ON LOCALLY NILPOTENT DERIVATIONS OF $\mathbf{k}[X_1, X_2, Y]/(\varphi(Y) - X_1X_2)$

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ABSTRACT. We investigate the locally nilpotent derivations of the $\mathbf{k}$-algebra

$$B = \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2),$$

where $\mathbf{k}$ is a field of characteristic zero, X_1, X_2, Y are indeterminates over $\mathbf{k}$ and $\varphi \in \mathbf{k}[Y]$ is a non-constant polynomial. We prove that a certain subgroup of $\mathrm{Aut}_{\mathbf{k}}(B)$ acts transivitely on the set of kernels of nonzero locally nilpotent derivations of B.

INTRODUCTION

Given a field $\mathbf{k}$ of characteristic zero and a nonconstant polynomial $\varphi \in \mathbf{k}[Y]$, consider the $\mathbf{k}$-algebra

$$B = \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2),$$

where X_1, X_2, Y are indeterminates over $\mathbf{k}$. The main result of this paper is the Transitivity Theorem (see 2.3), which asserts that a certain subgroup G of $\mathrm{Aut}_{\mathbf{k}}(B)$ acts transitively on the set of kernels of locally nilpotent derivations $0 \neq D : B \to B$.

Note that, in the special case where $\deg \varphi = 1$, the Transitivity Theorem gives the well-known result [6] of R. Rentschler (see 2.4).

The Transitivity Theorem will be used, in the subsequent paper [2], to describe the locally nilpotent derivations of B and of other rings. In fact this paper is one step in a bigger project: Understanding the locally nilpotent derivations of the polynomial ring $\mathbf{k}[X, Y, Z]$. In [3], Freudenburg described a way to modify a given locally nilpotent derivation of $\mathbf{k}[X, Y, Z]$ so as to obtain another one; he called this process the "local slice construction" and asked whether every locally nilpotent derivation D of $\mathbf{k}[X, Y, Z]$ could be constructed, starting from $\partial/\partial X$, by finitely many applications of the local slice construction. Note that we show in [1] that this is indeed the case whenever D is w-homogeneous, with $w(X)$, $w(Y)$, $w(Z)$ positive integers. It turns out that describing the locally nilpotent derivations of the two-dimensional ring B improves our understanding of the local slice construction in $\mathbf{k}[X, Y, Z]$, and in other rings as well, and this is our motivation for writing the present work. The connection between B and the local slice construction is not treated in this paper (it will be in [2]). Let us

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only mention that, for that application, we need the results of the present paper to be valid over a ground field $\mathbf{k}$ which is not necessarely algebraically closed.

In writing this paper, our aim was not to give the shortest possible proof of the Transitivity Theorem, but rather to develop some tools that are needed in [2]. This is the case, in particular, for the bidegree function, whose properties are carefully established in paragraph 2.7.

Section 1 gathers the definitions and basic facts which we need. The Transitivity Theorem is proved in the second section. The purpose of the third section is to prove Lemma 3.1, which is used in the proof of 2.8. Note that section 3 is essentially an exercise in the theory of affine plane curves; that section is called an appendix to reflect the fact that its topic and methods are independent of the rest of the text.

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Paper [4] gives generators for the group of automorphisms of B and [5] investigates automorphisms and locally nilpotent derivations of $\mathbf{k}[X_1, X_2, Y]/(\varphi(Y) - X_1^k X_2)$ where $k > 1$ (but we need the case $k = 1$). Although our results do not seem to follow from [4] and [5] as corollaries (at least we could not find how to do that), the two papers were still very helpful. In particular, the homogeneization technique used in 2.7.3 is borrowed from [5].

I also wish to mention that I recently received a note, from Makar-Limanov, which sketches a proof of the Transitivity Theorem.

1. Definitions and basic facts

1.1. **Conventions.**

- Throughout this paper, $\mathbf{k}$ denotes an arbitrary field of characteristic zero.
- The word *ring* means

  commutative ring with a unity element and of characteristic zero.

  The same convention applies to the words *algebra* and *field*. We stress that **all fields and rings are tacitly assumed to be of characteristic zero**.
- The group of units of a ring A is denoted A^*.
- If A is a subring of a ring B and $r \in \mathbb{N}$, the notation $B = A^{[r]}$ means that B is A-isomorphic to the polynomial ring in r variables over A. (So, $B = A^{[r]}$ and $B' = A^{[r]}$ do not imply that $B = B'$.)
- If L/K is a field extension and $r \in \mathbb{N}$, the notation $L = K^{(r)}$ means that L is a purely transcendental extension of K, of transcendence degree r.
- If A is an integral domain then $\operatorname{Frac} A$ is its field of fractions. If $A \subseteq B$ are domains then $\operatorname{trdeg}_A(B)$ is the transcendence degree of $\operatorname{Frac} B$ over $\operatorname{Frac} A$.

1.2. An *inert subring* of a domain B is a subring A of B satisfying:

$$\forall\, x, y \in B, \qquad xy \in A \setminus \{0\} \implies x, y \in A.$$

If A is an inert subring of B then the following hold.

(1) $A^* = B^*$.
(2) A is algebraically closed in B.
(3) For each $a \in A$, a is irreducible in A if and only if it is irreducible in B.
(4) If B is a UFD then so is A.
(5) $S^{-1}A$ is an inert subring of $S^{-1}B$, for any multiplicatively closed set $S \subseteq A \setminus \{0\}$.

1.3. A *locally nilpotent derivation* of a domain B is a derivation $D : B \to B$ satisfying $\forall_{x \in B}\ \exists_{e \in \mathbb{N}}\ D^e(x) = 0$.

If $D : B \to B$ is any derivation, we write

$$\ker D = \{\, x \in B \mid D(x) = 0 \,\},$$

which is a subring of B. We use the following notations:

$$\begin{aligned} \text{LND}(B) &= \text{set of } \textbf{nonzero} \text{ locally nilpotent derivations of } B \\ \text{KLND}(B) &= \{\, \ker D \mid D \in \text{LND}(B) \,\}. \end{aligned}$$

If E is a subset of B,

$$\begin{aligned} \text{LND}_E(B) &= \{\, D \in \text{LND}(B) \mid D(E) = \{0\} \} \\ \text{KLND}_E(B) &= \{\, \ker D \mid D \in \text{LND}_E(B) \,\}. \end{aligned}$$

1.4. **Basic properties of locally nilpotent derivations.** Let B be an integral domain, let $D : B \to B$ be a nonzero derivation of B, and let $A = \ker D$. The following facts are well-known.

(1) If D is locally nilpotent then A is an inert subring of B. In particular, $B^* = A^*$ and if B is a UFD then so is A. Note, also, that if K is any field contained in B then $K \subseteq A$, so D is a K-derivation.
(2) Let S be a multiplicatively closed subset of $B \setminus \{0\}$, and consider the derivation $S^{-1}D : S^{-1}B \to S^{-1}B$. Then:
   (a) $S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subset A$.
   (b) If $S \subset A$ then $\ker S^{-1}D = S^{-1}A$.
(3) Assume that $\mathbb{Q} \subseteq B$. If D is locally nilpotent, and if $s \in B$ satisfies $D(s) \in B^*$, then $B = A[s] = A^{[1]}$.
(4) Assume that $\mathbb{Q} \subseteq B$. If D is locally nilpotent, choose any $s \in B$ such that $Ds \neq 0$ and $D^2 s = 0$ (such an s exists, and is called a *preslice* of D), and let $S = \{1, Ds, (Ds)^2, \dots\} \subset A$. Then $S^{-1}D(s) \in (S^{-1}B)^*$ so, by (3), $S^{-1}B = (S^{-1}A)[s] = (S^{-1}A)^{[1]}$.
(5) If D is locally nilpotent, let $S = A \setminus \{0\}$, then (4) implies $S^{-1}B = (\operatorname{Frac} A)^{[1]}$.

(6) Let $a \in B \setminus \{0\}$. The derivation $aD : B \to B$ is locally nilpotent if and only if D is locally nilpotent and $a \in A$.

We stress the following consequence of part (5) of 1.4:

1.5. *If B is a domain and $A \in \text{KLND}(B)$ then* $\operatorname{trdeg}_A B = 1$.

1.6. If $B = \oplus_{i\in\mathbb{Z}} B_i$ is a $\mathbb{Z}$-graded domain, a derivation $D : B \to B$ is *homogeneous* if there exists $d \in \mathbb{Z}$ satisfying $\forall_{i\in\mathbb{N}}\ D(B_i) \subseteq B_{i+d}$. If this is the case then $\ker D$ is a *homogeneous subring* of B, i.e., $\ker D = \oplus_{i\in\mathbb{Z}}(\ker(D) \cap B_i)$.

2. The Transitivity Theorem

Throughout this section, $B = \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2)$ where X_1, X_2, Y are indeterminates over $\mathbf{k}$ and $\varphi \in \mathbf{k}[Y]$ is a polynomial of degree $n \geq 1$. We also write $B = \mathbf{k}[x_1, x_2, y]$, where x_1, x_2, y are the images of X_1, X_2, Y under the canonical epimorphism $\mathbf{k}[X_1, X_2, Y] \to B$.

2.1. It is easy to see that, for each $i \in \{1, 2\}$, there is a unique locally nilpotent derivation $D_i : B \to B$ satisfying $D_i(x_i) = 0$ and $D_i(y) = x_i$. Moreover, $\ker D_i = \mathbf{k}[x_i]$, so $\mathbf{k}[x_1]$ and $\mathbf{k}[x_2]$ are two elements of $\text{KLND}(B)$.

2.2. **Notations.** (See [4].)

- Define $\tau \in \operatorname{Aut}_{\mathbf{k}}(B)$ by $\tau(x_1) = x_2$, $\tau(x_2) = x_1$ and $\tau(y) = y$.
- For each $f \in \mathbf{k}[x_1]$, define $\Delta_f \in \operatorname{Aut}_{\mathbf{k}}(B)$ by $\Delta_f(x_1) = x_1$ and $\Delta_f(y) = y + x_1 f$. (Then $\Delta_f(x_2) = x_1^{-1}\varphi(y + x_1 f)$.)
- Let G be the subgroup of $\operatorname{Aut}_{\mathbf{k}}(B)$ generated by $\{\tau\} \cup \{\Delta_f \mid f \in \mathbf{k}[x_1]\}$. (Perhaps one should call G the "tame subgroup" of $\operatorname{Aut}_{\mathbf{k}}(B)$, see part (2) of 2.4.)

The assignment $(\alpha, A) \longmapsto \alpha(A)$, where $\alpha \in \operatorname{Aut}_{\mathbf{k}}(B)$ and $A \in \text{KLND}(B)$, is a left-action of the group $\operatorname{Aut}_{\mathbf{k}}(B)$ on the set $\text{KLND}(B)$. We restrict this action to the subgroup G of $\operatorname{Aut}_{\mathbf{k}}(B)$ defined in 2.2. Then:

2.3. **Transitivity Theorem.** *The action of G on $\text{KLND}(B)$ is transitive.*

2.4. **Remarks.**

(1) For each $f \in \mathbf{k}[x_1]$, Δ_f is an exponential automorphism of B. In fact, $\Delta_f = e^{fD_1}$ where D_1 is as in 2.1. Also, if $f, g \in \mathbf{k}[x_1]$ then $\Delta_f \circ \Delta_g = \Delta_{f+g}$.

(2) Consider the special case where $\varphi(Y) = Y$. Then $B = \mathbf{k}[x_1, x_2] = \mathbf{k}^{[2]}$ and G is the tame subgroup of $\operatorname{Aut}_{\mathbf{k}}(B)$, so the well-known result [6] of Rentschler is a corollary of 2.3.

The proof of the Transitivity Theorem is divided into several parts. Paragraphs 2.5–2.7 cover some preliminaries; then we prove Proposition 2.8, which is a homogeneous version of the theorem; finally we prove the theorem.

Preliminaries to the proof of 2.3
(p,q)-gradings, filtrations and bidegree

Note that if $\varphi, \psi \in \mathbf{k}[Y]$ are associates, then

$$\mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2) \cong \mathbf{k}[X_1, X_2, Y]/(\psi - X_1X_2).$$

Thus we will assume throughout that $\varphi \in \mathbf{k}[Y]$ is a monic polynomial of degree $n \geq 1$.

2.5**.** It is clear that B is a subalgebra of $R = \mathbf{k}[x_1, 1/x_1, y]$. Given a pair (p,q) of relatively prime positive integers, we shall now define a graded ring $B^{p,q}$ which is also a subalgebra of R, but which is usually distinct from B.

As a $\mathbf{k}$-algebra, we set $B^{p,q} = \mathbf{k}[x_1, y^n/x_1, y]$, i.e.,

$$B^{p,q} \cong \mathbf{k}[X_1, X_2, Y]/(Y^n - X_1X_2).$$

In order to define the grading of $B^{p,q}$, consider first the $\mathbb{Z}$-grading of R determined by the conditions

$$R = \bigoplus_{d\in\mathbb{Z}} R_d, \qquad \mathbf{k} \subseteq R_0, \quad x_1 \in R_p, \quad y \in R_q.$$

Then note that $B^{p,q}$ is a homogeneous subring of R, i.e., if we set

$$B^{p,q}_d = R_d \cap B^{p,q}$$

for each $d \in \mathbb{Z}$, then

$$B^{p,q} = \bigoplus_{d\in\mathbb{Z}} B^{p,q}_d.$$

The ring $B^{p,q}$ is always equipped with this grading. Note:

(1) The set $\left\{ x_1^i y^j \mid (i,j) \in \mathbb{Z}\times\mathbb{N} \text{ and } ni + j \geq 0 \right\}$ is a basis of $B^{p,q}$ over $\mathbf{k}$.

We also define a map $H_{p,q} : B \to B^{p,q}$ as follows. Let $f \in B$. If $f = 0$, define $H_{p,q}(f) = 0$. If $f \neq 0$, regard f as an element of the graded ring $R = \oplus_{d\in\mathbb{Z}} R_d$, write

$$f = \sum_{d\in\mathbb{Z}} f_d \qquad (f_d \in R_d)$$

and define $H_{p,q}(f) = f_M$ where $M = \max\left\{ d \in \mathbb{Z} \mid f_d \neq 0 \right\}$. The fact that

$$H_{p,q}(f) \in B^{p,q} \tag{2}$$

can be proved directly, but we will obtain it as a trivial consequence of 2.6, below.

Note that $H_{p,q}(f) = 0$ iff $f = 0$ and that $H_{p,q}(fg) = H_{p,q}(f)H_{p,q}(g)$ for all $f, g \in B$.

2.6. **Associated graded ring.** Let B, R and (p,q) be as in 2.5 and let $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ denote the degree function associated with the grading $\oplus_{d\in\mathbb{Z}} R_d$ (i.e., $\deg(x_1^i y^j) = pi+qj$). For each $d \in \mathbb{Z}$, let

$$B_{(d)} = \{ f \in B \mid \deg(f) \le d \},$$

then

$$\cdots \subseteq B_{(d-1)} \subseteq B_{(d)} \subseteq B_{(d+1)} \subseteq \cdots$$

is a $\mathbb{Z}$-filtration of B. Consider the associated graded ring and its "symbol map"

$$\mathrm{gr}(B) = \bigoplus_{d\in\mathbb{Z}} B_{(d)}/B_{(d-1)} \qquad \text{and} \qquad B \xrightarrow{\mathrm{gr}} \mathrm{gr}(B).$$

Since the degree function of R satisfies $\deg(fg) = \deg(f) + \deg(g)$ for all $f, g \in R$, it follows that $\mathrm{gr}(B)$ is an integral domain and that the symbol map preserves multiplication. Write

$$\bar{x}_1 = \mathrm{gr}(x_1), \quad \bar{x}_2 = \mathrm{gr}(x_2), \quad \bar{y} = \mathrm{gr}(y)$$

and consider the subalgebra $\mathbf{k}[\bar{x}_1, \bar{x}_2, \bar{y}]$ of $\mathrm{gr}(B)$.

2.6.1. *The* $\mathbf{k}$*-homomorphism* $\pi : \mathbf{k}[X_1, X_2, Y] \to \mathrm{gr}(B)$ *defined by*

$$\pi(X_1) = \bar{x}_1, \quad \pi(X_2) = \bar{x}_2, \quad \pi(Y) = \bar{y}$$

is surjective and has kernel $(Y^n - X_1X_2)\mathbf{k}[X_1, X_2, Y]$.

Proof. The proof of surjectivity is straightforward and is left to the reader. Since $x_1x_2 = \varphi(y)$ holds (in B), and since by assumption $\varphi \in \mathbf{k}[Y]$ is monic of degree n, we have $\deg(x_1x_2) = \deg(y^n) > \deg(y^n - x_1x_2)$ and consequently $\bar{x}_1\bar{x}_2 = \bar{y}^n$ in $\mathrm{gr}(B)$. Thus the height one prime ideal $(Y^n - X_1X_2)$ of $\mathbf{k}[X_1, X_2, Y]$ is included in $\ker(\pi)$. We know that $\mathrm{gr}(B)$ is a domain and (since π is surjective) a finitely generated $\mathbf{k}$-algebra, so we will be done if we can show that $\bar{x}_1$, $\bar{y}$ are algebraically independent. Consider $F \in \mathbf{k}[X_1, Y] \setminus \{0\}$ and write $F = \sum_{(i,j)\in S} a_{ij} X_1^i Y^j$ where $S \neq \varnothing$ is a finite subset of $\mathbb{N}^2$ and $a_{ij} \in \mathbf{k}^*$ for all $(i,j) \in S$. It suffices to prove that $F(\bar{x}_1, \bar{y}) \neq 0$; moreover, we may assume that F is homogeneous in the sense that $pi+qj$ is the same number $d \in \mathbb{N}$ for all $(i,j) \in S$. Then by definition of the degree function of R we have

$$\deg\Big(\sum_{(i,j)\in S} a_{ij} x_1^i y^j \Big) = \max\{ pi + qj \mid (i,j) \in S \} = d,$$

so, in B, $\sum_{(i,j)\in S} a_{ij} x_1^i y^j$ has the same degree as each one of its terms $a_{ij} x_1^i y^j$. This condition implies that

$$\mathrm{gr}\,\big(\sum_{(i,j)\in S} a_{ij} x_1^i y^j \big) = \sum_{(i,j)\in S} \mathrm{gr}(a_{ij} x_1^i y^j),$$

where the right hand side is equal to $F(\bar{x}_1, \bar{y})$ because the symbol map preserves multiplication, and where the left hand side is nonzero because $\sum_{(i,j)\in S} a_{ij} x_1^i y^j$ is a nonzero element of B ($x_1, y \in B$ are algebraically independent). So we are done. $\square$

The above fact allows us to identify the graded ring $\mathrm{gr}(B)$ with the graded ring $B^{p,q}$ defined in 2.5. Moreover, under that identification, the map $\mathrm{gr} : B \to \mathrm{gr}(B)$ corresponds to $H_{p,q} : B \to B^{p,q}$ (which proves the claim (2), in 2.5).

2.6.2. *Every* $\mathbf{k}$*-derivation* $D : B \to B$ *satisfies*

$$\sup \{ \deg(Df) - \deg(f) \mid f \in B \setminus \{0\} \} < \infty.$$

Proof. Let $D : B \to B$ be a $\mathbf{k}$-derivation. Since R is equal to the localization B_{x_1}, D extends to a $\mathbf{k}$-derivation $R \to R$. Since R is a graded domain and a finitely generated $\mathbf{k}$-algebra, it is well-known that every $\mathbf{k}$-derivation $\partial : R \to R$ satisfies

$$\sup \{ \deg(\partial f) - \deg(f) \mid f \in R \setminus \{0\} \} < \infty.$$

The desired result follows. □

Consequently, any nonzero $\mathbf{k}$-derivation $D : B \to B$ determines a nonzero homogeneous $\mathbf{k}$-derivation $\mathrm{gr}(D) : \mathrm{gr}(B) \to \mathrm{gr}(B)$. Indeed, let

$$\delta = \sup \{ \deg(Df) - \deg(f) \mid f \in B \setminus \{0\} \} \in \mathbb{Z}.$$

Then, for each $i \in \mathbb{Z}$, D maps $B_{(i)}$ into $B_{(i+\delta)}$ and consequently there is an induced linear map $D_i : B_{(i)}/B_{(i-1)} \to B_{(i+\delta)}/B_{(i+\delta-1)}$. These define a homogeneous linear map $0 \neq \mathrm{gr}(D) : \mathrm{gr}(B) \to \mathrm{gr}(B)$ of degree δ which is easily seen to be a $\mathbf{k}$-derivation. Given $f \in B \setminus \{0\}$, we have

$$\mathrm{gr}(D)(\mathrm{gr}\, f) = \begin{cases} \mathrm{gr}(Df), & \text{if } \deg(Df) = \delta + \deg(f), \\ 0, & \text{if } \deg(Df) < \delta + \deg(f) \end{cases}$$

and it follows that $\mathrm{gr}(D)$ is locally nilpotent whenever D is, and that the symbol map $B \xrightarrow{\mathrm{gr}} \mathrm{gr}(B)$ maps $\ker(D) \setminus \{0\}$ into $\ker(\mathrm{gr}\, D) \setminus \{0\}$. We may rephrase these facts as follows:

2.6.3. *Each nonzero locally nilpotent derivation* $D : B \to B$ *determines a* ***nonzero homogeneous*** *locally nilpotent derivation* $D_{p,q} : B^{p,q} \to B^{p,q}$*. Moreover,* $H_{p,q} : B \to B^{p,q}$ *maps* $\ker(D) \setminus \{0\}$ *into* $\ker(D_{p,q}) \setminus \{0\}$*.*

2.7. **Bidegree.** Recall that B is a subalgebra of $R = \mathbf{k}[x_1, 1/x_1, y]$.

Given $f \in R$, we may write

$$f = \sum_{(i,j) \in \mathbb{Z} \times \mathbb{N}} \lambda_{i\,j} x_1^i y^j \qquad (\lambda_{i\,j} \in \mathbf{k})$$

and define the finite set

$$\mathrm{supp}(f) = \{ (i,j) \in \mathbb{Z} \times \mathbb{N} \mid \lambda_{i\,j} \neq 0 \}.$$

Then we define the "bidegree" map

$$\begin{array}{rccc} \mathrm{bideg} : & R & \longrightarrow & \mathbb{N} \times \mathbb{N} \\ & f & \longmapsto & (a, b) \end{array}$$

by declaring that a and b are the following integers:

$$a = \max\big[\{0\} \cup \{\, i \in \mathbb{N} \mid \ (i,0) \in \operatorname{supp}(f) \}\,\big]$$
$$b = \max\big[\{0\} \cup \{\, j \in \mathbb{N} \mid \ j(-1,n) \in \operatorname{supp}(f) \}\,\big].$$

Warning. If $f, g \in R$, then $\operatorname{bideg}(fg)$ is not always equal to $\operatorname{bideg} f + \operatorname{bideg} g$: Let $f = (x_1^{-2} + 1)y^n \in R$, then $\operatorname{bideg} f = (0,0)$ and $\operatorname{bideg}(f^2) = (0,2)$. So the properties of the bidegree function require careful proofs.

Note that there is an important case where the formula $\operatorname{bideg}(f^N) = N \operatorname{bideg} f$ is valid: See 2.7.4.

2.7.1. *If $P \in \mathbf{k}[T] \setminus \{0\}$ has degree d, then*

$$\operatorname{bideg} P(x_1) = (d,0) \quad \textit{and} \quad \operatorname{bideg} P(x_2) = (0,d).$$

Proof. Left to the reader. The second equality requires a little bit of work. □

2.7.2. *Let $f \in R$ and $(a,b) = \operatorname{bideg} f$. Then:*

$$a > 0 \implies (a,0) \in \operatorname{supp} f \quad \textit{and} \quad b > 0 \implies (-b, bn) \in \operatorname{supp} f.$$

Proof. Trivial. □

Given $f \in R$, let $C(f)$ be the unique subset of $\mathbb{R}^2$ which is closed, convex and has boundary $H_a \cup E \cup H_b$, where $(a,b) = \operatorname{bideg}(f)$ and:

$$E = \text{line segment joining } (-b, bn) \text{ and } (a,0),$$
$$H_a = \text{horizontal half line } \{\,(s,0) \mid s \le a\,\},$$
$$H_b = \text{horizontal half line } \{\,(s,bn) \mid s \le -b\,\}.$$

Note in particular:

(3) $$\forall\, (i,j) \in \operatorname{supp}(f), \quad (j = 0 \text{ or } ni + j = 0) \implies (i,j) \in C(f).$$

A better statement is true in the special case where $f \in A \in \text{KLND}(B)$:

2.7.3. **Lemma.** *If $f \in A$ for some $A \in \text{KLND}(B)$, then $\operatorname{supp}(f) \subset C(f)$.*

Proof. Suppose the contrary and consider a point $P = (i_0, j_0)$ in $\operatorname{supp}(f) \setminus C(f)$. Let $D \in \text{LND}(B)$ be such that $\ker D = A$.

Because $\operatorname{supp}(f)$ is a finite set, only finitely many $r \in \mathbb{Q}$ satisfy:

(4) *Some line of slope r contains more than one point of* $\operatorname{supp}(f)$.

On the other hand, the shape of $C(f)$ and the fact that P is is a point of

$$\big\{\,(s,t) \in \mathbb{R}^2 \;\big|\; t \ge 0\big\} \setminus C(f)$$

make it clear that there exist infinitely many $r \in \mathbb{Q}$ such that

(5) *$r < 0$ and the line going through P and of slope r is disjoint from $C(f)$.*

So we may choose $r \in \mathbb{Q}$ satisfying (5) but not (4). Let p, q be relatively prime positive integers such that $r = -p/q$ and let $\deg : \mathbf{k}[x_1, 1/x_1, y] \to \mathbb{Z} \cup \{-\infty\}$ be the degree function given by $\deg(x_1) = p$ and $\deg(y) = q$ (as in 2.6). Then

$$\forall\, (i, j) \in \mathbb{Z}^2 \cap C(f) \qquad \deg(x_1^i y^j) < \deg(x_1^{i_0} y^{j_0}). \tag{6}$$

Consider $H_{p,q} : B \to B^{p,q}$ and $D_{p,q} : B^{p,q} \to B^{p,q}$ (see 2.5, 2.6) and recall that $D_{p,q}$ is locally nilpotent **and nonzero**. Let $h = H_{p,q}(f)$, then $h \in \ker(D_{p,q}) \setminus \{0\}$ by 2.6.3. Since $r = -p/q$ does not satisfy (4), $\mathrm{supp}(h)$ is a single point, so $h = \mu x_1^{i_1} y^{j_1}$ for some $\mu \in \mathbf{k}^*$ and $(i_1, j_1) \in \mathrm{supp}(f)$. Since $\deg(x_1^{i_1} y^{j_1}) = \deg(h) = \deg(f) \geq \deg(x_1^{i_0} y^{j_0})$, (6) implies that $(i_1, j_1) \notin C(f)$, so (3) gives

$$j_1 > 0 \quad \text{and} \quad ni_1 + j_1 > 0$$

(note that j_1 and $ni_1 + j_1$ cannot be negative because of (1) together with $h \in B^{p,q}$). Now this and (1) imply that $x_1^{i_1} y^{j_1 - 1} \in B^{p,q}$. Since $(x_1^{i_1} y^{j_1-1})y \in \ker(D_{p,q})$, it follows that $y \in \ker(D_{p,q})$ because $\ker(D_{p,q})$ is an inert subring of $B^{p,q}$ (see 1.4); then $x_1(y^n/x_1) = y^n \in \ker(D_{p,q})$, so $x_1 \in \ker(D_{p,q})$ for the same reason. But this is impossible because $D_{p,q} \neq 0$ and x_1, y generate $\mathrm{Frac}\, B^{p,q}$ over $\mathbf{k}$. □

2.7.4. **Lemma.** *If $f \in A$ for some $A \in \mathrm{KLND}(B)$, then* $\mathrm{bideg}(f^N) = N\, \mathrm{bideg}(f)$ *holds for all positive integers N.*

Proof. By 2.7.3, it suffices to show that the formula is valid for all $f \in R$ satisfying

$$\mathrm{supp}(f) \subset C(f). \tag{7}$$

So consider $f \in R$ satisfying (7) and let N be a positive integer. Let $(a, b) = \mathrm{bideg}(f)$ and $(a', b') = \mathrm{bideg}(f^N)$. Since

$$a = \max\big(0, \deg_{x_1} f(x_1, 0)\big) \quad \text{and} \quad a' = \max\big(0, \deg_{x_1}(f(x_1, 0)^N)\big),$$

it is clear that $a' = Na$. So it suffices to show that $b' = Nb$.

If $b = 0$ then $f \in \mathbf{k}[x_1, x_1^{-1}]$ by (7), so $f^N \in \mathbf{k}[x_1, x_1^{-1}]$ and consequently $b' = 0$. So $b' = Nb$ holds true in this case.

So we may assume that $b > 0$. By 2.7.2 we have $(-b, bn) \in \mathrm{supp}\, f$, so (7) gives

$$\mathrm{supp}\, f = \{(-b, bn)\} \cup S \cup T, \tag{8}$$

where $S = \{\, (i, j) \in \mathrm{supp}\, f \mid ni + j < 0 \text{ and } j = nb\}$ and $T = \{\, (i, j) \in \mathrm{supp}\, f \mid j < nb\}$. Observe that if (α, β) is any element of $\mathrm{supp}(f^N)$, then $(\alpha, \beta) = u_1 + \cdots + u_N$ for some elements $u_1, \ldots, u_N$ of $\mathrm{supp}\, f$. So, to complete the proof, it suffices to show:

(9) If $u_1, \ldots, u_N \in \mathrm{supp}\, f$ satisfy $u_1 + \cdots + u_N = (-k, kn)$ for some $k \geq Nb$, then $u_\nu = (-b, bn)$ for all ν.

Write $u_\nu = (i_\nu, j_\nu)$ for all ν. Then $j_\nu \le nb$ for all ν by (8), so the condition $k \ge Nb$ implies that no u_ν can belong to T:

$$u_1, \dots, u_N \in \{(-b, bn)\} \cup S.$$

So we have $ni_\nu + j_\nu \le 0$ for all ν and

$$\sum_{\nu=1}^{N}(ni_\nu + j_\nu) = n\sum_{\nu=1}^{N} i_\nu + \sum_{\nu=1}^{N} j_\nu = n(-k) + kn = 0$$

gives $ni_\nu + j_\nu = 0$ for all ν. So no u_ν can belong to S and we are done. □

2.7.5. **Lemma.** *Suppose that $f \in A$ for some $A \in \text{KLND}(B)$ and let $(a, b) = \operatorname{bideg} f$.*

(1) $a = 0 \Leftrightarrow f \in \mathbf{k}[x_2]$ *and* $b = 0 \Leftrightarrow f \in \mathbf{k}[x_1]$.
(2) *Assuming that $f \notin \mathbf{k}$,* $a = 0 \Leftrightarrow A = \mathbf{k}[x_2]$ *and* $b = 0 \Leftrightarrow A = \mathbf{k}[x_1]$.

Proof. Clearly, it suffices to prove assertion (1). Moreover, the two implications "$\Leftarrow$" are trivial.

If $b = 0$ then the line segment E joins $(0, 0)$ to $(a, 0)$, so $C(f) = (-\infty, a] \times \{0\}$; by 2.7.3, each $(i, j) \in \operatorname{supp}(f)$ satisfies $j = 0$, so $f \in \mathbf{k}[x_1, x_1^{-1}]$. Since $\mathbf{k}[x_1] \in \text{KLND}(B)$, $\mathbf{k}(x_1) \cap B = \mathbf{k}[x_1]$. So $f \in \mathbf{k}[x_1]$.

If $a = 0$ then the line segment E joins $(-b, bn)$ to the origin, so by 2.7.3

$$\text{each } (i, j) \in \operatorname{supp}(f) \text{ satisfies } ni + j \le 0. \tag{10}$$

Let us argue that $f \in \mathbf{k}[x_2]$.

Given $d \in \mathbb{N}$, let H_d be the set of elements of B which can be written in the form

$$\sum_{i=0}^{d-1} h_i^0(y) x_1^{d-i} + \sum_{j=0}^{e} h_d^j(y) x_2^j \tag{11}$$

for some $e \in \mathbb{N}$ and for some $h_i^j \in \mathbf{k}[Y]$ satisfying $\deg_Y(h_i^j) \le ni$, where $n = \deg_Y(\varphi)$. Observe that $H_0 = \mathbf{k}[x_2]$.

By (10), $f = \sum_{i=0}^{r} h_i^0(y) x_1^{-i}$ for some $r \in \mathbb{N}$ and for some $h_i^0 \in \mathbf{k}[Y]$ satisfying $\deg_Y(h_i^0) \le ni$. Thus

$$x_1^r f = \sum_{i=0}^{r} h_i^0(y) x_1^{r-i} = \sum_{i=0}^{r-1} h_i^0(y) x_1^{r-i} + \sum_{j=0}^{0} h_r^j(y) x_2^j,$$

i.e., the condition

$$x_1^d f \in H_d \tag{12}$$

is true for $d = r$. We show by descending induction that (12) is true for all $d \in \{0, 1, \dots, r\}$. Assume that $d \geq 0$ is such that $x_1^{d+1} f \in H_{d+1}$, i.e.,

$$x_1^{d+1} f = \sum_{i=0}^{d} h_i^0(y) x_1^{d+1-i} + \sum_{j=0}^{e} h_{d+1}^j(y) x_2^j. \tag{13}$$

Then $\sum_{j=0}^{e} h_{d+1}^j(y) x_2^j \in x_1 B$, or equivalently:

$$\sum_{j=0}^{e} h_{d+1}^j X_2^j \in (X_1, \varphi - X_1X_2)\mathbf{k}[X_1, X_2, Y] = (X_1, \varphi)\mathbf{k}[X_1, X_2, Y],$$

which implies that $\varphi \mid h_{d+1}^j$ in $\mathbf{k}[Y]$ for all $j \in \{0, 1, \dots, e\}$. Write $h_{d+1}^j = \varphi h_d^{j+1}$ (for $0 \leq j \leq e$) and notice that $\deg_Y(h_{d+1}^j) \leq (d+1)n$ implies $\deg_Y(h_d^{j+1}) \leq dn$. We have $h_{d+1}^j(y) x_2^j = \varphi(y) h_d^{j+1}(y) x_2^j = h_d^{j+1}(y) x_1 x_2^{j+1}$, so, dividing (13) by x_1 yields

$$\begin{aligned} x_1^d f &= \sum_{i=0}^{d} h_i^0(y) x_1^{d-i} + \sum_{j=0}^{e} h_d^{j+1}(y) x_2^{j+1} \\ &= \sum_{i=0}^{d-1} h_i^0(y) x_1^{d-i} + h_d^0(y) + \sum_{j=0}^{e} h_d^{j+1}(y) x_2^{j+1} = \sum_{i=0}^{d-1} h_i^0(y) x_1^{d-i} + \sum_{j=0}^{e+1} h_d^j(y) x_2^j, \end{aligned} \tag{14}$$

i.e., $x_1^d f \in H_d$. Thus by induction (12) holds for all $d \in \{0, 1, \dots, r\}$; in particular, it holds for $d = 0$, which means that $f \in H_0 = \mathbf{k}[x_2]$. $\square$

See 2.2 for the element τ of G.

2.7.6. **Lemma.** *Suppose that $f \in A$ for some $A \in \text{KLND}(B)$ and let $(a, b) = \operatorname{bideg}(f)$. Then $\operatorname{bideg}\big(\tau(f)\big) = (b, a)$.*

Proof. We may assume that $f \neq 0$. If $a = 0$ then, by 2.7.5 and 2.7.1, $f = P(x_2)$ for some $P \in \mathbf{k}[T]$ with $\deg_T(P) = b$. Then $\tau(f) = P(x_1)$ has bidegree $(b, 0) = (b, a)$ by 2.7.1. Similarly, if $b = 0$ then the result is trivial. So we may assume that

$$a, b > 0.$$

Consider the embeddings $B \subset R = \mathbf{k}[x_1, 1/x_1, y] \subset R' = K[x_1, 1/x_1]$, where $K = \mathbf{k}((y^{-1})) = \operatorname{Frac} \mathbf{k}[[y^{-1}]]$. Note that $\tau \in \operatorname{Aut}_{\mathbf{k}}(B)$ extends to $\tau \in \operatorname{Aut}_K(R')$.

Define an equivalence relation $\sim$ on $R' \setminus \{0\}$ by declaring that

$$\alpha \sim \beta \text{ if and only if } \alpha/\beta \in \mathbf{k}[[y^{-1}]]^*.$$

This relation is useful because:

$$\forall_{\substack{\beta \in R' \setminus \{0\} \\ i,j \in \mathbb{Z}}} \qquad \beta \sim x_1^i y^j \quad \iff \quad (i, j) \in \operatorname{supp}(\beta) \subset \{i\} \times (-\infty, j]. \tag{15}$$

Note that $\sim$ is a congruence for multiplication, i.e., if $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ then $\alpha_1\beta_1 \sim \alpha_2\beta_2$. Also, let $\xi_2 = y^n/x_1$; then clearly

$$x_2 \sim \xi_2.$$

Results 2.7.2 and 2.7.3 and the fact that $b > 0$ imply:

$$(-b, nb) \in \operatorname{supp}(f) \subseteq \{(-b, nb)\} \cup I,$$

where $I = \{\, (i,j) \in \mathbb{Z} \times \mathbb{N} \mid \ (j = nb \text{ and } i < -b) \text{ or } j < nb \,\}$; thus we have

$$f = \lambda(x_1^{-1}y^n)^b + \sum_{(i,j)\in I} \lambda_{ij} x_1^i y^j, \tag{16}$$

where $\lambda \in \mathbf{k}^*$ and $\lambda_{ij} \in \mathbf{k}$. Similarly,

$$(a, 0) \in \operatorname{supp}(f) \subseteq \{(a, 0)\} \cup I',$$

where $I' = \{\, (i,j) \in \mathbb{Z} \times \mathbb{N} \mid \ i < a \text{ and } nbi + (a+b)j \le nab \,\}$; so we may also write f as

$$f = \mu x_1^a + \sum_{(i,j)\in I'} \mu_{ij} x_1^i y^j, \tag{17}$$

where $\mu \in \mathbf{k}^*$ and $\mu_{ij} \in \mathbf{k}$. Applying τ to (16) and (17) gives

$$\tau(f) = \lambda(x_2^{-1}y^n)^b + \sum_{(i,j)\in I} \lambda_{ij} x_2^i y^j \qquad \text{and} \qquad \tau(f) = \mu x_2^a + \sum_{(i,j)\in I'} \mu_{ij} x_2^i y^j. \tag{18}$$

Note that

$$(x_2^{-1}y^n)^b \sim (\xi_2^{-1}y^n)^b = x_1^b, \qquad x_2^a \sim \xi_2^a = x_1^{-a}y^{na} \tag{19}$$

and

$$x_2^i y^j \sim (\xi_2^i y^j) = x_1^{-i} y^{ni+j}. \tag{20}$$

Let us now verify that

$$\forall\, (i,j) \in I, \qquad -i < b \text{ or } ni + j < 0. \tag{21}$$

Assume that $(i,j) \in I$ and that $-i \ge b$; we have to show that $ni+j < 0$. By definition of I, there are two cases to consider: (i) If $j = nb$ and $i < -b$, then $ni+j < -nb+j = 0$. (ii) If $j < nb$, then

$$-i \ge b \Rightarrow ni \le -nb \Rightarrow ni + j \le -nb + j < 0.$$

This proves (21). Similarly, one can prove

$$\forall\, (i,j) \in I', \qquad ni + j < na. \tag{22}$$

For each $(i,j) \in I$ we have

$$\begin{aligned}(s,t) \in \operatorname{supp}(x_2^i y^j) &\implies s = -i \text{ and } t \le ni + j && \text{by (20) and (15)}\\ &\implies s < b \text{ or } t < 0 && \text{by (21).}\end{aligned}$$

Hence,

$$(s,t) \in \operatorname{supp}\Big(\sum_{(i,j)\in I} \lambda_{ij} x_2^i y^j\Big) \quad\implies\quad s < b \text{ or } t < 0. \tag{23}$$

On the other hand, (19) and (15) imply that $(b,0) \in \operatorname{supp}\big[(x_2^{-1}y^n)^b\big]$; this, (23) and (18) imply that the first component of $\operatorname{bideg}(\tau(f))$ is b.

Using (22) in place of (21), a similar argument shows that

$$(s,t) \in \operatorname{supp}\Big(\sum_{(i,j)\in I'} \mu_{ij} x_2^i y^j\Big) \quad\implies\quad t < na$$

and it follows that the second component of $\operatorname{bideg}(\tau(f))$ is a. □

2.7.7. To compare bidegrees, let $\mathbb{N}\times\mathbb{N}$ be endowed with the reverse lexicographic order, i.e.,

$$(a,b) < (a',b') \iff b < b' \text{ or } (b = b' \text{ and } a < a').$$

Since this is a well-ordering of $\mathbb{N}\times\mathbb{N}$, given $A \in \text{KLND}(B)$ it makes sense to define

$$\operatorname{bideg}(A) = \min\{\operatorname{bideg} f \mid f \in A \setminus \mathbf{k}\} \in \mathbb{N}\times\mathbb{N}.$$

Homogeneous case

The following result is needed in the proof of 2.3. It is concerned with the special case where $\varphi = Y^n$.

2.8. **Proposition.** *Let $B = \mathbf{k}[X_1, X_2, Y]/(Y^n - X_1X_2)$, where $n \ge 1$, and write $B = \mathbf{k}[x_1, x_2, y]$ where x_1, x_2, y are the images of X_1, X_2, Y via the canonical epimorphism $\mathbf{k}[X_1, X_2, Y] \to B$. Let p, q be relatively prime positive integers and let B be endowed with the (p,q)-grading:*

$$B = \bigoplus_{d\in\mathbb{Z}} B_d, \qquad \mathbf{k} \subseteq B_0, \quad x_1 \in B_p, \quad y \in B_q.$$

Suppose that $D \neq 0$ is a homogeneous locally nilpotent derivation of B whose kernel is neither $\mathbf{k}[x_1]$ nor $\mathbf{k}[x_2]$. Then there exists a homogeneous and irreducible element h of $\ker D$ and, given any such h, we have $\ker D = \mathbf{k}[h]$ and one of the following holds:

(1) *For some $\lambda, \mu \in \mathbf{k}^*$, $h = \mu x_1^{-1}(y - \lambda x_1^q)^n$*

(2) *For some $\lambda, \mu \in \mathbf{k}^*$, $h = \mu x_2^{-1}(y - \lambda x_2^q)^n$.*

2.9. **Proof of 2.8.**

2.9.1. **Set-up.** Throughout 2.9, $B = \mathbf{k}[x_1, x_2, y]$ and p, q are as in the statement of 2.8 (so $x_1x_2 = y^n$) and we regard B as a subalgebra of $R = \mathbf{k}[x_1, x_1^{-1}, y]$. Let $R = \oplus_{d\in\mathbb{Z}} R_d$ be the (p,q)-grading, as in 2.5, and let $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ denote the corresponding degree function ($\deg(x_1) = p$ and $\deg(y) = q$, so $\deg(x_2) = nq - p$). Note that $B_d = B \cap R_d$ for all $d \in \mathbb{Z}$.

2.9.2. We define a valuation

$$\operatorname{ord} : \mathbf{k}(x_1, y) \to \mathbb{Z} \cup \{\infty\}$$

as follows. As a first step, define a map $\operatorname{ord} : R \to \mathbb{Z} \cup \{\infty\}$ by

$$\operatorname{ord}(f) = \begin{cases} \min\{\, ni + j \mid (i,j) \in \operatorname{supp}(f)\,\}, & \text{if } f \in R \setminus \{0\} \\ \infty, & \text{if } f = 0 \end{cases}$$

and note that, for all $f, g \in R$, the following conditions hold:

- $\operatorname{ord}(f) = \infty \iff f = 0$
- $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$
- $\operatorname{ord}(f+g) \geq \min(\operatorname{ord}(f), \operatorname{ord}(g))$.

Then extend ord to a valuation of $\mathbf{k}(x_1, y)$. Note that $\operatorname{ord}(x_1) = n$, $\operatorname{ord}(y) = 1$, $\operatorname{ord}(x_2) = 0$ and:

$$B = \{\, f \in R \mid \operatorname{ord}(f) \geq 0\,\}. \tag{24}$$

2.9.3. **Lemma.** *Suppose that $f_1, \dots, f_k \in R$ satisfy $\operatorname{ord}(f_1) = \cdots = \operatorname{ord}(f_k)$.*

1. *If $\prod_{i=1}^k f_i$ belongs to B then $f_1, \dots, f_k \in B$.*
2. *If $\prod_{i=1}^k f_i$ belongs to A (for some $A \in \text{KLND}(B)$) then $f_1, \dots, f_k \in A$.*

Proof. Assertion (1) is an immediate consequence of (24). Assertion (2) follows from (1) and the fact that A is an inert subring of B. □

2.9.4. *If $D \in \text{LND}(B)$ is (p,q)-homogeneous, then there exists a homogeneous and irreducible element h of* $\ker D$.

Proof. Let $A = \ker(D)$, then A is a homogeneous subring of B and B has transcendence degree one over A. So we can choose a homogeneous element H of $A = \ker(D)$ which is neither zero nor a unit. Since B is affine over $\mathbf{k}$, there exists a factorization of H into irreducible elements of B. Let $h \in B$ be an irreducible factor of H; then h is homogeneous, and $h \in A$ since A is inert in B. Finally, h is irreducible in A because $A^* = B^*$. □

2.9.5. **Reduction.** *If* 2.8 *holds in the special case where* $\mathbf{k}$ *is algebraically closed, then it holds in general.*

Proof. Suppose that $D \neq 0$ is a (p, q)-homogeneous locally nilpotent derivation of B whose kernel is neither $\mathbf{k}[x_1]$ nor $\mathbf{k}[x_2]$. Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$ and define $\bar{B} = \bar{\mathbf{k}} \otimes_{\mathbf{k}} B$. Note that $\bar{B} \cong \bar{\mathbf{k}}[X_1, X_2, Y]/(Y^n - X_1X_2)$ and let $\bar{D} : \bar{B} \to \bar{B}$ be the unique extension of D to a $\bar{\mathbf{k}}$-derivation of $\bar{B}$. Then $\bar{D}$ is locally nilpotent, (p, q)-homogeneous and such that $\ker \bar{D}$ is neither $\bar{\mathbf{k}}[x_1]$ nor $\bar{\mathbf{k}}[x_2]$. Thus 2.8 can be applied to $\bar{D}$: There exists a homogeneous and irreducible element $\bar{h}$ of $\ker \bar{D}$, this $\bar{h}$ satisfies

$$\ker \bar{D} = \bar{\mathbf{k}}[\bar{h}]$$

and one of the following holds:

$$\bar{h} = \mu x_1^{-1}(y - \lambda x_1^q)^n, \quad \text{for some } \lambda, \mu \in \bar{\mathbf{k}}^*, \tag{25}$$

$$\bar{h} = \mu x_2^{-1}(y - \lambda x_2^q)^n, \quad \text{for some } \lambda, \mu \in \bar{\mathbf{k}}^*. \tag{26}$$

We may assume that $\mu = 1$ in (25) and (26), because $\ker \bar{D} = \bar{\mathbf{k}}[\mu^{-1}\bar{h}]$. Then we claim that $\bar{h} \in B$. To prove this, we first observe that if (25) (resp. (26)) holds then $\deg(\bar{h}) = (qn - 1)\deg(x_1)$ (resp. $\deg(\bar{h}) = (qn - 1)\deg(x_2)$); since $\bar{h}$ is homogeneous, we also have $\deg(y) = \deg(x_1^q)$ (resp. $\deg(y) = \deg(x_2^q)$), so $\deg(x_1) = 1$ (resp. $\deg(x_2) = 1$). Hence, $\deg(\bar{h}) = nq - 1$. We consider two cases.

If $\deg(\bar{h}) = 0$ then $n = 1 = q$, so (25) (resp. (26)) simplifies to $\bar{h} = x_2 - \lambda$ (resp. $\bar{h} = x_1 - \lambda$), which is impossible since $\ker \bar{D}$ is neither $\bar{\mathbf{k}}[x_1]$ nor $\bar{\mathbf{k}}[x_2]$.

So we must have $\deg(\bar{h}) > 0$. Let h be a homogeneous and irreducible element of $\ker D$ (such an h exists by 2.9.4). Since both h and $\bar{h}$ are homogeneous, the condition $\deg(\bar{h}) > 0$ implies:

$$h = \epsilon \bar{h}^N$$

for some $\epsilon \in \bar{\mathbf{k}}^*$ and some positive integer N.

Suppose that (25) holds. Then $h \in \mathbf{k}[x_1, x_1^{-1}, y]$ and

$$\begin{aligned} h = \epsilon x_1^{-N}(y - \lambda x_1^q)^{nN} &= \epsilon x_1^{-N}\left[y^{nN} - Nn\lambda x_1^q y^{nN-1} + \textstyle\sum_{i=2}^{nN} \binom{nN}{i} y^{nN-i}(-\lambda x_1^q)^i\right] \\ &= \epsilon x_1^{-N} y^{nN} - Nn\epsilon\lambda x_1^{q-N} y^{nN-1} + \cdots, \end{aligned}$$

so $\epsilon \in \mathbf{k}$ and consequently $\lambda \in \mathbf{k}$. Thus $\bar{h} \in B$.

If (26) holds then, by a similar argument, $\bar{h} \in B$.

Finally, it is easy to see that $\bar{h} \in B$ implies that $\ker D = \mathbf{k}\big[\bar{h}\big]$. This proves 2.9.5. □

2.9.6. **From now-on, we assume that k is algebraically closed.**

We now proceed to prove 2.8, under assumption 2.9.6.

Consider a (p, q)-homogeneous derivation $D \in \text{LND}(B)$, let $A = \ker D$ and suppose that A is neither $\mathbf{k}[x_1]$ nor $\mathbf{k}[x_2]$. Let h be a homogeneous irreducible element of A and define

$$(a, b) = \operatorname{bideg}(h).$$

Then $\min(a,b)>0$ by 2.7.5 and, by 2.7.2,

$$(a,0),(-b,bn)\in\operatorname{supp}(h).$$

Since h is (p,q)-homogeneous we must have $\deg(x_1^a)=\deg(x_1^{-b}y^{nb})$, so

$$p=\frac{nb}{\delta}\quad\text{and}\quad q=\frac{a+b}{\delta},\quad\text{where}\quad\delta=\gcd(nb,a+b).\tag{27}$$

Homogeneity of h also implies that the convex hull of $\operatorname{supp}(h)$ is the line segment joining $(-b,nb)$ and $(a,0)$; so the convex hull of $\operatorname{supp}(x_1^bh)$ is the line segment joining $(0,nb)$ and $(a+b,0)$. Thus $x_1^bh\in\mathbf{k}[x_1,y]$ and, using the assumption that $\mathbf{k}$ is algebraically closed,

$$x_1^bh=\mu\prod_{i=1}^{\delta}(y^p-\lambda_ix_1^q)$$

for some $\mu,\lambda_1,\dots,\lambda_\delta\in\mathbf{k}^*$. Equivalently,

$$h=\mu x_1^{-b}\prod_{i=1}^{\delta}(y^p-\lambda_ix_1^q).\tag{28}$$

Then we have

$$A\ni h^\delta=(\mu x_1^{-b})^\delta\prod_{i=1}^{\delta}(y^p-\lambda_ix_1^q)^\delta=\prod_{i=1}^{\delta}\mu x_1^{-b}(y^p-\lambda_ix_1^q)^\delta=\prod_{i=1}^{\delta}h_i,$$

where $h_i=\mu x_1^{-b}(y^p-\lambda_ix_1^q)^\delta$. Note that $\operatorname{ord}(h_i)=0$ for all i; so 2.9.3 implies that $h_1,\dots,h_\delta$ all belong to A. We claim:

$$h_1=\dots=h_\delta.\tag{29}$$

Indeed, it is easy to see that $\operatorname{ord}(h_i-h_j)\ge 1$, so

$$\operatorname{ord}\big((h_i-h_j)^n\big)\ge n\implies\operatorname{ord}\big(x_1^{-1}(h_i-h_j)^n\big)\ge 0\implies x_1^{-1}(h_i-h_j)^n\in B,$$

i.e., x_1 divides $(h_i-h_j)^n$ in B. Assuming that $h_i\neq h_j$, we have the two conditions

$$(h_i-h_j)^n\in A\setminus\{0\}\text{ and }x_1\mid(h_i-h_j)^n\text{ in }B,$$

and these imply that $x_1\in A$, contradicting the assumption $A\neq\mathbf{k}[x_1]$.

This proves (29); thus (28) simplifies to

$$h=\mu x_1^{-b}(y^p-\lambda x_1^q)^\delta\quad\text{for some }\mu,\lambda\in\mathbf{k}^*.\tag{30}$$

2.9.7. $\gcd(b,\delta)=1$

Proof. Let $d=\gcd(b,\delta)$ and let $h_0=x_1^{-b/d}(y^p-\lambda x_1^q)^{\delta/d}$. Then $h_0\in R$ and $h_0^d=\mu^{-1}h\in A$ imply (by 2.9.3) that $h_0\in A$. Since h is irreducible in A, it follows that $d=1$. □

2.9.8. $A=\mathbf{k}[h]$

Proof. By (27) and 2.9.7, (b, δ) is the unique pair of relatively prime positive integers satisfying $b/\delta = p/n$. So (b, δ) is independent of our choice of h.

Consequently, if h' is any homogeneous irreducible element of A then, by (30) applied to h', we have $h' = \nu x_1^{-b}(y^p - \epsilon x_1^q)^\delta$ for some $\nu, \epsilon \in \mathbf{k}^*$. Clearly, the proof of (29) also proves that $\nu^{-1}h' = \mu^{-1}h$. Assertion 2.9.8 follows. □

2.9.9. $\gcd(a, b) = 1$ *and* $b \mid p$.

Proof. $\gcd(a, b) = \gcd(b, a+b) = \gcd(b, nb, a+b) = \gcd(b, \gcd(nb, a+b)) = \gcd(b, \delta) = 1$. Since $\delta p = nb$, we have $b \mid \delta p$; so the assertion follows from $\gcd(b, \delta) = 1$. □

Define the positive integer $k = p/b$ and rewrite (30) as $h = \mu x_1^{-b}(y^{kb} - \lambda x_1^q)^\delta$. In order to prove 2.8, there remains to show that h satisfies one of conditions (1), (2) displayed in the statement of 2.8. For that purpose, we may assume that $\mu = 1$. So we write h as

$$h = x_1^{-b}(y^{kb} - \lambda x_1^q)^\delta, \quad \text{for some } \lambda \in \mathbf{k}^*. \tag{31}$$

Define

$$K = \operatorname{Frac} A = \mathbf{k}(h).$$

Since $A \in \text{KLND}(B)$, 1.4 implies that $\operatorname{Frac} B = (\operatorname{Frac} A)^{(1)}$, i.e.,

$$\mathbf{k}(x_1, y) = K^{(1)}. \tag{32}$$

On the other hand, let us prove the following:

2.9.10. $\mathbf{k}(x_1, y)$ *is K-isomorphic to the field of fractions of*

$$K[T, Y]/\big(Y^{kb} + T^b(\lambda_1 T^a + \mu_1)\big)$$

for some $\mu_1, \lambda_1 \in K^*$ (where T and Y are indeterminates over K).

Proof. By 2.9.7, there exist $u, v \in \mathbb{Z}$ such that $\delta u + bv = 1$. Define

$$w = y^{kb} - \lambda x_1^q \quad \text{and} \quad t = w^v x_1^u. \tag{33}$$

Now (31) gives $h = x_1^{-b}w^\delta$ and the following are easily obtained:

$$t^\delta = w^{v\delta}x_1^{u\delta} = w^{v\delta}x_1^{1-bv} = x_1(w^\delta x_1^{-b})^v = x_1 h^v,$$
$$t^b = w^{vb}x_1^{ub} = w^{1-\delta u}x_1^{ub} = w(w^{-\delta}x_1^b)^u = wh^{-u},$$

which we rewrite as

$$x_1 = h^{-v}t^\delta \quad \text{and} \quad w = h^u t^b. \tag{34}$$

In particular we have $x_1 \in K(t, y)$, so

$$\mathbf{k}(x_1, y) = K(t, y). \tag{35}$$

By (33), (34) and the fact that $q\delta = a + b$,

$$y^{kb} = w + \lambda x_1^q = (h^u)t^b + (\lambda h^{-vq})t^{a+b} = t^b\big((\lambda h^{-vq})t^a + h^u\big),$$

which gives

$$y^{kb} + t^b\big(\lambda_1 t^a + \mu_1\big) = 0, \tag{36}$$

where $\lambda_1 = -\lambda h^{-vq} \in K^*$ and $\mu_1 = -h^u \in K^*$. Finally, note that $Y^{kb} + T^b\big(\lambda_1 T^a + \mu_1\big)$ is an irreducible polynomial in $K[Y,T]$ (this is proved in the Appendix, see 3.1). So 2.9.10 follows from (35) and (36). $\square$

In view of 2.9.10 and of the fact, observed in (32), that $\mathbf{k}(x_1, y)$ is rational over K, 3.1 implies that the triple (k, a, b) satisfies one of the following conditions:

(a) $k = 1$ and $\min(a, b) = 1$
(b) $k = 1$ and $a = 2 = b$
(c) $k = 2$ and $a = 1 = b$.

However, we have $\gcd(a, b) = 1$ by 2.9.9, so (b) cannot hold. We shall now see that (c) cannot hold either and that (a) gives the desired result.

2.9.11. *If* (k, a, b) *satisfies* (a), *then* h *satisfies one of conditions* (1), (2) *displayed in the statement of* 2.8.

Proof. Assume that (a) holds and recall that $k = p/b = n/\delta$. Thus $k = 1$ gives $\delta = n$ and, by (31),

$$h = x_1^{-b}(y^b - \lambda x_1^q)^n. \tag{37}$$

We have $\min(a, b) = 1$, so there are two cases to consider. If $b = 1$ then (37) shows that h satisfies condition (1) of 2.8. Suppose now that $a = 1$. Keeping in mind that $b + 1 = a + b = q\delta = qn$, we manipulate (37) as follows:

$$\begin{aligned} h = x_1^{-b}(y^b - \lambda x_1^q)^n &= x_1^{-b}y^{-n}(y^{b+1} - \lambda x_1^q y)^n = x_1^{-b}y^{-n}(y^{qn} - \lambda x_1^q y)^n \\ &= x_1^{-b}y^{-n}((x_1x_2)^q - \lambda x_1^q y)^n = x_1^{-b}x_1^{qn}y^{-n}(x_2^q - \lambda y)^n = x_1^{-b}x_1^{qn}(x_1x_2)^{-1}(x_2^q - \lambda y)^n \\ &= x_2^{-1}(x_2^q - \lambda y)^n = (-\lambda)^n x_2^{-1}(y - \lambda^{-1}x_2^q)^n, \end{aligned}$$

so h satisfies condition (2) of 2.8. So 2.9.11 is proved. $\square$

Finally we show that (c) cannot hold.

Assume that (c) holds. Then $2 = k = n/\delta$, so $n = 2\delta$; on the other hand, $\delta = \gcd(nb, a + b) = \gcd(n, 2) = \gcd(2\delta, 2) = 2$, so $n = 2\delta = 4$ and $q = (a + b)/\delta = 1$. Hence, in view of (31),

$$h = x_1^{-1}(y^2 - \lambda x_1)^2 \quad \text{and} \quad n = 4.$$

Define $\omega \in \operatorname{Aut}_{\mathbf{k}}(B)$ by $\omega(x_1) = -\lambda^{-1}x_1$, $\omega(x_2) = -\lambda x_2$ and $\omega(y) = y$; then $\omega(h) = -\lambda x_1^{-1}(y^2 + x_1)^2$. In other words, it suffices to prove the following statement:

2.9.12. *Let* $h = x_1^{-1}(y^2 + x_1)^2$ *and assume that* $n = 4$. *Then* $\mathbf{k}[h]$ *is not the kernel of a locally nilpotent derivation of* B.

Proof. Suppose that $\mathbf{k}[h] \in \text{KLND}(B)$. Then, by part (4) of 1.4, there exists $\alpha \in \mathbf{k}[h] \setminus \{0\}$ such that $B_\alpha = \big(\mathbf{k}[h]_\alpha\big)^{[1]}$. This implies:

$$B/(h-\mu)B \cong \mathbf{k}^{[1]} \quad \text{for general } \mu \in \mathbf{k}. \tag{38}$$

However, let us compute $B/(h-\mu)B$. Recall that $n = 4$, so $x_1x_2 = y^4$ and

$$h = x_1^{-1}(y^2 + x_1)^2 = x_1^{-1}y^4 + 2y^2 + x_1 = x_2 + 2y^2 + x_1.$$

Consequently,

$$\begin{aligned} B/(h-\mu)B &\cong \mathbf{k}[X_1, X_2, Y]/(X_2 + 2Y^2 + X_1 - \mu,\ Y^4 - X_1X_2) \\ &\cong \mathbf{k}[X_1, Y]/\big(Y^4 + X_1(2Y^2 + X_1 - \mu)\big) \cong \mathbf{k}[X, Y]/(F), \end{aligned}$$

where we define $F = Y^4 + 2XY^2 + X^2 - \mu X = (Y^2 + X)^2 - \mu X$. It is easy to see that $\mathbf{k}[X, Y]/(F)$ is not isomorphic to $\mathbf{k}^{[1]}$. One way to see it is to consider $\theta \in \operatorname{Aut}_{\mathbf{k}} \mathbf{k}[X, Y]$ defined by $\theta(X) = X - Y^2$ and $\theta(Y) = Y$, and to observe that $\theta(F) = X^2 + \mu Y^2 - \mu X$. Since $X^2 + \mu Y^2$ is not a power of a linear form, we see that $\mathbf{k}[X, Y]/(F)$ has two places at infinity, and consequently is not isomorphic to $\mathbf{k}^{[1]}$. This contradicts (38), so 2.9.12 is proved. □

The proof of 2.8 is now complete. □

Proof of the Transitivity Theorem

We return to the general assumptions of this section, i.e.,

$$B = \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2),$$

where $\varphi \in \mathbf{k}[Y]$ is a nonconstant monic polynomial; let $n = \deg_Y(\varphi)$ and write $B = \mathbf{k}[x_1, x_2, y]$ as usual.

Recall the map $\text{bideg} : B \to \mathbb{N}\times\mathbb{N}$ defined in 2.7. From 2.7.7, recall that $\mathbb{N}\times\mathbb{N}$ is well-ordered, with the reverse lexicographic order; and that we defined $\text{bideg}(A) \in \mathbb{N}\times\mathbb{N}$ for each $A \in \text{KLND}(B)$.

See 2.2 for the definition of Δ_u.

2.10. **Proposition.** *Let $A \in \text{KLND}(B) \setminus \{\mathbf{k}[x_1]\}$, let $(a, b) = \text{bideg}(A)$ and suppose that $a \geq b$. Then there exists $(\lambda, s) \in \mathbf{k}^* \times \mathbb{N}$ such that if we set $u = \lambda x_1^s$ then the ring $A' = \Delta_u(A)$ satisfies $\text{bideg}(A') = (a', b)$ and $a' < a$. Moreover, $s = (a+b)/\gcd(nb, a+b) - 1$.*

Proof. Let $A \in \text{KLND}(B) \setminus \{\mathbf{k}[x_1]\}$, let $(a, b) = \text{bideg}(A)$ and suppose that $a \geq b$. Note that $A \neq \mathbf{k}[x_2]$, since $a \geq b$. By 2.7.5, we have $\min(a, b) > 0$, so:

$$a \geq b > 0.$$

Let

$$\delta = \gcd(bn, a+b), \qquad p = \frac{bn}{\delta} \qquad \text{and} \qquad q = \frac{a+b}{\delta},$$

then p, q are relatively prime positive integers. As in 2.5, consider the (p, q)-graded rings[1] $R = \mathbf{k}[x_1, 1/x_1, y] = \oplus_{d\in\mathbb{Z}} R_d$ and $B^{p,q} = \oplus_{d\in\mathbb{Z}} B^{p,q}_d$ as well as the corresponding degree function $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ (which satisfies $\deg(x_1) = p$ and $\deg(y) = q$).

Pick $f \in A \setminus \mathbf{k}$ such that $\operatorname{bideg}(f) = (a, b)$. By 2.7.2 and 2.7.3,

$$(a, 0), (-b, bn) \in \operatorname{supp}(f) \subset C(f);$$

since our choice of p, q is such that $\deg(x_1^a) = \deg(x_1^{-b} y^{bn})$, this implies that

$$(a, 0), (-b, bn) \in \operatorname{supp}(h') \tag{39}$$

where we define $h' = H_{p,q}(f)$. Now (39) implies that $\operatorname{bideg}(h') = (a, b)$, which implies that $h' \notin \mathbf{k}[x_1] \cup \mathbf{k}[x_2]$ (by 2.7.5, because $a, b > 0$).

Choose $D \in \operatorname{LND}(B)$ such that $\ker(D) = A$ and also consider $D_{p,q} : B^{p,q} \to B^{p,q}$. We know from 2.6.3 that $h' = H_{p,q}(f)$ belongs to $\ker(D_{p,q})$, so $\ker(D_{p,q})$ is neither $\mathbf{k}[x_1]$ nor $\mathbf{k}[x_2]$ (because $h' \notin \mathbf{k}[x_1] \cup \mathbf{k}[x_2]$). By 2.8 applied to $D_{p,q}$, there exists a homogeneous irreducible element h of $\ker(D_{p,q})$, that element satisfies

$$\ker(D_{p,q}) = \mathbf{k}[h] \tag{40}$$

and one of the following holds:

$$h = \mu x_1^{-1}(y - \lambda x_1^q)^n, \quad \text{for some } \lambda, \mu \in \mathbf{k}^*, \tag{41}$$

$$h = \mu \xi_2^{-1}(y - \lambda \xi_2^q)^n, \quad \text{for some } \lambda, \mu \in \mathbf{k}^*, \tag{42}$$

where we define $\xi_2 = y^n/x_1$ (this element cannot be called x_2, because $x_2 = \varphi(y)/x_1$).

Recall that $h' \in \ker(D_{p,q}) = \mathbf{k}[h]$ and that $\deg(h') > 0$ by (39); so $\deg(h) > 0$ as well. Since both h and h' are homogeneous, we obtain

$$h' = \epsilon h^N \tag{43}$$

for some $\epsilon \in \mathbf{k}^*$ and some positive integer N.

We claim that if (42) holds then so does (41). Assume that (42) holds; then

$$h = \mu\xi_2^{-1}(y - \lambda\xi_2^q)^n = \mu(y^n/x_1)^{-1}(y - \lambda(y^n/x_1)^q)^n = \mu x_1^{-(nq-1)}(x_1^q - \lambda y^{nq-1})^n,$$

from which we see that $\operatorname{bideg}(h) = (1, nq-1)$. By (43) and 2.7.4 we have $N(1, nq-1) = \operatorname{bideg}(h') = (a, b)$; since we have $a \geq b > 0$, we must also have $1 \geq nq - 1 > 0$, so $nq = 2$ and

$$h = \mu x_1^{-(nq-1)}(x_1^q - \lambda y^{nq-1})^n = \mu x_1^{-1}(x_1^q - \lambda y)^n,$$

so (41) holds.

From now-on, assume that (41) holds: $h = \mu x_1^{-1}(y - \lambda x_1^q)^n$. Let $u = \lambda x_1^{q-1} \in \mathbf{k}[x_1]$ and consider $\Delta_u \in G$ (see 2.2), i.e., $\Delta_u(x_1) = x_1$ and $\Delta_u(y) = y + \lambda x_1^q$. Note:

$$\Delta_u(h) = \mu\xi_2. \tag{44}$$

[1]We remind the reader that $B^{p,q}$ is not necessarely equal to B, see 2.5 for details.

From (41) and the fact that h is homogeneous, we see that $\Delta_u(y) = y + \lambda x_1^q$ is homogeneous; so Δ_u extends to a homogeneous, degree-preserving automorphism $\Delta'_u : R \to R$ and consequently

$$H_{p,q}(\Delta'_u(f)) = \Delta'_u(H_{p,q}(f)) = \Delta'_u(h') = \epsilon\Delta'_u(h)^N = \epsilon\mu^N\xi_2^N.$$

So, if we define $f' = \Delta_u(f)$, we obtain that $\text{supp}(H_{p,q}(f')) = \{(-b, bn)\}$. This shows that $\text{bideg}(f') = (a', b)$ for some $a' < a$, so 2.10 is proved. □

Proof of Theorem 2.3.

It suffices to prove that, given any $A \in \text{KLND}(B) \setminus \{\mathbf{k}[x_1]\}$,

(45) There exists $\theta \in G$ such that $\text{bideg}(\theta A) < \text{bideg}(A)$.

Let $(a, b) = \text{bideg}(A)$.

If $a \geq b$ then by 2.10 there exists $u \in \mathbf{k}[x_1]$ such that $\text{bideg}(\Delta_u A) = (a', b)$, where $a' < a$, so $\text{bideg}(\Delta_u A) < \text{bideg}(A)$.

If $a < b$, pick $f \in A \setminus \mathbf{k}$ such that $\text{bideg}(f) = (a, b)$. Then by 2.7.6 we have $\text{bideg}[\tau(f)] = (b, a) < (a, b) = \text{bideg}(A)$, so $\text{bideg}\big[\tau(A)\big] \leq \text{bideg}[\tau(f)] < \text{bideg}(A)$.

Hence, (45) is true in all cases. This completes the proof of 2.3. □

2.11. **Remark.** As a corollary to the Transitivity Theorem, one obtains that each $A \in \text{KLND}(B)$ satisfies $A = \mathbf{k}^{[1]}$ (but of course there are quicker ways to prove this). Clearly, if f is such that $A = \mathbf{k}[f]$, then $\text{bideg}(A) = \text{bideg}(f)$. Consequently, if $\text{bideg}(A) = (a, b)$ then $\text{bideg}\big(\tau(A)\big) = (b, a)$ by 2.7.6.

3. APPENDIX: Rationality of certain affine plane curves

The aim of this section is to establish the following fact, which is used in the proof of 2.8 (see between 2.9.10 and 2.9.11).

3.1. **Lemma.** *Let $\mathbf{k}$ be a field (of characteristic zero), X and Y indeterminates over $\mathbf{k}$ and let $F = Y^{kb} + X^b(\lambda X^a + \mu) \in \mathbf{k}[X, Y]$, where $\lambda, \mu \in \mathbf{k}^*$ and a, b, k are positive integers. Then F is an irreducible element of $\mathbf{k}[X, Y]$. Consider the field extension $L/\mathbf{k}$, where L is the field of fractions of $\mathbf{k}[X, Y]/F\mathbf{k}[X, Y]$. Then $L = \mathbf{k}^{(1)}$ if and only if one of the following holds:*

(a) *$k = 1$ and $\min(a, b) = 1$*
(b) *$k = 1$ and $a = 2 = b$*
(c) *$k = 2$ and $a = 1 = b$.*

3.2. **Lemma.** *If 3.1 holds in the special case where $\mathbf{k}$ is algebraically closed, then it holds in general.*

Proof. Assume that 3.1 holds in the special case where $\mathbf{k}$ is algebraically closed. Let $\mathbf{k}$ be arbitrary and consider

$$F = Y^{kb} + X^b(\lambda X^a + \mu) \in \mathbf{k}[X, Y],$$

where $a, b, k \geq 1$ and $\lambda, \mu \in \mathbf{k}^*$.

Let $\bar{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$; the first step is to prove that

$$F \text{ is irreducible in } \bar{\mathbf{k}}[X, Y]. \tag{46}$$

Write $f = -X^b(\lambda X^a + \mu) \in \mathbf{k}[X]$ and $K = \bar{\mathbf{k}}(X)$, then $F = Y^{kb} - f \in K[Y]$ where $f \in K$. By Gauss Lemma, (46) is equivalent to F being irreducible in $K[Y]$; and it is well-known that the latter condition is a consequence of

$$f^{1/d} \notin \bar{\mathbf{k}}(X) \text{ for all } d > 1. \tag{47}$$

The verification of (47) is straightforward, and is left to the reader. So (46) is proved.

Let $A = \mathbf{k}[X, Y]/F\mathbf{k}[X, Y]$ and $\bar{A} = \bar{\mathbf{k}}[X, Y]/F\bar{\mathbf{k}}[X, Y]$. Then (46) implies that $\operatorname{Frac} A$ is a subfield of $\operatorname{Frac} \bar{A}$. In fact, it is easy to see that $\operatorname{Frac} \bar{A}$ can be obtained by adjoining the elements of $\bar{\mathbf{k}}$ to $\operatorname{Frac} A$; in other words, the function field

$$\operatorname{Frac} \bar{A} \text{ over } \bar{\mathbf{k}} \tag{48}$$

is obtained from the function field

$$\operatorname{Frac} A \text{ over } \mathbf{k} \tag{49}$$

by "constant field extension". Since the characteristic is zero, it follows that the function fields (48) and (49) have the same genus. Finally, note that the curve "$F = 0$" goes through the origin ($(X, Y) = (0, 0)$); this implies that the function field (49) has at least one place of degree one. Hence, (49) is rational if and only if (48) is rational. □

Until the end of this section, we assume that k is algebraically closed.

3.3. **Notations.** If $G \in \mathbf{k}[X_1, \dots, X_n]$ then $\frac{\partial G}{\partial X_i}$ is abbreviated G_{X_i}. If $G \in \mathbf{k}[X, Y]$ is a nonconstant polynomial, let $V(G) \subset \mathbb{A}^2$ be the curve "$G = 0$". If $H \in \mathbf{k}[X, Y, Z]$ is a homogeneous polynomial, let $V(H) \subset \mathbb{P}^2$ be the curve "$H = 0$".

3.4. **Genus formula.** If $C \subset \mathbb{P}^2$ is an irreducible projective curve, its geometric genus g can be computed by the formula

$$g = \frac{(\deg C - 1)(\deg C - 2)}{2} - \sum_{P \in S} \frac{\mu(P)(\mu(P) - 1)}{2}, \tag{50}$$

where S is the set of singular points of C—including the infinitely near ones—and, given $P \in S$, $\mu(P)$ is the multiplicity of P on (a suitable strict transform of) C.

Given a subset E of $\mathbb{P}^2$, it is convenient to write

$$g'(E) = \sum_{P \in S_E} \frac{\mu(P)(\mu(P)-1)}{2},$$

where $S_E = \{P \in S \mid P \in E \text{ or } P \text{ is infinitely near a point of } E\}$. If the set E is a singleton, say $E = \{P_0\}$, $g'(\{P_0\})$ is abbreviated $g'(P_0)$.

For instance, we may partition $\mathbb{P}^2$ as $\mathbb{A}^2 \cup L_\infty$, where $\mathbb{A}^2$ consists of the points $(x:y:1)$ and L_∞ is the "line at infinity", containing the points $(x:y:0)$ [we say that the points of $\mathbb{A}^2$ are "at finite distance" and that those of L_∞ are "at infinity"]. Then we may rewrite the genus formula (50) as

$$g = \frac{(\deg C - 1)(\deg C - 2)}{2} - g'(\mathbb{A}^2) - g'(L_\infty). \tag{51}$$

3.5. **Lemma.** *Let P_0 be a singular point of a curve $C \subset \mathbb{P}^2$ and let $F_0 \in \mathbf{k}[[U,V]]$ be a local equation of C at P_0. Make the following assumptions:*

1. *$F_0(U,0) \neq 0$ and $F_0(0,V) \neq 0$*
2. *Let $\alpha = \operatorname{ord}_U F_0(U,0)$ and $\beta = \operatorname{ord}_V F_0(0,V)$; then the only elements (i,j) of $\operatorname{supp}(F_0)$ satisfying $\beta i + \alpha j \le \alpha\beta$ are $(\alpha, 0)$ and $(0,\beta)$.*

Then

$$g'(P_0) = \frac{(\alpha-1)(\beta-1)+(\delta-1)}{2},$$

where $\delta = \gcd(\alpha,\beta)$ and where $g'(P_0)$ is defined in 3.4.

Proof. Recall that $\operatorname{supp}(F_0) = \{(i,j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}$, where $F_0 = \sum_{(i,j)\in\mathbb{N}^2} a_{ij}U^iV^j$ ($a_{ij} \in \mathbf{k}$).

If $\alpha = \beta$ then the assumptions imply that the initial form of F_0 is $a_{\alpha\,0}U^\alpha + a_{0\,\beta}V^\alpha$ (with $a_{\alpha\,0}, a_{0\,\beta} \in \mathbf{k}^*$ and $\alpha > 0$). So P_0 is an ordinary singularity with multiplicity α, in which case it is well-known that $g'(P_0) = \alpha(\alpha-1)/2$. Since

$$\frac{(\alpha-1)(\beta-1)+(\delta-1)}{2} = \frac{(\alpha-1)(\alpha-1)+(\alpha-1)}{2} = \frac{\alpha(\alpha-1)}{2},$$

we are done in this case.

Assume that $\alpha \neq \beta$ and let $d = \max(\alpha,\beta)$ and $r_0 = \min(\alpha,\beta)$. Consider the euclidean algorithm of the pair (d, r_0):

$$\begin{array}{rcll}
d &=& q_0 r_0 + r_1 & (q_0, r_1 \in \mathbb{N},\ r_1 < r_0)\\
r_0 &=& q_1 r_1 + r_2 & (q_1, r_2 \in \mathbb{N},\ r_2 < r_1)\\
&\vdots&&\\
r_{k-2} &=& q_{k-1} r_{k-1} + r_k & (q_{k-1}, r_k \in \mathbb{N},\ r_k < r_{k-1})\\
r_{k-1} &=& q_k r_k + r_{k+1} & (q_k, r_{k+1} \in \mathbb{N},\ r_{k+1} < r_k,\ r_{k+1} = 0),
\end{array}$$

note that $r_k = \delta$ and define $N = q_0 + \cdots + q_k$. Then we inductively define morphisms

$$S_N \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_1} S_0 = \mathbb{P}^2 \tag{52}$$

as follows:

(a) Let $\pi_1 : S_1 \to S_0 = \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at the given point P_0.

(b) If $i \in \{1, \dots, N\}$ is such that $S_i \xrightarrow{\pi_i} \cdots \xrightarrow{\pi_1} S_0 = \mathbb{P}^2$ have been defined, set $\hat{\pi}_i = \pi_1 \circ \cdots \circ \pi_i : S_i \to S_0$ and let $C_i \subset S_i$ be the strict transform of C with respect to $\hat{\pi}_i$. If $i < N$ then assumptions (1) and (2) on F_0 ensure that $C_i \cap \hat{\pi}_i^{-1}(P_0)$ is a single point, which we denote $P_i \in S_i$; let $\pi_{i+1} : S_{i+1} \to S_i$ be the blowing-up of S_i at P_i.

This defines (52). It is known that the multiplicity sequence satisfies:

(c) $\big(\mu(P_0), \mu(P_1), \dots, \mu(P_{N-1})\big) = \big(\underbrace{r_0, \dots, r_0}_{q_0}, \underbrace{r_1, \dots, r_1}_{q_1}, \dots, \underbrace{r_k, \dots, r_k}_{q_k}\big)$.

The fact that $C_i \cap \hat{\pi}_i^{-1}(P_0)$ is a singleton for all $i = 1, \dots, N-1$ (see (b)) implies that the restriction of $\hat{\pi}_{N-1} : S_{N-1} \to S_0$ to a map $C_{N-1} \to C$ is bijective, but note that the restriction $C_N \xrightarrow{\pi_N} C_{N-1}$ is not necessarely injective. However, our assumptions have the following consequence: Let $F_{N-1} \in \mathbf{k}[[U, V]]$ be the local equation of C_{N-1} at P_{N-1}, then the initial form of F_{N-1} is $a_{\alpha\,0} U^\delta + a_{0\,\beta} V^\delta$ (with $a_{\alpha\,0}, a_{0\,\beta} \in \mathbf{k}^*$ and $\delta > 0$). Hence, P_{N-1} is either a smooth point or an ordinary singularity of C_{N-1}. Consequently, every point of $C_N \cap \hat{\pi}_N^{-1}(P_0)$ is a smooth point of C_N, or in other words:

(d) The N blow-ups (52) completely resolve the singularity of C at P_0.

By (d) and the definition of $g'(P_0)$,

$$g'(P_0) = \sum_{i=0}^{N-1} \frac{\mu(P_i)(\mu(P_i) - 1)}{2}.$$

Finally, it is an exercise of arithmetic to deduce the following formula from (c):

$$\sum_{i=0}^{N-1} \frac{\mu(P_i)(\mu(P_i) - 1)}{2} = \frac{(d-1)(r_0 - 1) + (\delta - 1)}{2},$$

so the proof is complete. □

3.6. **Lemma.** *Let $F = Y^b + \lambda X^a + \mu \in \mathbf{k}[X, Y]$, where $\lambda, \mu \in \mathbf{k}^*$ and*

$$\min(a, b) \geq 2 \quad \textit{and} \quad \max(a, b) \geq 3.$$

Then the curve $V(F)$ is not rational.

Proof. It is clear that the ideal (F_X, F_Y, F) contains 1, so the curve is smooth at finite distance, i.e., $g'(\mathbb{A}^2) = 0$. Consider the closure $V(F^*)$ of $V(F)$ in $\mathbb{P}^2$, where $F^* \in \mathbf{k}[X, Y, Z]$ is the homogeneization of F.

If $a = b$ then $F^* = Y^a + \lambda X^a + \mu Z^a$, so $V(F^*)$ meets the line at infinity L_∞ in a distinct points, so $V(F^*)$ is a smooth projective curve. Since its degree is $a = \max(a, b) \geq 3$, it is not rational.

There remains the case $a \neq b$; clearly, we may assume that

$$a < b.$$

Then $F^* = Y^b + \lambda X^a Z^{b-a} + \mu Z^b$, so $V(F^*)$ meets L_∞ in one point: $P_0 = (1 : 0 : 0)$. Consequently, the polynomial

$$F_0 = F^*(1, Y, Z) = Y^b + \lambda Z^{b-a} + \mu Z^b$$

is a local equation of $V(F^*)$ at P_0 (and P_0 is now the origin, $(y, z) = (0, 0)$). By 3.5,

$$g'(L_\infty) = g'(P_0) = \frac{(b-1)(b-a-1) + (\delta - 1)}{2}$$

where $\delta = \gcd(b, b-a)$. By the genus formula (51),

$$\begin{aligned} g &= \frac{(b-1)(b-2)}{2} - \frac{(b-1)(b-a-1) + (\delta-1)}{2} = \frac{(b-1)(a-1)}{2} - \frac{\delta - 1}{2} \\ &> \frac{(b-1)(a-1)}{2} - \frac{b-a}{2} = \frac{b(a-2)+1}{2} > 0, \end{aligned}$$

where we used $\delta - 1 < b - a$. So $V(F^*)$ is not rational. □

3.7. **Corollary.** *Let $F = Y^b + \lambda X^a + \mu \in \mathbf{k}[X, Y]$, where $\lambda, \mu \in \mathbf{k}^*$ and $a, b \geq 1$. Then the curve $V(F)$ is rational if and only if:*

$$\min(a, b) = 1 \quad \text{or} \quad a = b = 2.$$

Proof. Follows immediately from 3.6 □

Proof of 3.1. We may write $L = \mathbf{k}(x, y)$ where $y^{kb} + x^b(\lambda x^a + \mu) = 0$. Set $z = y^k/x \in \mathbf{k}(x, y)$, then

$$\mathbf{k}(x, z) = \mathbf{k}(x, y^k) \subseteq \mathbf{k}(x, y)$$

and the field $\mathbf{k}(x, z)$ is defined by the equation

$$z^b + \lambda x^a + \mu = 0. \tag{53}$$

If (a) or (b) holds, then $k = 1$ (so $\mathbf{k}(x, z) = \mathbf{k}(x, y)$) and 3.7 gives $\mathbf{k}(x, z) = \mathbf{k}^{(1)}$, so $\mathbf{k}(x, y) = \mathbf{k}^{(1)}$. If (c) holds then $V(F)$ is a curve of degree 2, and so is rational.

Hence, if one of (a), (b), (c) holds then $\mathbf{k}(x, y) = \mathbf{k}^{(1)}$.

Conversely, assume that $\mathbf{k}(x, y) = \mathbf{k}^{(1)}$. Then $\mathbf{k}(x, z) = \mathbf{k}^{(1)}$ by Lüroth Theorem and it follows that either $\min(a, b) = 1$ or $a = 2 = b$, by applying 3.7 to (53). Thus, if $k = 1$ then (a) or (b) holds. For this reason, we will assume from now-on that $k \geq 2$. Let g be the genus of the curve $V(F)$, or equivalently the genus of the function field $\mathbf{k}(x, y)/\mathbf{k}$. Since we assumed that $\mathbf{k}(x, y) = \mathbf{k}^{(1)}$, we have $g = 0$. The proof consists in showing that the conditions

$$k \geq 2 \text{ and } g = 0$$

imply that (c) holds.

We begin by noting that consideration of

$$\begin{aligned} F &= Y^{kb} + \lambda X^{a+b} + \mu X^b = Y^{kb} + X^b(\lambda X^a + \mu) \\ F_Y &= kbY^{kb-1} \\ F_X &= \lambda(a+b)X^{a+b-1} + \mu b X^{b-1} = X^{b-1}\big[\lambda a X^a + b(\lambda X^a + \mu)\big] \end{aligned}$$

shows that the singular locus of $V(F)$ (at finite distance) is

$$\begin{cases} \{(0,0)\} & \text{if } b > 1, \\ \varnothing & \text{if } b = 1. \end{cases} \tag{54}$$

Now $F = Y^{kb} + \lambda X^{a+b} + \mu X^b$ may be regarded as a local equation of $V(F)$ at $(0,0)$; so the hypothesis of 3.5 is satisfied, with $\alpha = b$, $\beta = kb$ and $\delta = b$; so

$$g'(\mathbb{A}^2) = g'\big((0,0)\big) = \frac{(b-1)(kb-1) + (b-1)}{2} = \frac{kb(b-1)}{2}. \tag{55}$$

Observe that this formula is valid in the two cases of (54), i.e., in all cases. In order to compute $g'(L_\infty)$, we have to consider several cases. Let $d = \deg(F) = \max(kb, a+b)$.

Case $d = kb = a + b$. Then $F^* = Y^{kb} + \lambda X^{kb} + \mu X^b Z^{(k-1)b}$, so $V(F^*)$ is a curve of degree d meeting L_∞ in d distinct points; so $V(F^*)$ is smooth at infinity, i.e., $g'(L_\infty) = 0$. Consequently,

$$\begin{aligned} 2g = (d-1)(d-2) - 2g'(\mathbb{A}^2) - 2g'(L_\infty) &= (kb-1)(kb-2) - kb(b-1) \\ &= kb(kb - b - 2) + 2 = kb(a-2) + 2. \end{aligned}$$

Since $g = 0$, we must have $a = 1$; then $kb = a + b$ implies that $b = 1$ and that $k = 2$, i.e., condition (c) holds.

Case $kb \neq a + b$. If $d = kb > a + b$ then $F^* = Y^{kb} + \lambda X^{a+b} Z^{kb-a-b} + \mu X^b Z^{kb-b}$, so $V(F^*) \cap L_\infty = \{P_0\}$ where $P_0 = (1:0:0)$. Then

$$F_0 = F^*(1, Y, Z) = Y^{kb} + \lambda Z^{kb-a-b} + \mu Z^{kb-b}$$

is a local equation of $V(F^*)$ at P_0 and the hypothesis of 3.5 is satisfied, with $\{\alpha, \beta\} = \{kb, kb - a - b\}$ and $\delta = \gcd(kb, a+b)$. Observe that $(d-1)(d-2) - (\alpha-1)(\beta-1) = (kb-1)(kb-2) - (kb-1)(kb-a-b-1)$, so

$$(d-1)(d-2) - (\alpha-1)(\beta-1) = (a+b-1)(kb-1). \tag{56}$$

Now

$$\begin{aligned} 2g = (d-1)(d-2) - 2g'(L_\infty) - 2g'(\mathbb{A}^2) &= (d-1)(d-2) - 2g'(P_0) - 2g'(\mathbb{A}^2) \\ &= \big[(d-1)(d-2) - (\alpha-1)(\beta-1)\big] - (\delta-1) - kb(b-1), \end{aligned}$$

so (56) implies

$$2g = (a+b-1)(kb-1) - kb(b-1) - (\delta - 1). \tag{57}$$

We claim that (57) is also valid when $kb < a+b$. Indeed, if $kb < a+b$ then $F^* = Y^{kb}Z^{a+b-kb} + \lambda X^{a+b} + \mu X^b Z^a$, so $V(F^*) \cap L_\infty = \{P_0\}$ where $P_0 = (0:1:0)$. Then

$$F_0 = F^*(X, 1, Z) = Z^{a+b-kb} + \lambda X^{a+b} + \mu X^b Z^a$$

is a local equation of $V(F^*)$ at P_0, so the hypothesis of 3.5 is satisfied, with $\{\alpha, \beta\} = \{a+b, a+b-kb\}$ and $\delta = \gcd(kb, a+b)$. It follows that

$$\begin{aligned}(d-1)(d-2) - (\alpha-1)(\beta-1) &= (a+b-1)(a+b-2) - (a+b-1)(a+b-kb-1) \\ &= (a+b-1)(kb-1),\end{aligned}$$

i.e., (56) holds in this case as well. Since (57) follows from (56), (57) is therefore valid whenever $kb \neq a+b$. Also,

$$\delta = \gcd(kb, a+b) \leq a+b$$

holds whenever $kb \neq a+b$.

Now an algebraic manipulation of (57) yields

$$2g = (k-2)\big[ab + (b-1)^2\big] + 2(b-1)(a+b-2) + (a+b-\delta), \tag{58}$$

i.e., $2g$ is expressed as the sum of three nonnegative integers. Since $g = 0$, we obtain the three conditions:

$$k = 2,\ b = 1 \text{ and } a+b = \delta.$$

The third condition reads $a+1 = \gcd(kb, a+b) = \gcd(2, a+1)$, so $a+1 \mid 2$. This gives $a = 1$, so condition (c) holds.

This completes the proof of 3.1. □

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