A COUNTEREXAMPLE TO HILBERT'S FOURTEENTH PROBLEM IN DIMENSION FIVE

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Abstract. The kernel of a certain triangular derivation of the polynomial ring $k[x_1, x_2, x_3, x_4, x_5]$ is shown to be non-finitely generated over k (a field of charactersitic zero), thus giving a new counterexample to Hilbert's Fourteenth Problem, in the lowest dimension to date. (Math. Subject Class. 13A50, 13E15)

1. Introduction

In a recent paper of the second author [2], a counterexample to Hilbert's Fourteenth Problem is given in dimension six, in the form of the kernel of a triangular derivation of $k^{[6]}$, the polynomial ring in six variables over a field k of characteristic zero. This dimension-six example has its roots in Roberts' well-known counterexample in dimension seven [5], which can be realized as the kernel of a triangular derivation of $k^{[7]}$. Roberts' example was generalized in both [1] and [3] to produce triangular counterexamples in every dimension $n \geq 7$.

In the present work, we use a quotient of the dimension-six example to produce a remarkably simple counterexample in dimension five. Like its predecessors, this new example is the kernel of a triangular derivation of a polynomial ring, namely, $D : k[a, b, x, y, z] \to k[a, b, x, y, z]$,

$$D = a^2 \frac{\partial}{\partial x} + (ax + b)\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

(see Section 3). Thus, the question of finite generation remains open only in dimension four, since it is known that no counterexample can be found in dimension less than four (see [6]).

Preliminaries. In what follows, k is a field of characteristic zero. The term *derivation* will mean a k-derivation of the form $D : B \to B$, where B is an integral k-domain. D is *locally nilpotent* if, to each $f \in B$, there is an integer $n \geq 0$ such that $D^n f = 0$.

For an integral k-domain R, the symbol $R^{[n]}$ will denote the polynomial ring in n variables over R. If $B = R[X_1, ..., X_n] = R^{[n]}$, an

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R-derivation T of B is called *triangular* (with respect to $X_1, ..., X_n$) if $TX_1 \in R$, and $TX_i \in R[X_1, ..., X_{i-1}]$ for $i = 2, ..., n$. Note that triangular derivations are locally nilpotent.

The proof of Proposition 2.5 uses the following well-known fact: Suppose that $D : B \to B$ is a locally nilpotent derivation and that $\sigma \in B$ satisfies $D\sigma \neq 0$ and $D^2\sigma = 0$. Then $b \mapsto \sum_{n=0}^{\infty} \frac{D^n b}{n!} \left(-\frac{\sigma}{D\sigma}\right)^n$ defines a ring homomorphism $B \to (\ker D)_{D\sigma}$, where $(\ker D)_{D\sigma}$ is the localization of $\ker D$ with respect to $\{1, D\sigma, (D\sigma)^2, \ldots\}$.

2. Kernel Sequences

Let k be a field of characteristic zero. Consider a pair (B, D), where $B = \oplus_{i\in\mathbf{N}} B_i$ is an $\mathbf{N}$-graded k-domain and $D : B \to B$ is a homogeneous locally nilpotent derivation. Let $A = \ker D$.

We are interested in elements $\alpha \in A$ such that:

(NFG) $\ker(\tilde{D})$ is not finitely generated as a k-algebra, where $\tilde{D} : B[T] \to B[T]$ is the unique derivation extending D which maps T on α (T is an indeterminate over B).

When this is the case, we say that "(B, D, α) satisfies (NFG) over k".

Lemma 2.1. *Suppose that the grading of B satisfies $B_0 = k$ and let $\mathcal{M}[T]$ be the extension to $B[T]$ of the maximal ideal $\mathcal{M} = \oplus_{i>0} B_i$ of B. If $\alpha \in A \setminus D(B)$ is a homogeneous element and $\tilde{D} : B[T] \to B[T]$ is the unique derivation extending D and satisfying $\tilde{D}(T) = \alpha$, then*

(1) $\ker \tilde{D} \subseteq k + \mathcal{M}[T]$.
(2) *Let $\{\phi_n\}$ be an infinite sequence in $\ker \tilde{D}$ and let $b_n \in B$ be the leading coefficient of ϕ_n. If the sequence $\{\deg(b_n)\}$ is bounded above but $\{\deg_T(\phi_n)\}$ is not, then the k-algebra $\ker \tilde{D}$ is not finitely generated.*

Proof. Assertion (2) is an immediate consequence of (1), so we prove only (1).

Let $\delta = \deg(\alpha) - \deg(D)$ and, for each $d \in \mathbf{Z}$, define

$$B[T]_d = \sum_{i\in\mathbf{N}} B_{d-\delta i} T^i.$$

Then $B[T] = \oplus_{d\in\mathbf{Z}} B[T]_d$ is a $\mathbf{Z}$-grading of $B[T]$ satisfying:

- $B_i \subseteq B[T]_i$, for all $i \in \mathbf{N}$.
- If $\phi = \sum_i \phi_i T^i$ is a homogeneous element of $B[T]$ (with $\phi_i \in B$) then, for all $i \in \mathbf{N}$, ϕ_i is a homogeneous element of B.
- $\tilde{D}$ is homogeneous, so $\ker \tilde{D}$ is a homogeneous subring of $B[T]$.

Let ϕ be a homogeneous element of $\ker \tilde{D}$. Write $\phi = \sum_{i\in\mathbf{N}} \phi_i T^i$, $\phi_i \in B$, $\phi_i = 0$ for $i \gg 0$. Since $0 = \tilde{D}(\phi) = \phi^{(D)}(T) + \alpha\phi'(T)$,

$$D\phi_{i-1} = -i\alpha\phi_i, \quad \text{for all } i > 0. \tag{1}$$

Thus $\phi_i \notin k^*$ for all $i > 0$, for otherwise equation (1) would give $\alpha \in D(B)$, a contradiction; since ϕ_i is a homogeneous element of B, it must belong to $\mathcal{M}$. So

$$\phi_0 \in k + \mathcal{M} \quad \text{and} \quad \phi_i \in \mathcal{M} \text{ for all } i > 0.$$

This means that $\phi \in k + \mathcal{M}[T]$, so (1) is proved. □

Definition 2.2. By a *kernel-sequence* of (B, D), we mean a sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ of nonzero homogeneous elements of A for which there exists an integer $p > 0$ satisfying

$$a_{n+p} = a_n, \quad \text{for all } n \geq 1.$$

Such an integer p is called a *period* of $\mathbf{a}$. Clearly, if p is the minimal period then the set of all periods of $\mathbf{a}$ is $\{p, 2p, 3p, \dots\}$. If p is a period and $\alpha \in A$ satisfies $a_1 \cdots a_p = \alpha^p$, we call α a *p-average* of $\mathbf{a}$. The phrase "α is an average of $\mathbf{a}$" means that α is a p-average of $\mathbf{a}$ for some period p.

Remark 2.3. Assume that k contains all roots of unity and let $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ be a kernel-sequence of (B, D).

1. If α is an average of $\mathbf{a}$ then so is $\zeta\alpha$ for every root of unity ζ.
2. If α is an average of $\mathbf{a}$ then there exists a root of unity ζ such that, for every period p, $\zeta\alpha$ is a p-average of $\mathbf{a}$.
3. If α_1 and α_2 are averages of $\mathbf{a}$, then $\alpha_1 = \zeta\alpha_2$ for some root of unity ζ.

Definition 2.4. A kernel-sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ of (B, D) is said to be *effective* if the following equivalent conditions are satisfied:

1. There exists a sequence $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$ in B, other than the zero sequence, satisfying $Db_n = a_n b_{n-1}$ for all $n > 0$.
2. There exists a sequence $\mathbf{b}$ satisfying condition (1) and such that every term b_n is nonzero and homogeneous.

If $\mathbf{a}$ is an effective kernel-sequence of (B, D), then a sequence $\mathbf{b}$ satisfying condition (2) is called an *integral sequence of (B, D) belonging to $\mathbf{a}$*.

A kernel-sequence $\mathbf{a}$ of (B, D) is *strictly effective* if: (i) $\mathbf{a}$ is effective; (ii) $\mathbf{a}$ has an average α; and (iii) $\alpha \notin D(B)$.

Proposition 2.5. *Suppose that $B_0 = k$ and that (B, D) admits a strictly effective kernel-sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$. Let $\alpha \in A$ be an average of* $\mathbf{a}$. *Then (B, D, α) satisfies* (NFG) *over k.*

Proof. Let p be a period of $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ such that α is a p-average of $\mathbf{a}$. Let $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$ be an integral sequence belonging to $\mathbf{a}$ and let $\nu \geq 0$ be the unique integer such that $D^{\nu}(b_0) \neq 0$ and $D^{\nu+1}(b_0) = 0$. Define the integer $e = \nu + p - 1 \geq 0$.

Let $\tilde{D} : B[T] \to B[T]$ be the extension of D satisfying $\tilde{D}(T) = \alpha$. Consider the ring homomorphism $\xi : B[T] \to (\ker \tilde{D})_{\alpha}$ defined by

$$\xi(f) = \sum_{j \in \mathbf{N}} \frac{\tilde{D}^j(f)}{j!} \left(-\frac{T}{\alpha} \right)^j \qquad (f \in B[T]),$$

where $(\ker \tilde{D})_{\alpha}$ is the localization of $\ker \tilde{D}$ with respect to $\{1, \alpha, \alpha^2, \dots\}$. Let $\phi_n = \xi(\alpha^e b_n)$ for $n \geq 0$. In view of Lemma 2.1, it suffices to show that the following two claims hold for all $n \geq 0$:

(1) $\phi_n \in \ker \tilde{D}$;

(2) the leading term of ϕ_n is $\dfrac{(-1)^{n+\nu}}{(n+\nu)!} \gamma_n T^{n+\nu}$, where $\{\gamma_n\}_{n=0}^{\infty}$ is a sequence in $A \setminus \{0\}$ with finite image.

We have

$$D^j(b_n) = a_n \dots a_{n-j+1} b_{n-j} \quad (0 \leq j \leq n)$$

and

$$D^{n+i}(b_n) = a_n \dots a_1 D^i(b_0) \quad (i \geq 0),$$

so

$$\begin{aligned} \phi_n &= \sum_{j \in \mathbf{N}} \frac{(-1)^j}{j!} \frac{\alpha^e D^j(b_n)}{\alpha^j} T^j \\ &= \sum_{j=0}^{n} \frac{(-1)^j}{j!} \left(\frac{\alpha^{p-1} a_n \dots a_{n-j+1}}{\alpha^j} \right) \alpha^{\nu} b_{n-j} T^j \\ &+ \sum_{i=1}^{\nu} \frac{(-1)^{n+i}}{(n+i)!} \left(\frac{\alpha^{p-1} a_n \dots a_1}{\alpha^n} \right) \alpha^{\nu-i} D^i(b_0) T^{n+i}. \end{aligned}$$

Since the product of any p consecutive terms of the sequence $\{a_n\}_{n=1}^{\infty}$ gives α^p,

$$S = \{ \frac{\alpha^{p-1} a_n \dots a_{n-j+1}}{\alpha^j} \mid (j, n) \in \mathbf{N}^2 \text{ and } j \leq n \}$$

is a finite subset of $A \setminus \{0\}$. In particular, $S \subset A$ implies that $\phi_n \in B[T] \cap (\ker \tilde{D})_{\alpha} = \ker \tilde{D}$, so (1) holds. Since the leading term of ϕ_n is

$$\frac{(-1)^{n+\nu}}{(n+\nu)!} \left(\frac{\alpha^{p-1} a_n \dots a_1}{\alpha^n} \right) D^{\nu}(b_0) T^{n+\nu},$$

and S is a finite subset of A, (2) holds. $\square$

Corollary 2.6. *Suppose that B is noetherian or factorial, and that (B, D) admits a strictly effective kernel-sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ with average $\alpha \in A$. Suppose that q is a homogeneous prime element of B satisfying: $q \in A$, and*

(1) *$q \nmid a_n$ for all $n \geq 1$;*
(2) *$\alpha \notin D(B) + qB$.*

Then $\overline{\mathbf{a}} = \{\overline{a}_n\}$ is a strictly effective kernel-sequence of $(\overline{B}, \overline{D})$ with average $\overline{\alpha}$, where $\overline{B} = B/qB$, $\overline{D} : \overline{B} \to \overline{B}$ is the induced derivation, $\overline{a}_n = a_n + qB$ and $\overline{\alpha} = \alpha + qB$. If, moreover,

(3) *$qB \cap B_0$ is a maximal ideal of B_0,*

then $(\overline{B}, \overline{D}, \overline{\alpha})$ satisfies (NFG) *over $\overline{B}_0 = B_0/(qB \cap B_0)$.*

Proof. It is immediate that $\overline{\mathbf{a}}$ is a kernel-sequence of $(\overline{B}, \overline{D})$ with average $\overline{\alpha}$.

Consider an integral sequence $\mathbf{b} = \{b_n\}_{n=0}^{\infty}$ of (B, D) belonging to $\mathbf{a}$. Observe that, if $q \mid b_n$ for all $n \geq 0$, then $\left\{\frac{b_n}{q}\right\}_{n=0}^{\infty}$ is another integral sequence belonging to $\mathbf{a}$; since B is noetherian or factorial we have $\bigcap_{n=1}^{\infty} q^n B = \{0\}$, so we may assume that some b_n is not divisible by q. Then $\overline{\mathbf{b}} = \{b_n + qB\}_{n=0}^{\infty}$ satisfies condition (1) of Definition 2.4, so $\overline{\mathbf{a}}$ is effective. Since we assumed that $\alpha \notin D(B) + qB$, we have $\overline{\alpha} \notin \overline{D}(\overline{B})$, i.e., $\overline{\mathbf{a}}$ is strictly effective. The last assertion follows from Proposition 2.5 because $\overline{B}_0$ is a field. □

3. A Non-Finitely Generated Kernel in Dimension Five

The following six-dimensional counterexample to Hilbert's Fourteenth Problem was given in [2]. We interpret it in terms of the concepts of Section 2.

Example 3.1. Let $R = k[a, b, s, t, u] = k^{[5]}$, with grading $R = \oplus_{i \in \mathbf{N}} R_i$ determined by $R_0 = k$, $a, b, s \in R_3$, $t \in R_6$ and $u \in R_9$. Define $d : R \to R$ by

$$d = a\frac{\partial}{\partial s} + bs\frac{\partial}{\partial t} + bt\frac{\partial}{\partial u}.$$

Then d is locally nilpotent and homogeneous of degree 0. Consider the kernel-sequence $\mathbf{a} = \{a_n\}_{n=1}^{\infty} = \{a, b, ab, a, b, ab, \dots\}$ of (R, d). Section 2 of [2] proves that $\mathbf{a}$ is effective (the proof of this is non-trivial!). However, $\mathbf{a}$ is not strictly effective because it does not have an average ($(a^2b^2)^{1/3} \notin R$).

Consider the overring $B = k[x, y, s, t, u] = k^{[5]}$ of R, where $x^3 = a$ and $y^3 = b$, with grading $B = \oplus_{i \in \mathbf{N}} B_i$ determined by $B_0 = k$, $x, y \in B_1$,

$s \in B_3$, $t \in B_6$ and $u \in B_9$. Define $D : B \to B$ by

$$D = x^3 \frac{\partial}{\partial s} + y^3 s \frac{\partial}{\partial t} + y^3 t \frac{\partial}{\partial u}.$$

Then D is locally nilpotent and homogeneous of degree 0. Since D extends d, $\mathbf{a}$ is an effective kernel-sequence of (B, D). Clearly, $\alpha = x^2 y^2$ is an average of $\mathbf{a}$; since $D(B)$ is contained in the ideal (x^3, y^3) of B, $\alpha \notin D(B)$, i.e., $\mathbf{a}$ is a strictly effective kernel-sequence of (B, D). Consequently, if $\mathcal{R} = B[v] = B^{[1]} = k^{[6]}$ and $\mathcal{D} : \mathcal{R} \to \mathcal{R}$ is the unique extension of D satisfying $\mathcal{D}(v) = x^2 y^2$, then Proposition 2.5 implies that $\ker \mathcal{D}$ is not finitely generated as a k-algebra.

Using Corollary 2.6, we deduce a counterexample in dimension five:

Example 3.2. Let B, D, $\mathbf{a}$ and α be as in Example 3.1 and consider a different grading $B = \oplus_{i \in \mathbf{N}} B_i$, where $B_0 = k[y]$, $x \in B_1$, $s, t, u \in B_3$. Then D is homogeneous of degree 0 and $\mathbf{a}$ is a strictly effective kernel sequence of (B, D) with average α. Note that $q = y - 1$ is a homogeneous prime element of B satisfying $Dq = 0$ and $q \not| a_n$ for all n. Let $\overline{B} = B/qB = k[x, s, t, u] = k^{[4]}$ then the induced derivation $\overline{D} : \overline{B} \to \overline{B}$ is:

$$\overline{D} = x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u}.$$

Since $\overline{\alpha} = x^2$ does not belong to the ideal (x^3, s, t) of $\overline{B}$, we have $\overline{\alpha} \notin \overline{D}(\overline{B})$ (or equivalently condition (2) of Corollary 2.6 holds). By 2.6, the kernel of $T : \overline{B}[v] \to \overline{B}[v]$ is not finitely generated as a k-algebra, where T is the extension of $\overline{D}$ which maps v to α. Explicitly, $T : k[x, s, t, u, v] \to k[x, s, t, u, v]$ is given by:

$$T = x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v} \ . \tag{2}$$

Observe that $T(s - xv) = 0$ and that $k[x, s, t, u, v] = k[x, s - xv, t, u, v]$ ($s - xv$ is a variable). Let σ be the polynomial automorphism of $k[x, s, t, u, v]$ fixing x, t, u, v and mapping $s \mapsto (s + xv)$. Then

$$\sigma T \sigma^{-1} = x^2 \frac{\partial}{\partial v} + (xv + s) \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} \ ,$$

which gives the following:

Theorem 3.3. *Let D be the triangular derivation of $k[a, b, x, y, z] = k^{[5]}$ defined by*

$$D = a^2 \frac{\partial}{\partial x} + (ax + b) \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \ . \tag{3}$$

Then the kernel of D is not finitely generated as a k-algebra.

Remarks

(1) By (2), there exists a triangular monomial derivation of $k^{[5]}$ whose kernel is not a finitely generated k-algebra. (See [4].)
(2) Let $R = k^{[2]}$ and let X, Y, Z be indeterminates over R. By (3), there exists a triangular R-derivation of $R[X, Y, Z]$ whose kernel is not a finitely generated R-algebra.

References

[1] A. van den Essen, T. Janssen, *Kernels of elementary derivations*, Univ. Nijmegen Report No. 9548 (Nijmegen, The Netherlands) 1995

[2] G. Freudenburg, *A counterexample to Hilbert's Fourteenth Problem in dimension six*, Transformation Groups (to appear)

[3] H. Kojima, M. Miyanishi, *On Roberts' counterexample to the fourteenth problem of Hilbert*, J. Pure Appl. Algebra **122** (1997) 277-292

[4] S. Maubach, *Triangular monomial derivations on $k[X_1, X_2, X_3, X_4]$ have kernel generated by at most four elements*, Univ. Nijmegen Report No. 9822 (Nijmegen, The Netherlands) 1998

[5] P. Roberts, *An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's Fourteenth Problem*, J. Algebra **132** (1990) 461-473

[6] O. Zariski, *Interpretations algebrico-geometriques du quatorzieme probleme de Hilbert*, Bull. Sci. Math. **78** (1954) 155-168

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