathbf{k}(f) = \mathrm{ML}(\mathcal{B})$ is algebraically closed in $\mathbf{k}(X_1, X_2, X_3)$, we must have $\mathbf{k}(f) = \mathbf{k}(f_0)$ and hence $\mathbf{k}[f] \subseteq \mathbf{k}[f_0] \subset \mathbf{k}(f)$; thus $\mathbf{k}[f] = \mathbf{k}[f_0] = A \cap A'$, so $\mathbf{k}[f]$ is factorially closed in $B$. As $f$ is irreducible in $\mathbf{k}[f]$, it therefore follows that it is irreducible in $B$. So 1.10(b) implies 1.10(a). $\square$

4.5. **Corollary.** *Suppose that $f$ is an element of $B$ satisfying:*

(i) *$f$ belongs to at least one element of* $\mathrm{KLND}(B)$

(ii) *for infinitely many $\lambda \in \mathbf{k}$, $B/(f-\lambda)$ is a domain and* $\mathrm{ML}\big(B/(f-\lambda)\big) = \mathbf{k}$.

*Then $f$ belongs to at least two elements of* $\mathrm{KLND}(B)$.

*Proof.* By (i), we may consider $A \in \mathrm{KLND}(B)$ such that $f \in A$. As $(f, A)$ satisfies the global assumptions 3.1, we can use 3.12 and obtain that $f$ satisfies condition (c) of 1.10; so $f$ satisfies condition (a) of the same result. $\square$

## 5. Families of Danielewski surfaces in $\mathbb{A}^3$

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero and let $B = \mathbf{k}^{[3]}$. Given a nonconstant polynomial $f \in B$, define the family of surfaces

$$S_\lambda = V_B(f - \lambda) \subset \operatorname{Spec} B = \mathbb{A}^3_{\mathbf{k}} \quad \text{(for each } \lambda \in \mathbf{k}).$$

We are interested in polynomials $f$ satisfying:

$(*)$ *For general $\lambda$, $S_\lambda$ is a Danielewski surface.*

To be precise, "$S_\lambda$ is a Danielewski surface" means that there exists a morphism $\pi_\lambda : S_\lambda \to \mathbb{A}^1_{\mathbf{k}}$ which satisfies definition 1.3. Example 5.1 presents a method for obtaining such polynomials $f$, but the method produces polynomials which satisfy a condition stronger than $(*)$; result 5.2 identifies exactly that condition. Note that we don't know any example showing that the "stronger" condition of 5.2 is indeed stronger than $(*)$.

5.1. **Example.** We show that if $A = \mathbf{k}[f,g]$ is any element of $\textsc{klnd}(B)$ satisfying:[4]

$$\psi_A = a(f)g^n \qquad (a(f) \in \mathbf{k}[f] \setminus \{0\}, \;\; n \geq 0), \tag{19}$$

then $f$ satisfies $(*)$. Note that $(f, A)$ satisfies the global assumptions of 3.1, so all results of Section 3 can be applied. Let $B_\lambda = B/(f - \lambda)$ and consider

$$\mathfrak{Q}_\lambda : \operatorname{Spec} B_\lambda \to \operatorname{Spec}\big(\ker D_\lambda\big)$$

defined as in 3.1. By 3.3, we have (for general $\lambda \in \mathbf{k}$) $\ker D_\lambda \cong A/(f - \lambda) = \mathbf{k}^{[1]}$ and $N(\mathfrak{Q}_\lambda) = |V_A(\psi_A, f - \lambda)| = |V_A(a(f)g^n, f - \lambda)| \leq 1$, so by 3.4:

*for general $\lambda$, $\mathfrak{Q}_\lambda$ defines a Danielewski surface.*

So $\operatorname{Spec} B_\lambda$ is a Danielewski surface; since $S_\lambda = V_B(f - \lambda) \cong \operatorname{Spec} B_\lambda$, $f$ satisfies $(*)$. (Note that $f$ also satisfies condition (a) of 5.2, below.)

Let us also prove:

- *if $n = 0$ then $f$ is a variable of $B$, so $B_\lambda = \mathbf{k}^{[2]}$ for all $\lambda$;*
- *if $n = 1$ then $f$ is a basic element of $B$ (cf.* 1.4*) and, for general $\lambda$, $B_\lambda \in \mathfrak{D}(\mathbf{k})$ and $B_\lambda \neq \mathbf{k}^{[2]}$;*
- *if $n > 1$ then* $\operatorname{ML}(B_\lambda) = \mathbf{k}^{[1]}$ *for general $\lambda$.*

*Proof.* If $n = 0$ then $\psi_A \in \mathbf{k}[f]$, so $I_A \cap \mathbf{k}[f] \neq \{0\}$; this implies that $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} B = (\mathbf{k}(f) \otimes_{\mathbf{k}[f]} A)^{[1]} = \mathbf{k}(f)^{[2]}$, so $f$ is a variable of $B$ by [9, Thm 3] and consequently $B_\lambda = \mathbf{k}^{[2]}$ for all $\lambda$.

Note that if $B_\lambda = \mathbf{k}^{[2]}$ for infinitely many $\lambda$ then $f$ is a variable of $B$ by [9, Thm 3] and consequently $\psi_A \in \mathbf{k}[f]$ (part (5) of [8, 3.2]), so $n = 0$. Consequently, $n = 1$ implies that $B_\lambda \neq \mathbf{k}^{[2]}$ for general $\lambda$. If $n = 1$ then 3.11 implies (as in the proof of 3.12) that $\mathbf{k}(f) \otimes_{\mathbf{k}[f]} B$ belongs to $\mathfrak{D}(\mathbf{k}(f))$, i.e., $f$ satisfies condition (c) of 1.10. Then $f$ belongs to at least two elements of $\textsc{klnd}(B)$, i.e., is a basic element of $B$; by 4.4, $B_\lambda \in \mathfrak{D}(\mathbf{k})$ for general $\lambda$.

If $\operatorname{ML}(B_\lambda) = \mathbf{k}$ for infinitely many $\lambda \in \mathbf{k}$ then (4.5) $f$ belongs to two elements of $\textsc{klnd}(B)$ and consequently part (c) of 4.4 implies that $n \leq 1$. In other words, if $n > 1$ then (for general $\lambda$) $\operatorname{ML}(B_\lambda) \neq \mathbf{k}$, so $\ker(D_\lambda)$ must be the unique element of $\textsc{klnd}(B_\lambda)$ and $\operatorname{ML}(B_\lambda) = \ker D_\lambda \cong A/(f - \lambda) = \mathbf{k}^{[1]}$, for general $\lambda$. $\square$

5.2. **Proposition.** *Let $f \in B \setminus \mathbf{k}$ and define $S_\lambda = V_B(f - \lambda) \subset \mathbb{A}^3_{\mathbf{k}}$ for each $\lambda \in \mathbf{k}$. For the polynomial $f$, conditions* (a) *and* (b) *are equivalent:*

(a) *The following two conditions hold:*
  (i) *For general $\lambda \in \mathbf{k}$, $S_\lambda$ is a Danielewski surface*
  (ii) *for some $\lambda_0 \in \mathbf{k}$ satisfying condition* (i)*, some nontrivial $G_a$-action on $S_{\lambda_0}$ can be extended to a $G_a$-action on $\mathbb{A}^3_{\mathbf{k}}$.*

(b) *There exists $g \in B$ such that if we set $A = \mathbf{k}[f,g]$ then $A \in \textsc{klnd}(B)$ and $\psi_A = a(f)g^n$, for some $a(f) \in \mathbf{k}[f] \setminus \{0\}$ and $n \geq 0$.*

[4] See 5.3 for examples of kernels satisfying (19).

*Proof.* We prove that (a) implies (b) (the converse is proved in 5.1). Suppose that $f$ satisfies (a). There is a $G_a$-action $\alpha$ on $\mathbb{A}^3_{\mathbf{k}}$ which restricts to a nontrivial $G_a$-action on $S_{\lambda_0}$. Let $D' : B \to B$ be the locally nilpotent derivation such that $\alpha = \alpha_{D'}$ (2.5). Using that $\alpha$ maps $S_{\lambda_0} = V(f - \lambda_0)$ into itself, it is easy to see that $D'(f - \lambda_0) = 0$. So $f \in A$, where we define $A = \ker D'$; hence the pair $(f, A)$ satisfies the global assumptions 3.1 and the results of Section 3 can be applied. Choose an irreducible locally nilpotent derivation $D : B \to B$ with kernel $A$ and for each $\lambda$ define $B_\lambda = B/(f - \lambda)$ and $D_\lambda : B_\lambda \to B_\lambda$ as in 3.1. If $\mathrm{ML}(B_\lambda) = \mathbf{k}$ for infinitely many $\lambda$ then 3.10 implies that $f$ satisfies condition (b) with $n \leq 1$. We may therefore assume that

$$\mathrm{ML}(B_\lambda) \neq \mathbf{k} \textit{ for general } \lambda. \tag{20}$$

By assumption, for general $\lambda$ there exists a morphism $\pi_\lambda : \mathrm{Spec}(B_\lambda) \to \mathbb{A}^1_{\mathbf{k}}$ which makes $\mathrm{Spec}(B_\lambda)$ a Danielewski surface. Algebraically, $\pi_\lambda$ corresponds to an injective homomorphism $\varphi : \mathbf{k}^{[1]} \to B_\lambda$ whose image is an element of $\mathrm{KLND}(B_\lambda)$, but (20) implies that $\ker(D_\lambda)$ is the unique element of $\mathrm{KLND}(B_\lambda)$, so $\varphi$ factors as

$$\mathbf{k}^{[1]} \xrightarrow{\cong} \ker(D_\lambda) \hookrightarrow B_\lambda$$

and in particular $\ker(D_\lambda) = \mathbf{k}^{[1]}$. It follows that $\pi_\lambda$ factors as

$$\mathrm{Spec}\, B_\lambda \xrightarrow{\mathcal{Q}_\lambda} \mathrm{Spec}(\ker D_\lambda) \xrightarrow{\cong} \mathbb{A}^1_{\mathbf{k}},$$

so $\mathcal{Q}_\lambda$ defines a Danielewski surface and consequently $N(\mathcal{Q}_\lambda) \leq 1$. As the conditions $\ker(D_\lambda) = \mathbf{k}^{[1]}$ and $N(\mathcal{Q}_\lambda) \leq 1$ hold for general $\lambda$, it follows from 3.7 that $f$ satisfies (b). $\square$

**5.3.** Here are some examples of kernels $A = \mathbf{k}[f, g] \in \mathrm{KLND}(B)$ satisfying (19).

(1) Let $f = X^n Y + Z^2 + Xq(X, Z) \in B = \mathbf{k}[X, Y, Z]$, then $A = \mathbf{k}[f, X]$ belongs to $\mathrm{KLND}(B)$ and $\psi_A = X^n$. In [21], the Danielewski surfaces $V_B(f - \lambda)$ in the case $n = 2$ are shown to have interesting pathologies (for instance automorphisms which do not extend to $\mathbb{A}^3$).

(2) Let $n \geq 1$ and let $A = \ker \Delta_n$ with $\Delta_n$ as in [14, 4.1]. Then $A = \mathbf{k}[G_n, F]$ and $\psi_A = G_n F^n$ ($G_n$ and $F$ are defined in [14] and are homogeneous polynomials of degrees $4n + 1$ and 2 respectively).

(3) (Generalization of (2).) Let $D : B \to B$ be any nonzero locally nilpotent derivation which is homogeneous with respect to a positive grading of $B$. Then $A = \ker D$ satisfies (19). Indeed, by [5, 3.10], for a suitable choice of $f, g$ we have $A = \mathbf{k}[f, g]$ and $\psi_A = f^i g^n$ for some $i, n \in \mathbb{N}$.

Note that if Conjecture 1.14 is true then 4.4 implies that for *every* element $A$ of $\mathrm{KLND}(B)$ there is at least one pair $(f, g)$ which satisfies $A = \mathbf{k}[f, g]$ and (19). In particular, every known example of kernel satisfies (19).

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