AFFINE RULINGS OF WEIGHTED PROJECTIVE PLANES

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Abstract. It is explained that the following two problems are equivalent: (i) Describing all affine rulings of any given weighted projective plane; (ii) describing all weighted-homogeneous locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$. Then the solution of (i) is outlined. (Outline of our joint work with Peter Russell.)

An "affine ruling" of an algebraic surface X is a morphism $p : U \to \Gamma$ where Γ is a curve, U is a nonempty open subset of X isomorphic to $\Gamma \times \mathbb{A}^1$ and p is the projection $\Gamma \times \mathbb{A}^1 \to \Gamma$ (note that we will modify this definition in Section 1). Then, *how can we find all affine rulings on a given surface X?* In our joint work with Peter Russell ([2] and [3]), we give a complete answer to that question in the case where X is a weighted projective plane, and a partial answer when X belongs to a larger class of surfaces.

The aim of this paper is to give an outline of [2] and [3] which is readable by a wider set of algebraists and geometers. To achieve this, we explain several notions which are well-known to geometers familiar with algebraic surfaces (but we do assume some familiarity with the notion of linear system on a surface); we also omit all proofs and present the material in an order which is quite different from that of [2] and [3].

Two distinct approaches are proposed in [2] and [3] but only the one via discrete data (section 5 of [2] and most of [3]) is outlined here. Although the other approach (via "X-immersions") is necessary for a full understanding of the subject, it can be omitted in this type of outline. Also note that [2] and [3] contain several nontrivial results in the theory of weighted graphs, but that none of these appears here.

Motivation

There are several reasons for studying affine rulings, but let us explain the connection with locally nilpotent derivations. Consider the polynomial ring $A = \mathbf{k}[X_0, X_1, X_2]$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. It is known that describing the locally nilpotent derivations $D : A \to A$ is equivalent to describing their kernels; by a result of Miyanishi [7], the kernel of such a derivation is a subalgebra $\mathbf{k}[f, g]$ of A, where f and g are algebraically independent over $\mathbf{k}$. So describing the locally nilpotent derivations $D : A \to A$ is equivalent to answering: *Which pairs of polynomials $f, g \in A$ have the property that $\mathbf{k}[f, g]$ is the kernel of a locally nilpotent derivation of*

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A? However, this question seems to be very difficult. If we restrict ourselves to the case where D is (or equivalently f and g are) homogeneous with respect to weights $w(X_i) = a_i$, where a_0, a_1, a_2 are relatively prime positive integers, then we can think of the zero sets of f and g as curves in the weighted projective plane $\mathbb{P}(a_0, a_1, a_2) = \operatorname{Proj} A$; then [1] gives the following result:

Theorem. *For w-homogeneous elements $f, g \in A$ satisfying $\gcd(w(f), w(g)) = 1$, the following are equivalent:*

(1) *There exists a locally nilpotent derivation D of A such that $\ker D = \mathbf{k}[f, g]$;*
(2) *there exists a w-homogeneous locally nilpotent derivation D of A such that $\ker D = \mathbf{k}[f, g]$;*
(3) *f and g are irreducible elements of A and the complement of the set $\{fg = 0\}$ in the weighted projective plane $\mathbb{P}(a_0, a_1, a_2)$ is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$ as an algebraic surface.*

(Here, $\mathbb{A}^1_*$ denotes the affine line minus one point.) Note that the case where $\gcd(w(f), w(g)) \neq 1$ turns out to be very special, and is completely described in [1]. Hence, describing homogeneous locally nilpotent derivations of A is equivalent to finding all pairs of curves C_1, C_2 on $\mathbb{P}(a_0, a_1, a_2)$ with the property that $\mathbb{P}(a_0, a_1, a_2) \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. Although this is not entirely obvious, it turns out that this is equivalent to finding all affine rulings of $\mathbb{P}(a_0, a_1, a_2)$ (see Problems 1 and 2 of 1.3). So,

Describing the affine rulings of $\mathbb{P}(a_0, a_1, a_2)$ is equivalent to describing the w-homogeneous locally nilpotent derivations of A.

The class of homogeneous locally nilpotent derivations of A is not well understood, and corresponds to an interesting class of G_a-actions on $\mathbb{A}^3$. In particular, our description of the affine rulings of $\mathbb{P}(a_0, a_1, a_2)$ might eventually enable one to produce families of automorphisms of $\mathbb{A}^3$ which are currently unknown.

Conventions and terminology

(1) The set of nonnegative (resp. positive) integers is denoted $\mathbb{N}$ (resp. $\mathbb{Z}^+$).
(2) All curves and surfaces considered in this paper are assumed to be algebraic varieties over an algebraically closed field $\mathbf{k}$ of characteristic zero. In particular, curves and surfaces are irreducible and reduced.
(3) If $f : X \to Y$ is a birational morphism of surfaces then the *center* of f (denoted $\operatorname{center}(f)$) is the set of points $y \in Y$ such that $f^{-1}(y)$ contains more than one point. The inverse image of the center is called the *exceptional locus* of f.
(4) If Λ is a linear system on a surface X, then by a *member* of Λ we mean a divisor D of X such that $D \in \Lambda$. Assume that D has irreducible support, i.e., that $D = nC$ for some $n \in \mathbb{Z}^+$ and some irreducible curve C; we call D a *reduced* member of Λ if $n = 1$, a *multiple* member if $n > 1$.

(5) Let S be a smooth projective surface. If D is a divisor of S then, by a *component* of D, we always mean an irreducible (or prime) component of D. If D and D' are divisors of S then $D \cdot D'$ denotes their intersection number and $D^2 = D \cdot D$. If $C \subset S$ is a smooth rational curve ($C \cong \mathbb{P}^1$) and $C^2 = r$, we call C an *r-curve*; by an *r-component* of a divisor D, we mean a component of D which is an r-curve. A divisor D of S has *strong normal crossings* if it is reduced, effective, and satisfies: (i) each component of D is a smooth curve; (ii) if D_i and D_j are distinct components of D then $D_i \cdot D_j \leq 1$; and (iii) if D_i, D_j and D_k are distinct components of D then $D_i \cap D_j \cap D_k$ is empty.

(6) Every graph considered in this paper is a finite undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices. (The words "graph" and "tree" are always used in this restricted sense.)

(7) A *weighted graph* is a graph in which each vertex is assigned an integer (called its weight).

(8) Let S be a smooth projective surface and D a divisor of S with strong normal crossings; let $D_1, \dots, D_n$ be the distinct components of D. The *dual graph* of (D, S) is the weighted graph $\mathcal{G} = \mathcal{G}(D, S)$ whose vertices are the components of D; distinct vertices D_i and D_j are joined by an edge if the curves D_i and D_j have nonempty intersection; and the weight of a vertex D_i is the self-intersection number D_i^2 of the curve D_i. We say that D_j is a *neighbor* of D_i if $i \neq j$ and $D_i \cap D_j \neq \emptyset$ (i.e., if the vertices D_i, D_j of $\mathcal{G}$ are neighbors); the number of neighbors of D_i is called its *branching number*; if this number is strictly greater than 2, we say that D_i is a *branching component* of D (or that the vertex D_i is a *branch point* of $\mathcal{G}$). We say that $\mathcal{G}$ is a *linear chain* (or a *linear tree*) if it is a tree without branch points; an *admissible chain* is a linear chain in which every weight is strictly less than -1; note that the empty graph is an admissible chain. We say that D is a tree (or a linear chain, or an admissible chain, etc) if $\mathcal{G}$ has the corresponding property.

(9) Let X and X^* be projective normal surfaces, β a birational isomorphism between them (either $X \stackrel{\beta}{\to} X^*$ or $X \stackrel{\beta}{\leftarrow} X^*$) and Λ a one-dimensional linear system on X without fixed components. In this situation, we will often use the fact that Λ and β determine, in a natural way, a one-dimensional linear system Λ^* on X^* without fixed components. The tacit understanding is that, for suitably chosen rational maps $X \stackrel{\lambda}{\to} \mathbb{P}^1$ and $X^* \stackrel{\lambda^*}{\to} \mathbb{P}^1$ determining Λ and Λ^* respectively, β, λ and λ^* form a commutative diagram.

For instance, if $X^* \to X$ is a blowing-up morphism, then this process gives the strict transform Λ^* of Λ. On the other hand, if $X \to X^*$ is a blowing-up then the Λ^* so obtained will simply be called the "image" of Λ (then Λ is the strict transform of Λ^*).

(10) We will need to consider the Nagata ruled surface $\mathbb{F}_m$ for positive values of m; Λ_m denotes the standard ruling of $\mathbb{F}_m$ and Σ_m the negative section of Λ_m (Σ_m is a curve on $\mathbb{F}_m$ isomorphic to a projective line, $\Sigma_m^2 = -m$ and $\Sigma_m \cdot D = 1$ for every $D \in \Lambda_m$).

1. Preliminaries

Definition of affine ruling

1.1. Let X be a projective normal rational surface. We claim that every "affine ruling" $p : U \to \Gamma$ of X (as defined in the introduction) determines a linear system Λ on X. Indeed, $U \cong \Gamma \times \mathbb{A}^1$ is normal and rational, so Γ is an open subset of $\mathbb{P}^1$. The morphism p extends to a rational map $X \to \mathbb{P}^1$ which, in turn, determines a unique linear system Λ on X without fixed components.

It is proved in Section 1 of [2] that any two affine rulings of X determining the same linear system Λ differ in a trivial way. Since our task is to enumerate all affine rulings, we should not distinguish between rulings which determine the same linear system; *so we adopt the viewpoint that Λ itself is the affine ruling:*

Definition. Let Λ be a one-dimensional linear system on X without fixed components. We say that Λ is an *affine ruling* of X if there exist nonempty open subsets $U \subset X$ and $\Gamma \subseteq \mathbb{P}^1$ such that $U \cong \Gamma \times \mathbb{A}^1$ and such that the projection morphism $\Gamma \times \mathbb{A}^1 \to \Gamma$ determines Λ.

From now-on, "affine ruling" is always understood as in the above definition.

If Λ is an affine ruling of X then the general member C of Λ satisfies $C \cap U \cong \mathbb{A}^1$; it follows:

- the general member of Λ is irreducible and reduced;
- Λ has at most one base point on X.

A class of surfaces and three problems

1.2. The symbol X always denotes a projective algebraic surface which is at least normal and rational. Following the terminology of [2] and [3], we say that an algebraic surface X satisfies (‡) if:

(‡) X is a projective normal rational surface, X is affine-ruled, $\text{rank}(\text{Pic } X_s) = 1$ and every singular point of X is a cyclic quotient singularity,

where X_s denotes the smooth locus of X. (*Remarks:* (i) "X is affine-ruled" means that there exists at least one affine ruling of X; (ii) the group $\text{Pic}(X_s)$ is the same as the divisor class group of X.)

If a, b, c are any positive integers, the weighted projective plane $\mathbb{P}(a, b, c)$ satisfies (‡). In particular, $\mathbb{P}^2$ satisfies (‡). In Section 1 of [2], it is proved that *if X satisfies (‡) then X has at most three singular points.* We also know that the class (‡) contains many surfaces other than the weighted projective planes, and we have a characterization (see 1.6) of weighted projective planes in the class (‡).

1.3. Given an algebraic surface X satisfying (‡), consider:

Problem 1. Find all affine rulings of X.

Problem 2. Find all pairs of curves C_1, C_2 on X such that $X \setminus (C_1 \cup C_2)$ is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$.

Problem 3. Find all curves C in X such that $\bar{\kappa}(X_s \setminus C) = -\infty$.

It is shown in [2] that Problems 1 and 2 are equivalent, and that a solution to Problem 1 contains, in particular, a solution to Problem 3; also, some references are given in [2] for Problem 3 in the case $X = \mathbb{P}^2$.

Our aim is to investigate Problem 1 for an arbitrary X satisfying (‡), and in particular for $X = \mathbb{P}(a_0, a_1, a_2)$.

Resolution graph of a surface

1.4**. Definition.** If X is a projective normal rational surface, then it makes sense to consider the minimal resolution of singularities $\rho : \hat{X} \to X$ of X. Then $\rho^{-1}(\operatorname{Sing} X)$ is the support of a divisor $\hat{E}$ of $\hat{X}$ with strong normal crossings and, moreover, each connected component of $\hat{E}$ is a tree of projective lines. The dual graph $\mathcal{G}(\hat{E}, \hat{X})$ is called the *resolution graph* of the surface X.

We recall a fact concerning resolution graphs:

1.5. *For X as in* 1.4*, the following are equivalent:*

(1) *Every singular point of X is a cyclic quotient singularity;*
(2) *every connected component of the resolution graph of X is an admissible chain.*

Remark. As far as this article is concerned, the reader unfamiliar with the notion of "cyclic quotient singularities" may use 1.5 as a definition.

The following result was obtained in [3] after essentially everything else had been proved:

1.6**. Theorem.** *Let X be a complete normal rational surface which is affine-ruled and satisfies* $\operatorname{rank}(\operatorname{Pic} X_s) = 1$*. If X has the same resolution graph as the weighted projective plane $\mathbb{P}(a, b, c)$, then X is isomorphic to $\mathbb{P}(a, b, c)$.*

In other words, 1.6 characterizes weighted projective planes among all surfaces satisfying (‡).

Resolution graph of a weighted projective plane

Given positive integers a_0, a_1, a_2, the weighted projective plane $\mathbb{P}(a_0, a_1, a_2)$ is defined by

$$\mathbb{P}(a_0, a_1, a_2) = \operatorname{Proj} A,$$

where $A = \mathbf{k}[X_0, X_1, X_2]$ is graded by assigning weight a_i to X_i. By 1.3.1 of [4], there exist *pairwise relatively prime* positive integers a'_0, a'_1, a'_2 such that $\mathbb{P}(a_0, a_1, a_2) \cong \mathbb{P}(a'_0, a'_1, a'_2)$. From now-on, whenever a weighted projective plane $\mathbb{P}(a_0, a_1, a_2)$ is under consideration, we will always assume that

$$a_0, a_1, a_2 \text{ are pairwise relatively prime.}$$

Let $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2)$ and consider the points $q_0 = (1 : 0 : 0)$, $q_1 = (0 : 1 : 0)$, $q_2 = (0 : 0 : 1) \in \mathbb{P}$. Then it is known that $\mathrm{Sing}(\mathbb{P}) \subseteq \{q_0, q_1, q_2\}$, where q_i is singular if and only if $a_i > 1$. Moreover, these are well understood cyclic quotient singularities, and the resolution graphs of such singularities were described in [5]. Consequently, the resolution graph of $\mathbb{P}$ is known. We now proceed to describe it (see Section 1 of [3] for details).

1.7. (1) Given an ordered triple (a, b, c) of pairwise relatively prime positive integers, we define an admissible chain $\mathcal{A}_{(a,b,c)}$ as follows. Write $r_0 = a$ and let r_1 be the unique integer satisfying $0 \leq r_1 < r_0$ and $br_1 \equiv c \pmod a$; consider the "outer" Euclidean algorithm on (r_0, r_1): $r_{i-1} = q_i r_i - r_{i+1}$ $(0 \leq r_{i+1} < r_i$, $i = 1, \dots, n)$ and $r_{n+1} = 0$; then $\mathcal{A}_{(a,b,c)}$ is:

$$\overset{-q_1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{-q_2}{\bullet}\!\!-\!\!-\ \cdots\ -\!\!-\!\!\overset{-q_{n-1}}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{-q_n}{\bullet}.$$

Note that $\mathcal{A}_{(a,b,c)}$ is the empty chain if and only if $a = 1$. Also, one can see that if b and c are interchanged then the only difference is that the integers $(q_1, \dots, q_n)$ are produced in the reverse order; so $\mathcal{A}_{(a,b,c)} = \mathcal{A}_{(a,c,b)}$.

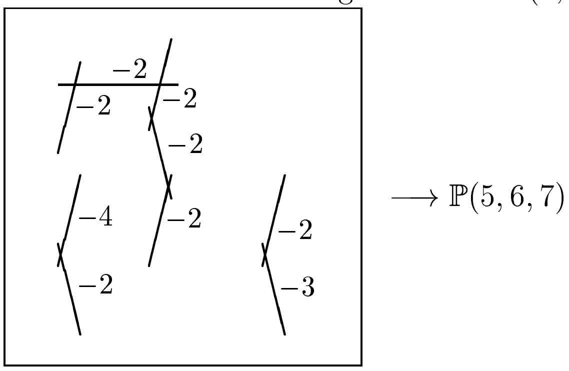
(2) Given an unordered triple $[a_0, a_1, a_2]$ of pairwise relatively prime positive integers, let $\mathcal{G}_{[a_0,a_1,a_2]}$ be the disjoint union of $\mathcal{A}_{(a_0,a_1,a_2)}$, $\mathcal{A}_{(a_1,a_2,a_0)}$ and $\mathcal{A}_{(a_2,a_0,a_1)}$.

1.8. **Proposition.** *Given pairwise relatively prime $a_0, a_1, a_2 \in \mathbb{Z}^+$, the resolution graph of $\mathbb{P}(a_0, a_1, a_2)$ is $\mathcal{G}_{[a_0,a_1,a_2]}$.*

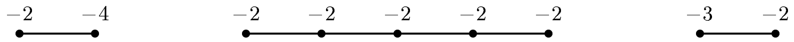
Remark. In [3], any surface X satisfying (‡) and with resolution graph $\mathcal{G}_{[a_0,a_1,a_2]}$ is called a surface "of type $[a_0, a_1, a_2]$". Then, near the end of the paper, it is shown that such a surface must be isomorphic to $\mathbb{P}(a_0, a_1, a_2)$ (this is 1.6 in the present work).

1.9. **Example.** To find the resolution graph of $\mathbb{P}(5, 6, 7)$ by using Definition 1.4, one has to minimally resolve the singularities and look at the resolution locus, i.e., the inverse image of the singular points (see Figure 1).

FIGURE 1. Resolution of singularities of $\mathbb{P}(5, 6, 7)$



Then the resolution graph of $\mathbb{P}(5,6,7)$ is the dual graph of the resolution locus:



However, it is quicker to use 1.8, which tells us that the resolution graph is $\mathcal{G}_{[5,6,7]}$. Note that $\mathcal{G}_{[5,6,7]}$ is exactly the above graph.

2. Blowing-up according to a tableau

Blowing-up, blowing-down and equivalence of weighted graphs

2.1. Let $\mathcal{G}$ be a weighted graph. We define three types of "blowing-up of $\mathcal{G}$":

(1) If v is a vertex of $\mathcal{G}$ then the *blowing-up of $\mathcal{G}$ at v* is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, adding one edge joining e to v, and decreasing the weight of v by 1. (This process is called a blowing-up "at a vertex", or a "sprouting" blowing-up.)

(2) If $\epsilon = \{v_1, v_2\}$ is an edge of $\mathcal{G}$ (so v_1, v_2 are distinct vertices of $\mathcal{G}$ joined by an edge), then the *blowing-up of $\mathcal{G}$ at ϵ* is the weighted graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by adding one vertex e of weight -1, deleting the edge $\epsilon = \{v_1, v_2\}$, adding the two edges $\{v_1, e\}$ and $\{e, v_2\}$, and decreasing the weights of v_1 and v_2 by 1. (This is called a blowing-up "at an edge", or a "subdivisional" blowing-up.)

(3) The *free blowing-up of $\mathcal{G}$* is the weighted graph $\mathcal{G}'$ obtained by taking the disjoint union of $\mathcal{G}$ and of a vertex e of weight -1.

We will use the symbol $\mathcal{G} \leftarrow \mathcal{G}'$ to indicate that $\mathcal{G}$ and $\mathcal{G}'$ are weighted graphs and that $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$. In each of the above three cases, we call e the vertex *created* by $\mathcal{G} \leftarrow \mathcal{G}'$. In reverse, we say that $\mathcal{G}$ is obtained by contracting (or blowing-down) $\mathcal{G}'$ at e.

2.2. More precisely, given a weighted graph $\mathcal{G}'$ and a vertex e of $\mathcal{G}'$, the blowing-down of $\mathcal{G}'$ at e is allowed if and only if the following three conditions hold: (i) e has weight -1; (ii) e has at most two neighbors in $\mathcal{G}'$; (iii) if v_1 and v_2 are distinct neighbors of e in $\mathcal{G}'$ then v_1, v_2 are not neighbors in $\mathcal{G}'$.

Given a sequence $\mathcal{G}_0 \leftarrow \cdots \leftarrow \mathcal{G}_n$ of blowings-up, we may also speak of the contraction (or blowing-down) "$\mathcal{G}_n \geq \mathcal{G}_0$" of weighted graphs.

2.3. Two weighted graphs are *equivalent* if one can be obtained from the other by a finite sequence of blowings-up and blowings-down.

2.4. Let S be a smooth projective surface and D a divisor of S with strong normal crossings. If $S' \to S$ is the blowing-up of S at a point P then the inverse image in S' of $\{P\} \cup \operatorname{Supp}(D)$ is the support of a divisor D' of S' with strong normal crossings. Then the dual graph $\mathcal{G}(D', S')$ is a blowing-up of $\mathcal{G}(D, S)$.

Weighted pairs

2.5. By a *weighted pair*, we mean an ordered pair $(\mathcal{G}, v)$ where $\mathcal{G}$ is a (nonempty) weighted graph and v is a vertex of $\mathcal{G}$. If $(\mathcal{G}, v)$ is a weighted pair, we call v its *distinguished vertex*.

2.6. Let $(\mathcal{G}, v)$ and $(\mathcal{G}', v')$ be weighted pairs. Suppose that $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ (i.e., $\mathcal{G} \leftarrow \mathcal{G}'$) and that the following hold: (i) The blowing-up $\mathcal{G} \leftarrow \mathcal{G}'$ is either at v or at an edge incident to v; and (ii) v' is the vertex of $\mathcal{G}'$ which is created by the blowing-up $\mathcal{G} \leftarrow \mathcal{G}'$. Then we say that $(\mathcal{G}', v')$ is a *blowing-up* of $(\mathcal{G}, v)$ and write $(\mathcal{G}, v) \leftarrow (\mathcal{G}', v')$.

2.7. A *tableau* is a matrix $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)$ whose entries are integers satisfying $c_i \geq p_i \geq 1$ and $\gcd(p_i, c_i) = 1$ for all $i = 1, \dots, k$. We allow $k = 0$, in which case we say that T is the *empty tableau* and write $T = \mathbf{1}$. The set of all tableaux is denoted $\mathcal{T}$.

2.8. Let $(\mathcal{G}_0, e_0)$ be a weighted pair and $\binom{p}{c} \in \mathcal{T}$ a one column tableau. By *blowing-up* $(\mathcal{G}_0, e_0)$ *according to* $\binom{p}{c}$, we mean producing the sequence $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ defined as follows.

(1) Let $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ be the blowing-up at e_0 and let e_1 be the vertex of $\mathcal{G}_1$ so created. Define $\left(\begin{smallmatrix} u_1 & x_1 \\ v_1 & y_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e_1 & p \\ e_0 & c-p \end{smallmatrix}\right)$.

(2) If $i \geq 1$ is such that $(\mathcal{G}_i, e_i)$ and $\left(\begin{smallmatrix} u_i & x_i \\ v_i & y_i \end{smallmatrix}\right)$ have been defined, then:
   (a) If $y_i = 0$ then we set $n = i$ and stop.
   (b) If $y_i \neq 0$ then let $\mathcal{G}_{i+1}$ be the blowing-up of $\mathcal{G}_i$ at the edge $\{u_i, v_i\}$, let e_{i+1} be the vertex of $\mathcal{G}_{i+1}$ so created and define

$$\left(\begin{smallmatrix} u_{i+1} & x_{i+1} \\ v_{i+1} & y_{i+1} \end{smallmatrix}\right) = \begin{cases} \left(\begin{smallmatrix} e_{i+1} & x_i \\ v_i & y_i - x_i \end{smallmatrix}\right) & \text{if } x_i \leq y_i, \\ \left(\begin{smallmatrix} u_i & x_i - y_i \\ e_{i+1} & y_i \end{smallmatrix}\right) & \text{if } x_i > y_i. \end{cases}$$

Remark. The sequence $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ of 2.8 "follows" the euclidean algorithm of (p, c) and satisfies:

(i) $n \geq 1$ and equality holds if and only if $\binom{p}{c} = \binom{1}{1}$.
(ii) $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ is the only blowing-up in $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ which is "at a vertex".

2.9. Let $(\mathcal{G}_0, e_0)$ be a weighted pair and $T = \left(\begin{smallmatrix} p_1 & \cdots & p_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right) \in \mathcal{T}$ a tableau.

We define *the sequence* $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ *obtained by blowing-up* $(\mathcal{G}_0, e_0)$ *according to* T by induction on k:

- If $k = 0$ (i.e., T is the empty tableau), then $n = 0$ (no blowing-up is performed).
- If $k = 1$, then $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ is defined in 2.8.
- If $k > 1$, then $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ is

$$(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_m, e_m) \leftarrow (\mathcal{G}_{m+1}, e_{m+1}) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n),$$

  where $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_m, e_m)$ is the sequence obtained by blowing-up $(\mathcal{G}_0, e_0)$ according to $\binom{p_1}{c_1}$ and $(\mathcal{G}_m, e_m) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ is obtained by blowing-up $(\mathcal{G}_m, e_m)$ according to $\left(\begin{smallmatrix} p_2 & \cdots & p_k \\ c_2 & \cdots & c_k \end{smallmatrix}\right)$.

2.10. Given a weighted pair $(\mathcal{G}_0, e_0)$, consider the set $\mathcal{S}_{(\mathcal{G}_0, e_0)}$ of sequences of blowings-up of weighted pairs $(\mathcal{G}_0, e_0) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n)$ satisfying: *if* $n > 0$ *then* $\mathcal{G}_0 \leftarrow \mathcal{G}_1$ *is the blowing-up of* $\mathcal{G}_0$ *at* e_0.

The above paragraphs show that each $T \in \mathcal{T}$ determines an element of $\mathcal{S}_{(\mathcal{G}_0, e_0)}$ (obtained by blowing-up $(\mathcal{G}_0, e_0)$ according to T). In fact, it is not difficult to see that this set map $\mathcal{T} \to \mathcal{S}_{(\mathcal{G}_0, e_0)}$ is bijective.

2.11. Consider a pair (π, G), where $\pi : S' \to S$ is a birational morphism of smooth projective surfaces and $G \subset S$ is a smooth curve. Assume:

(1) $G \cap \operatorname{center}(\pi)$ contains at most one point;

(2) if $G \cap \text{center}(\pi) = \{P\}$, then $\pi^{-1}(P)$ has exactly one (-1)-component.

Let us explain how (π, G) determines a tableau $\overline{\text{HN}}(\pi, G) \in \mathcal{T}$.

If $G \cap \text{center}(\pi)$ is empty, let $\overline{\text{HN}}(\pi, G)$ be the empty tableau.

If $G \cap \text{center}(\pi) = \{P\}$, let n be the number of irreducible components of $\pi^{-1}(P)$ and factor π as

$$S = S_0 \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_n} S_n \xleftarrow{\pi'} S',$$

where π_i is the blowing-up of S_{i-1} at a point $Q_i \in S_{i-1}$ infinitely near P (so $Q_1 = P$), and where the center of π' is disjoint from $(\pi_1 \cdots \pi_n)^{-1}(P) \subset S_n$. Let $E_i = \pi_i^{-1}(Q_i) \subset S_i$ (for $1 \le i \le n$) and let $E_0 = G$. Then E_n is a (-1)-component of $\pi^{-1}(P)$ and, by assumption (2), it is the only one. It follows:

(3) $P_i \in E_{i-1}$ for all $i = 1, \dots, n$,

so we have a sequence of blowings-up of weighted pairs

$$(\mathcal{G}(D_0, S_0), E_0) \leftarrow \cdots \leftarrow (\mathcal{G}(D_n, S_n), E_n),$$

where D_i is the inverse image of G in S_i (so $D_0 = G = E_0$). By 2.10, there is a unique tableau $T \in \mathcal{T}$ such that this sequence is the blowing-up of $(\mathcal{G}(D_0, S_0), E_0)$ according to T. This tableau T is denoted $\overline{\text{HN}}(\pi, G)$.

Remark. $\overline{\text{HN}}(\pi, G)$ is a simplified version of the Hamburger-Noether tableau ([8], [6]). Note that the Hamburger-Noether tableau contains more information than $\overline{\text{HN}}(\pi, G)$.

3. Construction of affine rulings

This section gives a method for constructing all pairs (X, Λ) where X is a surface satisfying (‡) and Λ is an affine ruling of X. See the introduction for the notations $\mathbb{F}_m$, Λ_m and Σ_m.

3.1. **Definition.** Fix a triple $(m, T_1, T_2) \in \mathbb{Z}^+ \times \mathcal{T} \times \mathcal{T}$.

(1) By a *blowing-up of* $\mathbb{F}_m$ *according to* (T_1, T_2), we mean a triple (π, P_1, P_2) where
   - (a) $\pi : Y \to \mathbb{F}_m$ is a birational morphism (with Y smooth and projective);
   - (b) P_1, P_2 are points of $\mathbb{F}_m \setminus \Sigma_m$ belonging to distinct members of Λ_m ($P_i \in Z_i \in \Lambda_m$, $Z_1 \neq Z_2$);
   - (c) $\text{center}(\pi) \subseteq \{P_1, P_2\}$ and, for each $i = 1, 2$, $\pi^{-1}(P_i)$ contains at most one (-1)-curve (so $\overline{\text{HN}}(\pi, Z_i)$ is defined);
   - (d) $\overline{\text{HN}}(\pi, Z_i) = T_i$ for $i = 1, 2$.

Warning. The blowings-up of $\mathbb{F}_m$ according to (T_1, T_2) always exist, but are not unique.

(2) Let $\beta = (\pi, P_1, P_2)$ be a blowing-up of $\mathbb{F}_m$ according to (T_1, T_2), with notation as in part 1. We define a divisor D_β of Y (with strong normal crossings) as follows. For each $i = 1, 2$, let

$$E_i = \begin{cases} \text{strict transform of } Z_i \text{ in } Y, & \text{if } T_i = \mathbf{1}; \\ \text{the } (-1)\text{-component of } \pi^{-1}(P_i), & \text{if } T_i \neq \mathbf{1}. \end{cases}$$

Then let D_β be the reduced effective divisor of Y whose support is

$$\pi^{-1}(Z_1 \cup \Sigma_m \cup Z_2) \text{ minus } E_1 \text{ and } E_2.$$

It is easy to see that the dual graph $\mathcal{G}(D_\beta, Y)$ depends only on the discrete data (m, T_1, T_2), i.e., is independent of the choice of β. This weighted graph is denoted $\mathcal{G}(m, T_1, T_2)$.

Remark. In [2] and [3], the notation $(\mathcal{G}_{(-m)} \ominus T_1) \ominus T_2$ was used in place of $\mathcal{G}(m, T_1, T_2)$, but using the same notation here would force us to define many concepts. Concretely, the graph $\mathcal{G}(m, T_1, T_2)$ can be computed as follows:

(1) Let $\mathcal{G}$ be the weighted graph $\overset{0}{\underset{z_1}{\bullet}}\!\!-\!\!\!-\!\!\!-\!\!\overset{-m}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\underset{z_2}{\bullet}}$ (dual graph of $(Z_1 + \Sigma_m + Z_2, \mathbb{F}_m)$).
(2) Let $(\mathcal{G}, z_1) \leftarrow \cdots \leftarrow (\mathcal{G}', e_1)$ be the blowing-up of $(\mathcal{G}, z_1)$ according to T_1, and note that z_2 is a vertex of $\mathcal{G}'$.
(3) Let $(\mathcal{G}', z_2) \leftarrow \cdots \leftarrow (\mathcal{G}'', e_2)$ be the blowing-up of $(\mathcal{G}', z_2)$ according to T_2; then $\mathcal{G}(m, T_1, T_2)$ is obtained from $\mathcal{G}''$ by removing e_1 and e_2.

3.2. Let $\mathbb{T}(\ddagger)$ be the set of triples $(m, T_1, T_2) \in \mathbb{Z}^+ \times \mathcal{T} \times \mathcal{T}$ such that:

(1) T_1 satisfies one of the following three conditions:
  (a) $T_1 = \mathbf{1}$ (the empty tableau);
  (b) $T_1 = \binom{p}{c}$, for some $\binom{p}{c} \neq \binom{1}{1}$;
  (c) $T_1 = \left(\begin{smallmatrix} p & 1 \\ c & N \end{smallmatrix}\right)$, for some $\binom{p}{c} \neq \binom{1}{1}$ and $N \geq 1$.
(2) If T_2 is nonempty then its first column is not $\binom{1}{1}$.
(3) Each connected component of the weighted graph $\mathcal{G}(m, T_1, T_2)$ shrinks to an admissible chain.

3.3. We also define a subset $\mathbb{T}_0(\ddagger)$ of $\mathbb{T}(\ddagger)$ by:

$$\mathbb{T}_0(\ddagger) = \{(m, T_1, T_2) \in \mathbb{T}(\ddagger) \mid T_2 \text{ also satisfies one of (a), (b), (c) in part 1 of 3.2}\}.$$

3.4. It is important to note that the sets $\mathbb{T}(\ddagger)$ and $\mathbb{T}_0(\ddagger)$ can be described explicitely. See “Some explicit computations” at the end of [2].

Given $(m, T_1, T_2) \in \mathbb{T}(\ddagger)$ and a blowing-up β of $\mathbb{F}_m$ according to (T_1, T_2), the next result defines a pair (X_β, Λ_β) where X_β is a surface satisfying ($\ddagger$) and Λ_β is an affine ruling of X_β. The notation (X_β, Λ_β) will always be used in this sense, i.e., the pair determined by β as in 3.5.

3.5. **Theorem.** *Let $\tau = (m, T_1, T_2) \in \mathbb{T}(\ddagger)$, let $\beta = (\pi, P_1, P_2)$ be any blowing-up of $\mathbb{F}_m$ according to (T_1, T_2) (with notation $\pi : Y \to \mathbb{F}_m$) and consider the divisor D_β of Y. Then the following hold:*

(1) *There exists a birational morphism $\sigma : Y \to X_\beta$ whose exceptional locus is exactly the support of D_β, and where X_β is a surface satisfying ($\ddagger$).*
(2) *If Λ_β is the linear system on X_β which is determined by Λ_m via π and σ (i.e., Λ_β is the image under σ of the strict transform of Λ_m), then Λ_β is an affine ruling of X_β.*
(3) *Up to isomorphism, X_β is completely determined by τ.*

3.6. **Theorem.** *Let X be a surface satisfying (‡) and Λ an affine ruling of X. Then there exist $(m, T_1, T_2) \in \mathbb{T}(\ddagger)$ and a blowing-up β of $\mathbb{F}_m$ according to (T_1, T_2) such that X is isomorphic to X_β and, under that isomorphism, Λ corresponds to Λ_β.*

The above two theorems show how to construct all pairs (X, Λ) where X is a surface satisfying (‡) and Λ is an affine ruling of X. Since the set $\mathbb{T}(\ddagger)$ can be described explicitely (3.4), this method is quite satisfactory. The next section will address the question of constructing all Λ on a given X.

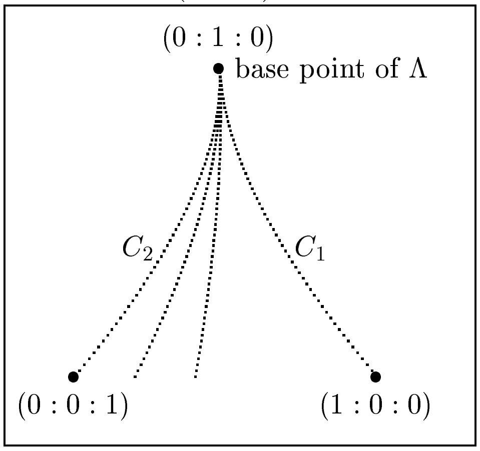
An example

For the remainder of this section, we consider the surface $X = \mathbb{P}(5, 6, 7)$ and a certain affine ruling Λ of X. Our aim is to illustrate how 3.6 can be proved, so we start from (X, Λ) and seek (m, T_1, T_2) and β.

We stress that Λ is a *specific* affine ruling of $\mathbb{P}(5, 6, 7)$, i.e., we could (but we won't) give explicit equations for it. The paragraphs below state several properties of Λ without justification, some of which are specific to this particular Λ (e.g. Figure 3). In other words, we claim that there exists an affine ruling Λ of $\mathbb{P}(5, 6, 7)$ having all the properties described in the following paragraphs.

It is proved in [2] that if X satisfies (‡) and Λ is an affine ruling of X then every member of Λ has irreducible support and Λ has at most two multiple members. In our case, Λ has exactly two multiple members, of the form $5C_1$ and $21C_2$ where C_1 and C_2 are irreducible curves (see Figure 2). The other dotted curves in Figure 2 represent some of the reduced members of Λ.

Figure 2. $X = \mathbb{P}(5, 6, 7)$, with affine ruling Λ



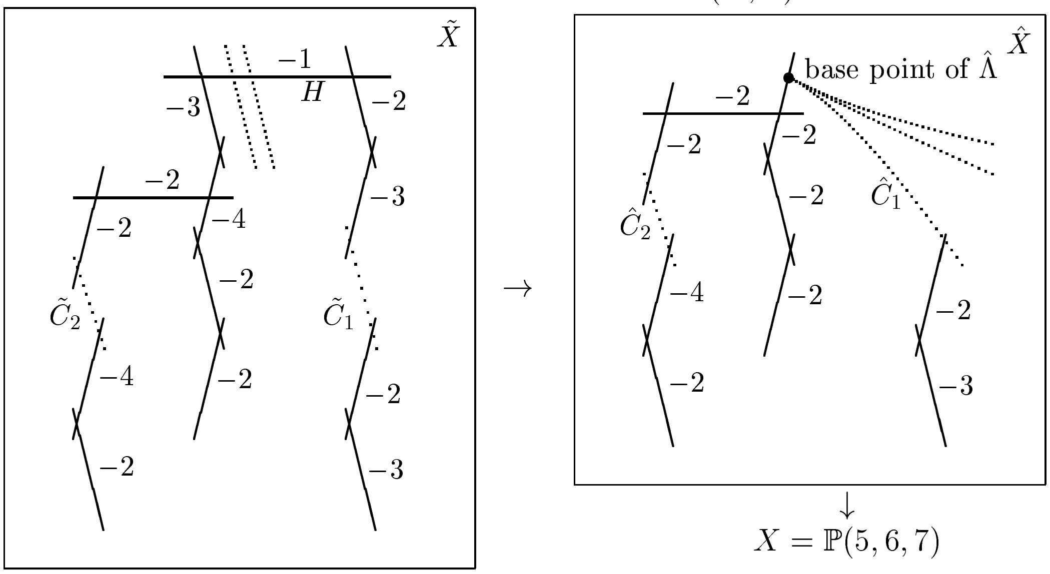
We begin by "resolving" (X, Λ), i.e., by constructing a pair $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^\sim$ as follows:

(1) Minimally resolve the singularities of X (write $\hat{X} \to X$). Let $\hat{\Lambda}$ be the strict transform of Λ on $\hat{X}$.

(2) Minimally resolve the base point of $\hat{\Lambda}$ (write $\tilde{X} \to \hat{X}$). Let $\tilde{\Lambda}$ be the strict transform of $\hat{\Lambda}$ on $\tilde{X}$.

Let $\rho : \tilde{X} \to X$ be the composition $\tilde{X} \to \hat{X} \to X$. The center of ρ is $\operatorname{Sing} X \cup \operatorname{Bs}\Lambda$ and $\rho^{-1}(\operatorname{Sing} X \cup \operatorname{Bs}\Lambda)$ is the support of a divisor D of $\tilde{X}$ with strong normal crossings. For the pair (X, Λ) under consideration, the resolution $(\tilde{X}, \tilde{\Lambda}) = (X, \Lambda)^{\sim}$ is as in Figure 3.

FIGURE 3. Resolution of (X, Λ)



In Figure 3, note the following:

(1) In $\tilde{X}$ (resp. $\hat{X}$), the divisor displayed in full lines is D (resp. the inverse image of $\operatorname{Sing}(X)$) and the dotted curves are the strict transforms of the dotted curves in Figure 2; in particular, $\tilde{C}_i$ (resp. $\hat{C}_i$) is the strict transform of C_i.

(2) Since $\tilde{\Lambda}$ is base-point-free and has $\mathbb{P}^1$ as general member, $\tilde{\Lambda}$ is a $\mathbb{P}^1$-ruling of $\tilde{X}$.

(3) The curve $H \subset \tilde{X}$ is the "last exceptional curve", i.e., if we factor $\tilde{X} \to \hat{X}$ as

$$\tilde{X} = Y_4 \xrightarrow{\mu_4} Y_3 \xrightarrow{\mu_3} Y_2 \xrightarrow{\mu_2} Y_1 \xrightarrow{\mu_1} Y_0 = \hat{X} \quad \text{(four monoidal transformations)},$$

then H is the exceptional curve of μ_4. Consequently, H is a section of $\tilde{\Lambda}$ (i.e., we have $H \cdot M = 1$ for every $M \in \tilde{\Lambda}$).

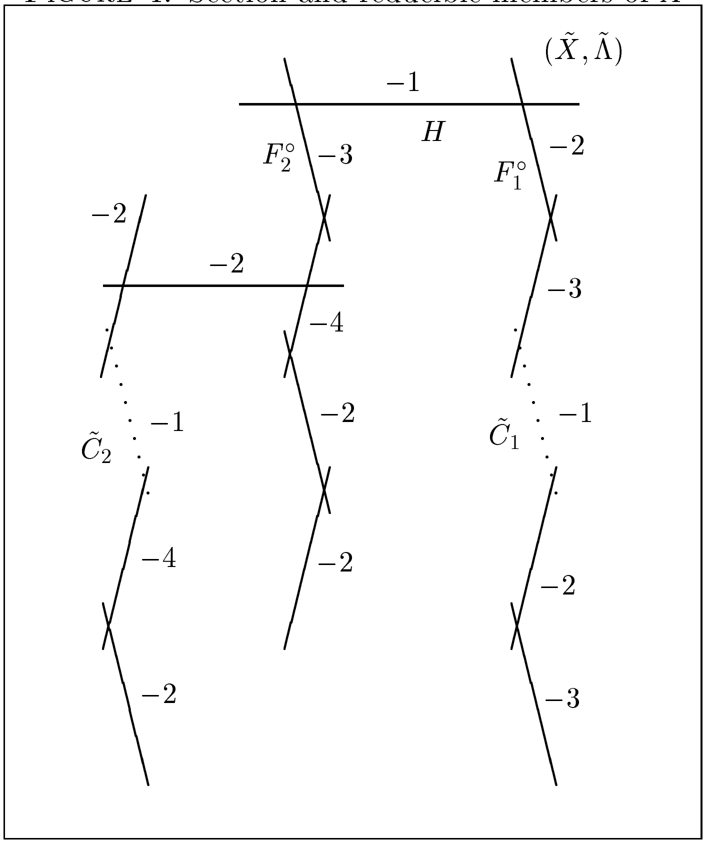
Next we claim:

(i) $\tilde{\Lambda}$ has exactly two reducible members, which we denote $F_1, F_2 \in \tilde{\Lambda}$;

(ii) each reducible member F_i has exactly one (-1)-component, namely, $\tilde{C}_i$;

(iii) $D = (F_1^{\#} - \tilde{C}_1) + H + (F_2^{\#} - \tilde{C}_2)$ where $F_i^{\#}$ is the reduced effective divisor of $\tilde{X}$ with same support as F_i. (See Figure 4.)

FIGURE 4. Section and reducible members of $\tilde{\Lambda}$



Note that, in Figure 4, if we consider only the full lines then we have a picture of D (the same picture as in Figure 3). But if we consider the full lines and also the two dotted lines $\tilde{C}_i$, the resulting divisor is $D + \tilde{C}_1 + \tilde{C}_2$, which is equal to $H + F_1^\# + F_2^\#$.

It is a well-known fact that, given a $\mathbb{P}^1$-ruling on a smooth projective rational surface, the reducible members may be shrunk to 0-curves and this produces one of the Nagata ruled surfaces together with its standard ruling. Applying this to $(\tilde{X}, \tilde{\Lambda})$ gives a morphism $\pi : \tilde{X} \to \mathbb{F}_1$ whose exceptional locus is the support of $(F_1^\# - F_1^\circ) + (F_2^\# - F_2^\circ)$, where F_i° is the unique irreducible component of F_i which meets H. Figure 5 shows the codomain of π.

In Figure 5, note the following:

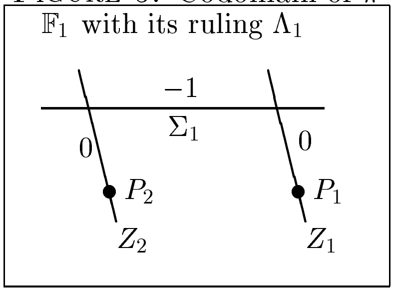
(1) $\pi(H) = \Sigma_1$, $\pi(F_i) = \pi(F_i^\circ) = Z_i \in \Lambda_1$ (for $i = 1, 2$) and $\operatorname{center}(\pi) = \{P_1, P_2\}$.

(2) For each $i = 1, 2$, $\pi^{-1}(P_i)$ has a unique (-1)-component (namely, $\tilde{C}_i$).

By the second observation, it makes sense to define $T_i = \overline{\mathrm{HN}}(\pi, Z_i)$ (see 2.11). In fact we have $T_1 = \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} 1 & 3 \\ 3 & 7 \end{smallmatrix}\right)$.

Now it is clear that $\beta = (\pi, P_1, P_2)$ is a blowing-up of $\mathbb{F}_1$ according to (T_1, T_2) and that $D_\beta = D$. Consequently, the weighted graph $\mathcal{G}\left(1, \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 3 \\ 3 & 7 \end{smallmatrix}\right)\right)$ (defined in

Figure 5. Codomain of π



3.1) is the dual graph of $(D, \tilde{X})$, which shrinks to the resolution graph of X; thus each connected component of $\mathcal{G}\,(1, \left(\begin{smallmatrix}3\\5\end{smallmatrix}\right), \left(\begin{smallmatrix}1&3\\3&7\end{smallmatrix}\right))$ shrinks to an admissible chain and it follows from definition 3.2 that $(m, T_1, T_2) = (1, \left(\begin{smallmatrix}3\\5\end{smallmatrix}\right), \left(\begin{smallmatrix}1&3\\3&7\end{smallmatrix}\right)) \in \mathbb{T}(\ddagger)$. Finally, note that $(X_\beta, \Lambda_\beta) = (X, \Lambda)$.

4. Affine rulings of a given surface

4.1. Given a surface X satisfying ($\ddagger$), we define a subset $\mathbb{T}(X)$ of $\mathbb{T}(\ddagger)$ by

$$\mathbb{T}(X) = \{(m, T_1, T_2) \in \mathbb{T}(\ddagger) \mid (m, T_1, T_2) \text{ satisfies conditions (1), (2)}\},$$

where (1) and (2) are the following equivalent conditions:

(1) For some blowing-up β of $\mathbb{F}_m$ according to (T_1, T_2), $X_\beta \cong X$.
(2) For all blowings-up β of $\mathbb{F}_m$ according to (T_1, T_2), $X_\beta \cong X$.

That (1) and (2) are equivalent is a consequence of part 3 of 3.5, which also implies that if $\mathbb{T}(X) \cap \mathbb{T}(X') \neq \emptyset$ then $X \cong X'$ and $\mathbb{T}(X) = \mathbb{T}(X')$ (so the sets $\mathbb{T}(X)$ form a partition of $\mathbb{T}(\ddagger)$).

We also define the subset $\mathbb{T}_0(X) = \mathbb{T}_0(\ddagger) \cap \mathbb{T}(X)$ of $\mathbb{T}(X)$.

The following fact is proved in [2]:

4.2. *For a surface X satisfying ($\ddagger$), the following problems are equivalent:*

(1) *Finding all affine rulings of X (i.e., solving Problem 1);*
(2) *describing the set $\mathbb{T}(X)$;*
(3) *describing the set $\mathbb{T}_0(X)$.*

In fact, 3.5 and 3.6 show that the first problem reduces to the second: Assume that the set $\mathbb{T}(X)$ is known and, for each $(m, T_1, T_2) \in \mathbb{T}(X)$, consider all blowings-up β of $\mathbb{F}_m$ according to (T_1, T_2); then the pairs (X_β, Λ_β) give all affine rulings of X.

The second problem reduces to the third because [2] gives a method for generating $\mathbb{T}(X)$ from its subset $\mathbb{T}_0(X)$ (see part 2 of 4.11, below).

For a general X satisfying ($\ddagger$), we don't know how to describe $\mathbb{T}(X)$ or $\mathbb{T}_0(X)$; however, thanks to 1.6, we can do it if X is a weighted projective plane. The idea is as follows.

4.3. Let X be a surface satisfying (‡) and let $\mathcal{R}$ denote the resolution graph of X. Given an arbitrary $\tau = (m, T_1, T_2) \in \mathbb{T}(\ddagger)$, consider the weighted graph $\mathcal{G}(\tau) = \mathcal{G}(m, T_1, T_2)$ defined as in 3.1. It is not difficult to see that the following conditions are equivalent:

(1) $\mathcal{G}(\tau) \sim \mathcal{R}$ (equivalence of weighted graphs);

(2) for every blowing-up β of $\mathbb{F}_m$ according to (T_1, T_2), the surface X_β has resolution graph $\mathcal{R}$.

By 1.6, if X is a weighted projective plane then the second condition is equivalent to $X_\beta \cong X$, and hence to $\tau \in \mathbb{T}(X)$. In other words,

$$\mathbb{T}(X) = \{\tau \in \mathbb{T}(\ddagger) \mid \mathcal{G}(\tau) \sim \mathcal{R}\} \quad \text{and} \quad \mathbb{T}_0(X) = \{\tau \in \mathbb{T}_0(\ddagger) \mid \mathcal{G}(\tau) \sim \mathcal{R}\}$$

hold whenever X is a weighted projective plane. Thus the determination of $\mathbb{T}_0(\mathbb{P}(a,b,c))$ reduces to solving:

Problem G. Given pairwise relatively prime $a, b, c \in \mathbb{Z}^+$, find all $\tau \in \mathbb{T}_0(\ddagger)$ such that $\mathcal{G}(\tau) \sim \mathcal{G}_{[a,b,c]}$.

Note that Problem G belongs to the theory of weighted graphs and makes no reference to geometry; it is solved in [3] using graph theory only. We will now state the solution.

DESCRIPTION OF $\mathbb{T}_0(\mathbb{P}(a,b,c))$

Given pairwise relatively prime positive integers a_0, a_1, a_2, paragraphs 4.4–4.7 define sets

$$\mathbb{T}_{\mathrm{I}}(a_0,a_1,a_2), \quad \mathbb{T}_{\mathrm{II.1}}(a_0,a_1,a_2), \quad \mathbb{T}_{\mathrm{II.2}}(a_0,a_1,a_2) \quad \text{and} \quad \mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2).$$

Then $\mathbb{T}_0(\mathbb{P}(a,b,c))$ is described in 4.8.

Remark. From now on, the 2×1 matrix $\binom{0}{1}$ should be interpreted as the empty tableau $\mathbf{1} \in \mathcal{T}$.

4.4. Given pairwise relatively prime positive integers a_0, a_1, a_2, it is clear that

$$\mathrm{Eq}(a_0,a_1,a_2)\textbf{:} \qquad a_0 = a_1 a_2 x_0 - a_2 x_1 - a_1 x_2$$

has a unique solution $(x_0, x_1, x_2) \in \mathbb{N}^3$ satisfying $0 \le x_1 < a_1$ and $0 \le x_2 < a_2$. Then $x_0 > 0$ and for $i = 1, 2$ we have $x_i = 0 \iff a_i = 1$ and $x_i \in \{0, 1\} \iff a_i \mid (a_0 + a_1 + a_2)$. For each $i = 1, 2$, there is a unique x_i' satisfying $x_i x_i' \equiv 1 \pmod{a_i}$ and $0 \le x_i' < a_i$; and a unique $x_i'' \in \mathbb{Z}$ satisfying $x_i x_i' - x_i'' a_i = 1$.

4.5. Let a_0, a_1, a_2 be pairwise relatively prime positive integers.

(1) The set $\mathbb{T}_{\mathrm{I}}(a_0,a_1,a_2)$ has exactly one element, namely

$$\left(x_0, \begin{pmatrix} x_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ a_2 \end{pmatrix} \right),$$

where (x_0, x_1, x_2) is the unique solution of $\mathrm{Eq}(a_0,a_1,a_2)$.

(2) The set $\mathbb{T}_{\mathrm{II.1}}(a_0,a_1,a_2)$ has at most one element, and is nonempty if and only if $\dfrac{a_0+a_1+a_2}{a_2}$ is a natural number strictly greater than 2. Moreover, if $\mathbb{T}_{\mathrm{II.1}}(a_0,a_1,a_2)$

is nonempty then let (x_0,x_1,x_2) be the unique solution to $\mathrm{Eq}(a_0,a_1,a_2)$, let x_1', x_1'' be as in 4.4 and define

$$\begin{pmatrix} p_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 - x_1 - x_1' + x_1'' \\ a_1 - x_1 \end{pmatrix} + (x_0 - x_2)\begin{pmatrix} a_1 - x_1' \\ a_1 \end{pmatrix};$$

then the unique element of $\mathbb{T}_{\mathrm{II}.1}(a_0,a_1,a_2)$ is

$$\left(1, \begin{pmatrix} x_1' \\ a_1 \end{pmatrix}, \begin{pmatrix} p_2 & 1 \\ c_2 & a_2 \end{pmatrix}\right).$$

(3) $\mathbb{T}_{\mathrm{II}.2}(a_0,a_1,a_2) = \{(m,T_2,T_1) \mid (m,T_1,T_2) \in \mathbb{T}_{\mathrm{II}.1}(a_0,a_1,a_2)\}$.

4.6. Let a_0,a_1,a_2 be pairwise relatively prime positive integers satisfying $a_1a_2 \mid a_0 + a_1 + a_2$, and write $\gamma = \frac{a_0+a_1+a_2}{a_1a_2}$. Then (a_0,a_1,a_2) determines two sets, $W_{(a_0,a_1,a_2)}$ and $W^{(a_0,a_1,a_2)}$, which we now proceed to define.

(1) Each 2×2 matrix M (with entries in $\mathbb{Z}$) determines a pair of sequences

$$s(M) = (s_0,s_1,s_2,\dots), \qquad t(M) = (t_0,t_1,t_2,\dots)$$

defined by $\begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix} = M$ and $\begin{cases} s_{n-1}+s_{n+1} = a_2\gamma t_n \\ t_{n-1}+t_{n+1} = a_1\gamma s_n \end{cases}$.

(2) Let $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and define $u_n = s_n(M)$ and $v_n = t_n(M)$.

(3) Let $M' = \left(\begin{smallmatrix} -\gamma-1+x_2 & x_1-1 \\ -\gamma-1+x_1 & x_2-1 \end{smallmatrix}\right)$, where (x_0,x_1,x_2) is the solution to $\mathrm{Eq}(a_0,a_1,a_2)$, and define $\xi_n = s_n(M')$ and $\eta_n = t_n(M')$.

(4) For every $n \in \mathbb{N}$, define

$$f_n = \left(1, \left(\begin{smallmatrix} \xi_n & 1 \\ u_n & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} v_{n+1}-\eta_{n+1} & 1 \\ v_{n+1} & a_2 \end{smallmatrix}\right)\right) \quad \text{and} \quad g_n = \left(1, \left(\begin{smallmatrix} u_{n+1}-\xi_{n+1} & 1 \\ u_{n+1} & a_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \eta_n & 1 \\ v_n & a_2 \end{smallmatrix}\right)\right).$$

Then define

$$W_{(a_0,a_1,a_2)} = \begin{cases} \{f_2,g_3,f_4,g_5,\dots\}, & \text{if } a_0 > a_1 - a_2, \\ \emptyset, & \text{else,} \end{cases}$$

and

$$W^{(a_0,a_1,a_2)} = \begin{cases} \{g_2,f_3,g_4,f_5,\dots\}, & \text{if } a_0 > a_2 - a_1, \\ \emptyset, & \text{else.} \end{cases}$$

4.7. Let a_0,a_1,a_2 be pairwise relatively prime positive integers. Then $\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)$ is nonempty if and only if $a_1a_2 \mid a_0+a_1+a_2$, in which case we have:

$$\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2) = W_{(a_0,a_1,a_2)} \cup W^{(a_0,a_1,a_2)}.$$

Remark. If $a_1a_2 \mid a_0+a_1+a_2$ and $a_0 > |a_1-a_2|$, then

$$\mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2) = \{f_2,f_3,f_4,\dots\} \cup \{g_2,g_3,g_4,\dots\}.$$

Also observe that $\mathbb{T}_{\mathrm{III}}(a_0,a_2,a_1) = \{(m,T_2,T_1) \mid (m,T_1,T_2) \in \mathbb{T}_{\mathrm{III}}(a_0,a_1,a_2)\}$ holds in all cases.

4.8. **Theorem.** *Let (a, b, c) be pairwise relatively prime positive integers. Then $\mathbb{T}_0(\mathbb{P}(a, b, c))$ is the union of the sets*

$$\mathbb{T}_{\mathrm{I}}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{II}.1}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{II}.2}(a_0, a_1, a_2) \cup \mathbb{T}_{\mathrm{III}}(a_0, a_1, a_2)$$

for all permutations (a_0, a_1, a_2) of (a, b, c).

4.9. **Example.** The following is a description of $\mathbb{T}_0(\mathbb{P}^2)$ (recall that $\mathbb{P}^2 = \mathbb{P}(1, 1, 1)$). First, 4.5 gives:

- $\mathbb{T}_{\mathrm{I}}(1, 1, 1) = \{(1, \mathbf{1}, \mathbf{1})\}$ (where $\mathbf{1}$ is the empty tableau);
- $\mathbb{T}_{\mathrm{II}.1}(1, 1, 1) = \{(1, \mathbf{1}, \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right))\}$;
- $\mathbb{T}_{\mathrm{II}.2}(1, 1, 1) = \{(1, \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right), \mathbf{1})\}$.

We have $\mathbb{T}_{\mathrm{III}}(1, 1, 1) = \{f_2, f_3, f_4, \dots\} \cup \{g_2, g_3, g_4, \dots\}$ by 4.7; by 4.6 (with $\gamma = 3$), we find that $u_n = v_n$ and $\xi_n = \eta_n$ for all n, and:

$$\begin{aligned} u_n &= 3u_{n-1} - u_{n-2}, \quad u_0 = 1, \quad u_1 = 1; \\ \xi_n &= 3\xi_{n-1} - \xi_{n-2}, \quad \xi_0 = -4, \ \xi_1 = -1. \end{aligned}$$

So,

$$\begin{aligned} \mathbb{T}_{\mathrm{III}}(1, 1, 1) = \{&(1, \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\5&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}4&1\\5&1\end{smallmatrix}\right), \left(\begin{smallmatrix}2&1\\13&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}11&1\\13&1\end{smallmatrix}\right), \left(\begin{smallmatrix}5&1\\34&1\end{smallmatrix}\right)), \dots\} \\ &\cup \{(1, \left(\begin{smallmatrix}1&1\\5&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}2&1\\13&1\end{smallmatrix}\right), \left(\begin{smallmatrix}4&1\\5&1\end{smallmatrix}\right)), (1, \left(\begin{smallmatrix}5&1\\34&1\end{smallmatrix}\right), \left(\begin{smallmatrix}11&1\\13&1\end{smallmatrix}\right)), \dots\} \end{aligned}$$

and $\mathbb{T}_0(\mathbb{P}^2)$ is the union of the above four sets.

4.10. **Example.** We now describe $\mathbb{T}_0(\mathbb{P}(2, 3, 5))$. By 4.5,

- $\mathbb{T}_{\mathrm{I}}(2, 3, 5) = \{(1, \binom{2}{3}, \binom{1}{5})\}$;
- $\mathbb{T}_{\mathrm{I}}(2, 5, 3) = \{(1, \binom{1}{5}, \binom{2}{3})\}$;
- $\mathbb{T}_{\mathrm{I}}(3, 2, 5) = \{(1, \binom{1}{2}, \binom{1}{5})\}$;
- $\mathbb{T}_{\mathrm{I}}(3, 5, 2) = \{(1, \binom{1}{5}, \binom{1}{2})\}$;
- $\mathbb{T}_{\mathrm{I}}(5, 2, 3) = \{(2, \binom{1}{2}, \binom{2}{3})\}$;
- $\mathbb{T}_{\mathrm{I}}(5, 3, 2) = \{(2, \binom{2}{3}, \binom{1}{2})\}$;
- $\mathbb{T}_{\mathrm{II}.1}(3, 5, 2) = \big\{\big(1, \binom{1}{5}, \left(\begin{smallmatrix}3&1\\4&2\end{smallmatrix}\right)\big)\big\}$;
- $\mathbb{T}_{\mathrm{II}.2}(3, 2, 5) = \big\{\big(1, \left(\begin{smallmatrix}3&1\\4&2\end{smallmatrix}\right), \binom{1}{5}\big)\big\}$;
- $\mathbb{T}_{\mathrm{II}.1}(5, 3, 2) = \big\{\big(1, \binom{2}{3}, \left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right)\big)\big\}$;
- $\mathbb{T}_{\mathrm{II}.2}(5, 2, 3) = \big\{\big(1, \left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right), \binom{2}{3}\big)\big\}$.

We have $\mathbb{T}_{\mathrm{III}}(3, 5, 2) = \{g_2, f_3, g_4, f_5, \dots\}$ by 4.7; by 4.6 (with $\gamma = 1$),

$$\begin{aligned} u_{n-2} + u_n &= 2v_{n-1}, \quad u_0 = 1, \quad u_1 = 1; \\ v_{n-2} + v_n &= 5u_{n-1}, \quad v_0 = 1, \quad v_1 = 1; \\ \xi_{n-2} + \xi_n &= 2\eta_{n-1}, \quad \xi_0 = -1, \ \xi_1 = 0; \\ \eta_{n-2} + \eta_n &= 5\xi_{n-1}, \quad \eta_0 = -1, \ \eta_1 = 0, \end{aligned}$$

so $\mathbb{T}_{\mathrm{III}}(3,5,2)=\{(1,\left(\begin{smallmatrix}5&1\\7&5\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right)),(1,\left(\begin{smallmatrix}2&1\\7&5\end{smallmatrix}\right),\left(\begin{smallmatrix}22&1\\31&2\end{smallmatrix}\right)),(1,\left(\begin{smallmatrix}39&1\\55&5\end{smallmatrix}\right),\left(\begin{smallmatrix}9&1\\31&2\end{smallmatrix}\right)),\dots\}$. Also,

$$\begin{aligned}\mathbb{T}_{\mathrm{III}}(3,2,5)&=\{(m,T_2,T_1)\mid (m,T_1,T_2)\in\mathbb{T}_{\mathrm{III}}(3,5,2)\}\\&=\{(1,\left(\begin{smallmatrix}1&1\\4&2\end{smallmatrix}\right),\left(\begin{smallmatrix}5&1\\7&5\end{smallmatrix}\right)),(1,\left(\begin{smallmatrix}22&1\\31&2\end{smallmatrix}\right),\left(\begin{smallmatrix}2&1\\7&5\end{smallmatrix}\right)),(1,\left(\begin{smallmatrix}9&1\\31&2\end{smallmatrix}\right),\left(\begin{smallmatrix}39&1\\55&5\end{smallmatrix}\right)),\dots\}.\end{aligned}$$

Then $\mathbb{T}_0(\mathbb{P}(2,3,5))$ is the union of the above twelve sets.

4.11. **Conclusion.** Let $\mathbb{P}=\mathbb{P}(a,b,c)$ where a,b,c are pairwise relatively prime positive integers. To describe the affine rulings of $\mathbb{P}$, one proceeds as follows.

(1) Using 4.4–4.8, describe $\mathbb{T}_0(\mathbb{P})$ explicitely (4.9 and 4.10 are examples of this).
(2) Generate $\mathbb{T}(\mathbb{P})$ from its subset $\mathbb{T}_0(\mathbb{P})$. (Each element τ of $\mathbb{T}_0(\mathbb{P})$ determines a set $[\tau,\infty)$ which can be described explicitely by using 5.39 of [2]; by 5.22 of [2], $\mathbb{T}(\mathbb{P})$ is the union of the sets $[\tau,\infty)$ for $\tau\in\mathbb{T}_0(\mathbb{P})$.)
(3) For each $\tau=(m,T_1,T_2)\in\mathbb{T}(\mathbb{P})$, consider all blowings-up β of $\mathbb{F}_m$ according to (T_1,T_2) and all corresponding pairs (X_β,Λ_β) (defined by 3.5). Then $X_\beta\cong\mathbb{P}$ and the Λ_β are all affine rulings of $\mathbb{P}$.

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