LOCALLY NILPOTENT DERIVATIONS AND DANIELEWSKI SURFACES

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Introduction

This work started as an attempt to understand the process known as the local slice construction. Introduced by Freudenburg in [3], this is a method for modifying a nonzero locally nilpotent derivation of $\mathbf{k}[X,Y,Z]$ so as to obtain another one (where $\mathbf{k}$ is a field of characteristic zero). Near the end of the cited paper, Freudenburg defines a graph Γ whose vertices are the kernels of the nonzero locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$ and where vertices $\ker(D)$ and $\ker(D')$ are joined by an edge whenever D' can be obtained from D by a local slice construction (in one step).

Over the years, it has become clear that the local slice construction is an interesting idea for studying the locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$. In particular, one would like to know if Γ is connected. Connectedness would mean that every locally nilpotent derivation can be obtained from one of them (say from $\partial/\partial X$) by a finite sequence of local slice constructions. In unpublished work, we have shown that this is indeed the case for derivations which are homogeneous with respect to positive weights.

In the hope of clarifying the local slice construction, we generalize it. Let B be an arbitrary integral domain of characteristic zero. In section 3 of the present paper, we define a graph $\underline{\text{KLND}}\,(B)$ which generalizes Freudenburg's graph Γ: The vertices of $\underline{\text{KLND}}\,(B)$ are the kernels of the nonzero locally nilpotent derivations of B and the edges, one might say, capture the essence of the local slice construction. Also, the graph $\underline{\text{KLND}}\,(B)$ is an invariant of the ring B and the group of automorphisms of B acts on it in a natural way. In the special case $B = \mathbf{k}[X,Y,Z]$, the two graphs Γ and $\underline{\text{KLND}}\,(B)$ have the same vertices and every edge of Γ is an edge of $\underline{\text{KLND}}\,(B)$; we don't know if every edge of $\underline{\text{KLND}}\,(B)$ is an edge of Γ.

This generalization produces new insight into the local slice construction. In particular, we find that that process is essentially a two-dimensional affair and that it is intimately related to Danielewski surfaces "$XY = P(Z)$".

We believe that $\underline{\text{KLND}}\,(B)$ is a suitable tool for studying polynomial rings ($B = \mathbf{k}^{[n]}$). For these rings, the graph $\underline{\text{KLND}}\,(B)$ seems to have just the right amount of edges to be interesting. This is not the case for all rings: One can find examples of rings B for which $\underline{\text{KLND}}\,(B)$ is the empty graph; or $\underline{\text{KLND}}\,(B)$ has only one vertex and no edges; or (see 6.2) $\underline{\text{KLND}}\,(B)$ has infinitely many vertices but no edges.

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In a subsequent paper, we intend to use the methods developed here to investigate the locally nilpotent derivations of $\mathbf{k}[X,Y,Z]$.

The material is organized as follows.

Section 1 gives the basic definitions and results that are needed in this paper.

Section 2 gives some algebraic properties of Danielewski surfaces. Note in particular results 2.5, 2.6 and 2.6.2, which characterize Danielewski surfaces in terms of locally nilpotent derivations.

Section 3 defines the graph $\underline{\text{KLND}}(B)$, where B is any integral domain of characteristic zero. In addition to $\underline{\text{KLND}}(B)$, two other graphs ($\underline{\text{KLND}}_*(B)$ and $\underline{\mathfrak{R}}(B)$) are defined in that section.

Section 4 describes the graph $\underline{\text{KLND}}(B)$ in the case where B is a two-dimensional ring.

Section 5 focuses on the subgraph $\underline{\text{KLND}}_*(B)$ of $\underline{\text{KLND}}(B)$ obtained by deleting all isolated vertices. If B is a factorial affine domain (of any dimension), Theorem 5.1 states that $\underline{\text{KLND}}_*(B)$ is a union of connected subgraphs G_i such that: (i) Each G_i is isomorphic to $\underline{\text{KLND}}(B_i)$ for some two-dimensional ring B_i (in fact a Danielewski surface); (ii) every edge of $\underline{\text{KLND}}_*(B)$ is an edge of exactly one G_i; and (iii) if $i \neq j$ then G_i and G_j have at most one vertex in common. So the local structure of $\underline{\text{KLND}}_*(B)$ is well understood, thanks to the thorough description of the two-dimensional case given in Section 4.

Section 6 gathers some remarks which conclude the paper.

1. Generalities

1.1. **Conventions.**

- **All fields and rings are tacitly assumed to be of characteristic zero.**
- Throughout, $\mathbf{k}$ denotes an arbitrary field (of characteristic zero).
- The set of units of a ring R is denoted R^*.
- If A is a subring of a ring B and $r \in \mathbb{N}$, the notation $B = A^{[r]}$ means that B is A-isomorphic to the polynomial ring in r variables over A. If L/K is a field extension, $L = K^{(r)}$ means that L is a purely transcendental extension of K, of transcendence degree r.
- If A is a domain then $\operatorname{Frac} A$ is its field of fractions. If $A \subseteq B$ are domains then $\operatorname{trdeg}_A(B)$ is the transcendence degree of $\operatorname{Frac} B$ over $\operatorname{Frac} A$.
- By a $\mathbf{k}$-*domain of transcendence degree* d, we mean an integral domain B containing $\mathbf{k}$ and satisfying $\operatorname{trdeg}_{\mathbf{k}}(B) = d$.
- If R is a subring of a domain A, then we write A_R as an abbreviation for the localized ring $S^{-1}A$, where $S = R \setminus \{0\}$; in particular, $A_A = \operatorname{Frac}(A)$; if $D : A \to A$ is a derivation, $S^{-1}D : S^{-1}A \to S^{-1}A$ is abbreviated $D_R : A_R \to A_R$.
- If $\alpha \in A$ then $A_\alpha = S^{-1}A$ where $S = \{1, \alpha, \alpha^2, \dots\}$.

1.2. **Definition.** An *inert subring* of a domain B is a subring R of B satisfying:

$$\forall_{x,y \in B} \quad xy \in R \setminus \{0\} \implies x, y \in R.$$

1.3. *If R is an inert subring of B then the following hold.*

(1) *$R^* = B^*$*
(2) *R is algebraically closed in B.*
(3) *If B is a UFD then so is R.*
(4) *$S^{-1}R$ is an inert subring of $S^{-1}B$, for any multiplicative subset $S \subseteq R \setminus \{0\}$.*

1.4. *A subring R of an integral domain B is inert if and only if ${B_R}^* = {R_R}^*$ and $B \cap R_R = R$.*

1.5. **Definitions.** Let B be a ring.

(1) A derivation $D : B \to B$ is
- *irreducible* if the only principal ideal of B which contains $D(B)$ is B;
- *locally nilpotent* if $\forall_{x \in B}\ \exists_{k>0}\ D^k(x) = 0$.

(2) Notations:

$$\begin{aligned} \text{LND}(B) &= \text{set of } \textbf{nonzero} \text{ locally nilpotent derivations } D : B \to B \\ \text{KLND}(B) &= \big\{ \ker D \mid D \in \text{LND}(B) \big\}. \end{aligned}$$

If E is a subset of B,

$$\begin{aligned} \text{LND}_E(B) &= \big\{ D \in \text{LND}(B) \mid D(E) = \{0\} \big\} \\ \text{KLND}_E(B) &= \big\{ \ker D \mid D \in \text{LND}_E(B) \big\}. \end{aligned}$$

1.6. **Basic properties of locally nilpotent derivations.** Let B be an integral domain, let $D : B \to B$ be a nonzero derivation of B, and let $A = \ker D$. The following facts are well-known.

(1) If D is locally nilpotent then A is an inert subring of B. In particular: $B^* = A^*$, $B \cap \operatorname{Frac} A = A$ and if B is a UFD then so is A. Note, also, that if K is any field contained in B then $K^* \subseteq B^* = A^*$, so D is a K-derivation.
(2) Let S be a multiplicatively closed subset of $B \setminus \{0\}$, and consider the derivation $S^{-1}D : S^{-1}B \to S^{-1}B$. Then:
  (a) $S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subset A$.
  (b) If $S \subset A$ then $\ker S^{-1}D = S^{-1}A$; consequently, $B \cap S^{-1}A = A$.
(3) Assume that $\mathbb{Q} \subseteq B$. If D is locally nilpotent, and if $s \in B$ satisfies $D(s) \in B^*$, then $B = A[s] = A^{[1]}$.
(4) Assume that $\mathbb{Q} \subseteq B$. If D is locally nilpotent, choose any $s \in B$ such that $Ds \neq 0$ and $D^2 s = 0$ (such an s exists, and is called a *preslice* of D), and let $S = \{1, Ds, (Ds)^2, \dots\} \subset A$. Then $S^{-1}D(s) \in (S^{-1}B)^*$ so, by (3), $S^{-1}B = (S^{-1}A)[s] = (S^{-1}A)^{[1]}$.
(5) If D is locally nilpotent, let $S = A \setminus \{0\}$, then (4) implies $S^{-1}B = (\operatorname{Frac} A)^{[1]}$.
(6) Let $a \in B \setminus \{0\}$. The derivation $aD : B \to B$ is locally nilpotent if and only if D is locally nilpotent and $a \in A$.

Note in particular the following consequence of part (5) of 1.6:

1.7. *If B is a domain and $A \in \text{KLND}(B)$ then* $\operatorname{trdeg}_A B = 1$.

1.8. **Rentschler's Theorem.** (See [5].) *Let $B = \mathbf{k}^{[2]}$, where $\mathbf{k}$ is a field of characteristic zero, and let $D : B \to B$ be a nonzero locally nilpotent derivation. Then there exist u, v such that $B = \mathbf{k}[u, v]$ and $\ker D = \mathbf{k}[u]$. Moreover, given any such u, v we have $D = f(u)\frac{\partial}{\partial v}$ for some $f(u) \in \mathbf{k}[u]$.*

1.9. **Simple derivations.** Let B be a $\mathbf{k}$-domain of transcendence degree two.

1.9.1. **Definition.** A derivation $D : B \to B$ is $\mathbf{k}$-*simple* if it is locally nilpotent, irreducible and satisfies

$$\exists\, y \in B \quad \ker D = \mathbf{k}[\, Dy\,].$$

Note that if this is the case then $\ker D = \mathbf{k}^{[1]}$. Consequently:

1.9.2. *If B admits a $\mathbf{k}$-simple derivation then $B^* = \mathbf{k}^*$.*

1.9.3. **Lemma.** *Suppose that $\Delta \in \mathrm{LND}(B)$ is $\mathbf{k}$-simple. If $D \in \mathrm{LND}(B)$ is irreducible and $\ker(D) = \ker(\Delta)$, then $D = \lambda\Delta$ for some $\lambda \in \mathbf{k}^*$. Consequently, D is $\mathbf{k}$-simple.*

Proof. Let $A = \ker D = \ker \Delta$ and choose $x, y \in B$ such that $\Delta(y) = x$ and $A = \mathbf{k}[x]$. Note that $\operatorname{Frac} B = \mathbf{k}(x, y)$ (by 1.7) and consider the partial derivative $\partial = \partial/\partial y : \mathbf{k}(x, y) \to \mathbf{k}(x, y)$. Extending Δ and D to derivations $\tilde{\Delta}$ and $\tilde{D}$ of $\mathbf{k}(x, y)$,

$$\tilde{\Delta} = x\partial \quad \text{and} \quad \tilde{D} = D(y)\partial.$$

It follows that $D(y)\Delta = xD$. Since D is locally nilpotent and $x \in \ker D$, xD is locally nilpotent; so $D(y)\Delta$ is locally nilpotent and it follows that $D(y) \in \ker \Delta$ by part (6) of 1.6. Hence, x and $D(y)$ are two elements of the ideal

$$I = \big\{\, \alpha \in A \;\mid\; \alpha\partial(B) \subseteq B \,\big\}$$

of A. Observe that $1 \notin I$, for otherwise $\partial(B) \subseteq B$, so $\Delta(B) \subseteq xB$, so $x \in B^*$ (because Δ is irreducible), so $x \in A^*$, but this is false because $A = \mathbf{k}[x] = \mathbf{k}^{[1]}$.

Since x is a prime element of A and $x \in I \neq A$, we have $I = xA$, so $x \mid D(y)$ in A. Then $D = \lambda\Delta$ where $\lambda = D(y)/x \in A$. Now $D(B) \subseteq \lambda B$, so $\lambda \in B^*$ by irreducibility of D. Since $B^* = \mathbf{k}^*$, $\mathbf{k}[\, Dy\,] = \mathbf{k}[\, \lambda x\,] = \mathbf{k}[x]$ and we are done. $\square$

On the number of kernels

Regarding the cardinality of the set $\mathrm{KLND}(B)$, we have the following elementary fact:

1.10. **Proposition.** *Let B be a domain of characteristic zero and suppose that $\mathbf{k} \subset B$ is a field such that $\operatorname{trdeg}_{\mathbf{k}}(B) < \infty$. Then the cardinality of $\mathrm{KLND}(B)$ is either 0, 1 or $|\mathbf{k}|$.*

Proof. As a first step, we show:

(1) *Let B be a $\mathbb{Q}$-domain and suppose that A, A' are distinct elements of $\mathrm{KLND}(B)$. Then $|\,\mathrm{KLND}(B)| \geq |A \cap A'|$.*

Let A and A' be distinct elements of $\mathrm{KLND}(B)$. Let $D, D' \in \mathrm{LND}(B)$ be such that $\ker D = A$ and $\ker D' = A'$. We first consider the case where:

$$D(A') \subseteq A' \quad \text{and} \quad D'(A) \subseteq A. \tag{2}$$

Then it follows that

$$D \circ D' = D' \circ D. \tag{3}$$

Indeed, let $\delta : B \to B$ denote the derivation $D \circ D' - D' \circ D$. Then by assumption (2), we have $A \cup A' \subseteq \ker \delta$. Since each of A, A' is algebraically closed in B, and since B has transcendence degree one over each of A, A', it follows that B is algebraic over $\ker \delta$, so $\delta = 0$ and (3) is true.

For each $\lambda \in A$, let $\Delta_\lambda : B \to B$ denote the derivation $D' + \lambda D$. Then (3) immediately implies that $\Delta_\lambda \in \text{LND}(B)$, so we have a map

$$\begin{array}{rcl} A & \longrightarrow & \text{KLND}(B) \\ \lambda & \longmapsto & \ker(\Delta_\lambda). \end{array} \tag{4}$$

We claim that the map (4) is injective. Indeed, consider distinct elements λ_1, λ_2 of A. Then for each $x \in \ker(\Delta_{\lambda_1}) \cap \ker(\Delta_{\lambda_2})$ we have

$$D'(x) + \lambda_1 D(x) = 0 = D'(x) + \lambda_2 D(x),$$

from which we deduce that $D(x) = 0 = D'(x)$, i.e., $x \in A \cap A'$. So $\ker(\Delta_{\lambda_1}) \cap \ker(\Delta_{\lambda_2}) \subseteq A \cap A'$ and consequently the transcendence degree of B over $\ker(\Delta_{\lambda_1}) \cap \ker(\Delta_{\lambda_2})$ is strictly greater than one. It follows that $\ker(\Delta_{\lambda_1}) \neq \ker(\Delta_{\lambda_2})$, so the map (4) is injective. Thus (1) holds under extra assumption (2).

There remains the case where (2) does not hold; without loss of generality, let us assume that

$$D(A') \not\subseteq A'. \tag{5}$$

For each $\lambda \in A$, let $\varepsilon_\lambda : B \to B$ be the automorphism of B defined by

$$\varepsilon_\lambda(x) = \sum_{i=0}^{\infty} \frac{\lambda^i D^i(x)}{i!} \qquad (x \in B)$$

i.e., ε_λ is the exponential of λD. As is well-known,

$$\varepsilon_{\lambda_1} \circ \varepsilon_{\lambda_2} = \varepsilon_{\lambda_1 + \lambda_2} \text{ for all } \lambda_1, \lambda_2 \in A. \tag{6}$$

Since $\varepsilon_\lambda(A') = \ker\left(\varepsilon_\lambda \circ D' \circ \varepsilon_\lambda^{-1}\right) \in \text{KLND}(B)$, the assignment $\lambda \mapsto \varepsilon_\lambda(A')$ is a map from A to $\text{KLND}(B)$. We claim that the restriction

$$\begin{array}{rcl} A \cap A' & \longrightarrow & \text{KLND}(B) \\ \lambda & \longmapsto & \varepsilon_\lambda(A') \end{array} \tag{7}$$

is an injective map. We begin by showing that (5) implies:

$$\text{If } \lambda \in A \cap A' \text{ satisfies } \varepsilon_\lambda(A') \subseteq A', \text{ then } \lambda = 0. \tag{8}$$

To see this, consider $\lambda \in A \cap A'$ satisfying $\varepsilon_\lambda(A') \subseteq A'$. By (5), we may pick an $x \in A'$ such that $D(x) \notin A'$. Fix such an x and let n be such that $D^i(x) = 0$ for all $i > n$; consider the polynomial $f(T) \in B[T]$ defined by

$$f(T) = \sum_{i=0}^{\infty} D'\left(\tfrac{D^i(x)}{i!}\right) T^i = \sum_{i=0}^{n} D'\left(\tfrac{D^i(x)}{i!}\right) T^i$$

and note that $f(T)$ is not the zero polynomial since the coefficient of T in $f(T)$ is $D'(D(x)) \neq 0$. Then for each $k \in \mathbb{N}$

$$f(k\lambda) = \sum_{i=0}^{n} D'\big(\tfrac{D^i(x)}{i!}\big)\,(k\lambda)^i = D'\left(\sum_{i=0}^{n} \big(\tfrac{D^i(x)}{i!}\big)\,(k\lambda)^i\right) = D'\big(\varepsilon_{k\lambda}(x)\big) = 0,$$

where the last equality follows from $\varepsilon_{k\lambda}(x) = \varepsilon_\lambda^k(x) \in \varepsilon_\lambda^k(A') \subseteq A'$. Now $f(T)$ cannot have infinitely many roots, so $\lambda = 0$ and (8) is proved.

Now (6) and (8) imply that the map (7) is injective, so (1) holds in this case as well. So (1) is proved.

To complete the proof of the proposition, suppose that $\mathbf{k} \subset B$ is a field such that $\operatorname{trdeg}_{\mathbf{k}}(B) < \infty$. Assuming that $|\operatorname{KLND}(B)| > 1$, we show that $|\operatorname{KLND}(B)| = |\mathbf{k}|$.

Consider distinct elements A and A' of $\operatorname{KLND}(B)$. Since $\mathbf{k} \subseteq A \cap A'$ by part (1) of 1.6, we have $|\operatorname{KLND}(B)| \geq |\mathbf{k}|$ by (1).

Consider a finite subset $\{x_1, \dots, x_n\}$ of B such that B is algebraic over $\mathbf{k}[x_1, \dots, x_n]$. The map

$$\begin{array}{rcl} \operatorname{LND}(B) & \longrightarrow & B^n \\ D & \longmapsto & (Dx_1, \dots, Dx_n) \end{array}$$

is injective, so $|\operatorname{LND}(B)| \leq |B^n| = |B| = |\mathbf{k}|$. Since $D \mapsto \ker D$ is a surjection from $\operatorname{LND}(B)$ to $\operatorname{KLND}(B)$, we have $|\operatorname{KLND}(B)| \leq |\operatorname{LND}(B)|$, so we are done. $\square$

Remark. It is possible to have $|\operatorname{KLND}(B)| > |B|$ if we don't assume that B has finite transcendence degree over some field. For instance, let $\mathbf{k}$ be a field of characteristic zero and let $B = \mathbf{k}[V]$ be a polynomial ring, where V is a set of indeterminates satisfying $|V| \geq |\mathbf{k}|$ (thus $|V| = |B|$). Fix a well-order on the set V. For each subset S of V other than $\varnothing$ and V, define a $\mathbf{k}$-derivation $D_S : B \to B$ by

$$D_S(X) = \begin{cases} 0, & \text{if } X \in S \\ \min S, & \text{if } X \notin S. \end{cases}$$

Then one can verify that $D_S \in \operatorname{LND}(B)$. Since $\ker(D_S) \cap V = S$, it follows that $|\operatorname{KLND}(B)| = |2^B|$.

2. Danielewski surfaces

2.1. Definition. Given a $\mathbf{k}$-algebra B, let $\Gamma_{\mathbf{k}}(B)$ denote the (possibly empty) set of ordered triples $(x_1, x_2, y) \in B \times B \times B$ satisfying:

> *The $\mathbf{k}$-homomorphism $\mathbf{k}[X_1, X_2, Y] \to B$ defined by*
>
> $$X_1 \mapsto x_1,\ X_2 \mapsto x_2 \text{ and } Y \mapsto y$$
>
> *is surjective and has kernel equal to $(\varphi - X_1X_2)\mathbf{k}[X_1, X_2, Y]$ for some nonconstant polynomial in one variable $\varphi \in \mathbf{k}[Y]$.*

If $\Gamma_{\mathbf{k}}(B) \neq \varnothing$ then we say that $(B, \mathbf{k})$ is a *Danielewski surface.* If this is the case then B is a $\mathbf{k}$-domain and $\operatorname{trdeg}_{\mathbf{k}}(B) = 2$.

Remark. The term "Danielewski surface" usually refers to hypersurfaces of $\mathbb{A}^3$ given by an equation of the form $xy = \varphi(z)$, or sometimes $x^n y = \varphi(z)$, because such surfaces were studied by Danielewski in connection with the cancellation problem (see [2]).

Remarks. Suppose that $(B, \mathbf{k})$ is a Danielewski surface and let $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$.

(1) Any two elements of $\{x_1, x_2, y\}$ are algebraically independent over $\mathbf{k}$.
(2) Once $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ is chosen, $\varphi \in \mathbf{k}[Y] \setminus \mathbf{k}$ is uniquely determined by the condition $\varphi(y) = x_1 x_2$.

2.2. **Lemma.** *Let X_1, X_2, Y be indeterminates over $\mathbf{k}$, let L be a field containing $\mathbf{k}$ and let $\pi : \mathbf{k}[X_1, X_2, Y] \to L$ be a $\mathbf{k}$-homomorphism with kernel $(\varphi - X_1X_2)\mathbf{k}[X_1, X_2, Y]$, where φ is some element of $\mathbf{k}[Y] \setminus \mathbf{k}$. Write $x_1 = \pi(X_1)$, $x_2 = \pi(X_2)$ and $y = \pi(Y)$, then the following hold:*

(a) *For each element β of the subring $\mathbf{k}[x_1, x_2, y]$ of L, there exists a unique $F \in \mathbf{k}[X_1, X_2, Y]$ satisfying $F(x_1, x_2, y) = \beta$ and $\deg_Y(F) < \deg_Y(\varphi)$.*
(b) $\mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y] = \mathbf{k}[x_1, x_2, y]$.

Proof. If we view $\Phi = \varphi - X_1X_2$ as a polynomial in Y with coefficients in $\mathbf{k}[X_1, X_2]$, then the leading coefficient of Φ belongs to $\mathbf{k}^*$. Thus assertion (a) follows from a straightforward application of the division algorithm in $\mathbf{k}[X_1, X_2][Y]$.

To prove (b), it suffices to show that $\mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y] \subseteq \mathbf{k}[x_1, x_2, y]$. Let $\beta \in \mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y]$, then

$$\beta = \frac{F(x_1, x_2, y)}{f(x_1)} = \frac{G(x_1, x_2, y)}{g(x_2)}$$

for some $F, G \in \mathbf{k}[X_1, X_2, Y]$, $f \in \mathbf{k}[X_1] \setminus \{0\}$ and $g \in \mathbf{k}[X_2] \setminus \{0\}$. By (a), we may arrange that $\deg_Y(F) < \deg_Y(\varphi)$ and $\deg_Y(G) < \deg_Y(\varphi)$. Then $g(x_2)F(x_1, x_2, y) = f(x_1)G(x_1, x_2, y)$ and the uniqueness part of (a) imply that $gF = fG$ in $\mathbf{k}[X_1, X_2, Y]$, so $f \mid F$ in $\mathbf{k}[X_1, X_2, Y]$. Let $Q \in \mathbf{k}[X_1, X_2, Y]$ be such that $F = Qf$, then $\beta = Q(x_1, x_2, y) \in \mathbf{k}[x_1, x_2, y]$. $\square$

The following result gathers the most basic properties of Danielewski surfaces. See 1.9.1 for $\mathbf{k}$-simple derivations.

2.3. **Proposition.** *Let $(B, \mathbf{k})$ be a Danielewski surface, fix an element $\gamma = (x_1, x_2, y)$ of $\Gamma_{\mathbf{k}}(B)$ and let φ be the unique element of $\mathbf{k}[Y] \setminus \mathbf{k}$ satisfying $\varphi(y) = x_1x_2$.*

(a) *B is a normal $\mathbf{k}$-domain and $B^* = \mathbf{k}^*$.*
(b) $B = \mathbf{k}^{[2]} \iff \deg_Y(\varphi) = 1$.
(c) *B is a UFD $\iff$ φ is irreducible in $\mathbf{k}[Y]$.*
(d) *For each $i = 1, 2$, there exists a unique $\mathbf{k}$-derivation $D_i^\gamma : B \to B$ satisfying $D_i^\gamma(x_i) = 0$ and $D_i^\gamma(y) = x_i$. Moreover, $\ker D_i^\gamma = \mathbf{k}[x_i]$ and D_i^γ is a $\mathbf{k}$-simple derivation of B.*

Proof. We shall prove assertions (a), (d), (c) and (b), in this order. It is immediate that B is a $\mathbf{k}$-domain and that

(9) Any two elements of $\{x_1, x_2, y\}$ are algebraically independent over $\mathbf{k}$.

By 2.2, there holds
$$\mathbf{k}[x_1, x_2, y] = \mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y]$$
where each $\mathbf{k}(x_i)[y] = \mathbf{k}(x_i)^{[1]}$ is a normal domain, so

(10) $$B \text{ is normal.}$$

Let us also record that $B_{x_1} = \mathbf{k}[x_1, 1/x_1, x_2, y] = \mathbf{k}[x_1, 1/x_1, y] = \mathbf{k}[x_1, 1/x_1]^{[1]}$ and similarly for B_{x_2}, i.e.,

(11) $$\text{For each } i = 1, 2, \quad B_{x_i} = \mathbf{k}[x_i, 1/x_i, y] = \mathbf{k}[x_i, 1/x_i]^{[1]}.$$

Suppose that $u \in B^*$. Then u is a unit of each of B_{x_1} and B_{x_2}, so (11) implies that $u \in \mathbf{k}[x_1, 1/x_1] \cap \mathbf{k}[x_2, 1/x_2]$. Since x_1, x_2 are algebraically independent over $\mathbf{k}$ by (9), we have $\mathbf{k}(x_1) \cap \mathbf{k}(x_2) = \mathbf{k}$ and $u \in \mathbf{k}$. This shows that $B^* = \mathbf{k}^*$. Together with (10), this proves assertion (a).

We shall now prove assertion (d). Let $(i, j) = (1, 2)$ or $(2, 1)$. Let $\delta_i : \mathbf{k}[X_1, X_2, Y] \to \mathbf{k}[X_1, X_2, Y]$ be the $\mathbf{k}$-derivation given by $\delta_i(X_i) = 0$, $\delta_i(Y) = X_i$ and $\delta_i(X_j) = \varphi'(Y)$. Then δ_i is triangular, hence locally nilpotent, and clearly $\delta_i(\Phi) = 0$, where $\Phi = \varphi - X_1X_2$. So we may define a locally nilpotent derivation $D_i : B \to B$ by taking $\delta_i \pmod{\Phi}$. Then $D_i(x_i) = 0$ and $D_i(y) = x_i$, thus proving the existence part of assertion (d). If $D : B \to B$ is any $\mathbf{k}$-derivation satisfying $D(x_i) = 0$ and $D(y) = x_i$, then $x_iD(x_j) = D(x_1x_2) = D(\varphi(y)) = \varphi'(y)x_i$, so $D(x_j) = \varphi'(y)$, which proves uniqueness of D_i.

It is easy to see that the kernel of the localization $B_{x_1} \to B_{x_1}$ of D_1 is $\mathbf{k}[x_1, 1/x_1]$, so $\ker D_1 = B \cap \mathbf{k}[x_1, 1/x_1]$. Consider an element β of $B \cap \mathbf{k}[x_1, 1/x_1]$. By 2.2,
$$\beta = F(x_1, x_2, y), \quad \text{for some } F \in \mathbf{k}[X_1, X_2, Y] \text{ such that } \deg_Y(F) < \deg_Y(\varphi).$$
Since $\beta \in \mathbf{k}[x_1, 1/x_1]$, there exists $m > 0$ such that $x_1^m F(x_1, x_2, y) \in \mathbf{k}[x_1]$, i.e.,
$$x_1^m F(x_1, x_2, y) = f(x_1), \quad \text{for some } f \in \mathbf{k}[X_1].$$
Then 2.2 implies that $X_1^m F = f$, so $X_1^m F \in \mathbf{k}[X_1]$, so $F \in \mathbf{k}[X_1]$ and $\beta \in \mathbf{k}[x_1]$. This shows that $\ker D_1 = \mathbf{k}[x_1] = \mathbf{k}^{[1]}$ (and by symmetry $\ker D_2 = \mathbf{k}[x_2] = \mathbf{k}^{[1]}$).

Next, we show that D_1 is irreducible. Let $g \in B$ be such that $D(B) \subseteq gB$. Since $D_1(y) = x_1$, we have $g \mid x_1$ in B, so $g \in \mathbf{k}[x_1]$ because $\mathbf{k}[x_1] = \ker D_1$ is inert in B. Hence, $g = G(x_1)$ for some $G \in \mathbf{k}[X_1]$. On the other hand, $D_1(x_2) = \varphi'(y)$, so $G(x_1) \mid \varphi'(y)$ in B and 2.2 allows us to write
$$G(x_1)F(x_1, x_2, y) = \varphi'(y),$$
where $F \in \mathbf{k}[X_1, X_2, Y]$ and $\deg_Y(F) < \deg_Y(\varphi)$. Now 2.2 implies that $GF = \varphi' \in \mathbf{k}[Y] \setminus \{0\}$, so $G \in \mathbf{k}[X_1] \cap \mathbf{k}[Y] = \mathbf{k}$. Hence D_1 is irreducible (and so is D_2 by symmetry). Thus assertion (d) is true.

Next, we prove assertion (c). Since x_1 is an irreducible element of $\mathbf{k}[x_1]$ and $\mathbf{k}[x_1] = \ker D_1$ is an inert subring of B, x_1 is an irreducible element of B. On the other hand,

$$
\begin{aligned}
(12)\quad B/x_1B &\cong \mathbf{k}[X_1, X_2, Y]/(X_1, \varphi - X_1X_2) \cong \mathbf{k}[X_1, X_2, Y]/(X_1, \varphi) \\
&\cong \mathbf{k}[X_2, Y]/(\varphi) \cong \Big(\mathbf{k}[Y]/(\varphi)\Big)^{[1]}
\end{aligned}
$$

shows that x_1 is a prime element of B if and only if φ is a prime element of $\mathbf{k}[Y]$. In particular, if B is a UFD then x_1 is prime in B, so φ is prime in $\mathbf{k}[Y]$.

Conversely, if φ is prime in $\mathbf{k}[Y]$ then x_1 is prime in B and, by (11), B_{x_1} is a UFD; so B is a UFD and assertion (c) is true.

For (b), note that if $\deg_Y(\varphi) = 1$ then it is obvious that $B = \mathbf{k}[x_1, x_2] = \mathbf{k}^{[2]}$. Conversely, assume that $B = \mathbf{k}^{[2]}$. By Rentschler's Theorem 1.8, $B = A^{[1]}$ for any $A \in \text{KLND}(B)$; in particular $B = \mathbf{k}[x_1]^{[1]}$, so $B/x_1B = \mathbf{k}^{[1]}$. By (12), $\deg_Y(\varphi) = 1$.

This completes the proof of 2.3. □

We also record the following simple fact:

2.4. **Lemma.** *Suppose that $(B, \mathbf{k})$ is a Danielewski surface and let $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$. Then x_1x_2 is a generator of the ideal $\mathbf{k}[y] \cap x_1B$ of $\mathbf{k}[y]$.*

Proof. We have $x_1x_2 = \varphi(y)$ for some nonconstant polynomial $\varphi \in \mathbf{k}[Y]$. Let $n = \deg_Y(\varphi)$. Given $\xi \in \mathbf{k}[y] \cap x_1B$, we may write $\xi = \psi(y)$, where $\psi \in \mathbf{k}[Y]$; by the division algorithm, $\psi = q\varphi + \rho$, with $q, \rho \in \mathbf{k}[Y]$ and $\deg \rho < n$. We have

$$(13)\qquad \rho(y) = \psi(y) - q(y)\varphi(y) = \xi - q(y)x_1x_2 \in x_1B,$$

so $\rho(y) = x_1F(x_1, x_2, y)$ for some $F \in \mathbf{k}[X_1, X_2, Y]$ such that $\deg_Y(F) < n$. Then $\rho = X_1F$ by 2.2, so $X_1 \mid \rho$ in $\mathbf{k}[X_1, X_2, Y]$, which implies that $\rho = 0$. Then (13) yields $\xi = q(y)x_1x_2 \in x_1x_2\mathbf{k}[y]$ and we are done. □

Remark. Applying 2.4 to $(x_2, x_1, y) \in \Gamma_{\mathbf{k}}(B)$ implies that x_1x_2 generates the ideal $\mathbf{k}[y] \cap x_2B$ of $\mathbf{k}[y]$. So: *The ideals $\mathbf{k}[y] \cap x_1B$ and $\mathbf{k}[y] \cap x_2B$ of $\mathbf{k}[y]$ are equal.*

Two characterizations of Danielewski surfaces

Results 2.5, 2.6 and 2.6.2 characterize Danielewski surfaces in terms of locally nilpotent derivations.

2.5. **Theorem.** *Let B be a $\mathbf{k}$-domain, let $y \in B$ and let $D_1, D_2 : B \to B$ be locally nilpotent derivations. Suppose that (y, D_1, D_2) satisfies:*

(i) $\ker D_1 \neq \ker D_2$

(ii) *for each $i = 1, 2$, $\ker D_i = \mathbf{k}^{[1]}$ and $D_i(y) \in \ker(D_i) \setminus \{0\}$.*

Then $(B, \mathbf{k})$ is a Danielewski surface. Moreover, if D_1 and D_2 are irreducible then exactly one of the following holds:

(2.5-1) *For each $i = 1, 2$, $D_i(y) \in \mathbf{k}^*$ and $B = (\ker D_i)^{[1]} = \mathbf{k}^{[2]}$.*

(2.5-2) *Let* $x_i = D_i(y)$ $(i = 1, 2)$, *then*

$$\ker D_1 = \mathbf{k}[x_1], \quad \ker D_2 = \mathbf{k}[x_2] \quad \textit{and} \quad (x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B).$$

For the proof of 2.5 we need the following simple observation, whose proof we leave to the reader:

2.5.1. *Let* X, Y *be indeterminates over the field* $\mathbf{k}$ *and let* $f \in \mathbf{k}[Y] \setminus \mathbf{k}$. *Then*

$$\mathbf{k}(X)[Y] \cap \mathbf{k}(X + f)[Y] = \mathbf{k}[X, Y],$$

where the intersection is taken in $\mathbf{k}(X, Y)$.

Proof of 2.5. Note that assumption (ii) and 1.7 imply that B has transcendence degree two over $\mathbf{k}$. More precisely, write $\ker(D_i) = \mathbf{k}[t_i]$ for each $i = 1, 2$. Since y is a preslice of D_i,

$$B \subseteq \mathbf{k}(t_i) \otimes_{\mathbf{k}[t_i]} B = \mathbf{k}(t_i)[y] = \mathbf{k}(t_i)^{[1]}. \tag{14}$$

In particular $\mathbf{k}(y, t_1) = \operatorname{Frac} B = \mathbf{k}(y, t_2)$, so (for each i) t_i, y are algebraically independent over $\mathbf{k}$. Since $\mathbf{k}(y, t_1) = \mathbf{k}(y, t_2)$,

$$t_2 = \frac{a_2 t_1 + b}{c t_1 + a_1} \tag{15}$$

where $a_1, a_2, b, c \in \mathbf{k}(y)$ and $a_1 a_2 - bc \neq 0$; in fact, we may arrange that

$$a_1, a_2, b, c \in \mathbf{k}[y], \quad a_1 a_2 - bc \neq 0 \ \text{ and } \ \gcd_{\mathbf{k}[y]}(a_1, a_2, b, c) = 1. \tag{16}$$

Consider the subring $R = \mathbf{k}[t_1, y]$ of B and note that $R = \mathbf{k}^{[2]}$. We claim:

$$a_2 t_1 + b \text{ and } c t_1 + a_1 \text{ are relatively prime in } R. \tag{17}$$

Indeed, let $\delta = \gcd_R(a_2 t_1 + b, c t_1 + a_1)$. If $\deg_{t_1}(\delta) > 0$ then we easily obtain a contradiction with the condition $a_1 a_2 - bc \neq 0$. So $\deg_{t_1}(\delta) = 0$, i.e., $\delta \in \mathbf{k}[y]$. It follows that δ is a common divisor of a_1, a_2, b, c, so (17) is a consequence of (16).

Since $B \subseteq \mathbf{k}(t_1)[y]$ by (14), we have $t_2 \in \mathbf{k}(t_1)[y]$ so

$$t_2 = f/\zeta, \qquad \text{where } f \in R \text{ and } \zeta \in \mathbf{k}[t_1] \setminus \{0\}.$$

This and (15) give:

$$(a_2 t_1 + b)\zeta = (c t_1 + a_1) f \qquad \text{(equation in } R\text{)},$$

so $(c t_1 + a_1) \mid (a_2 t_1 + b)\zeta$ in R; in view of (17), we obtain $(c t_1 + a_1) \mid \zeta$ in R. Since $\zeta \in \mathbf{k}[t_1] \setminus \{0\}$ and $\mathbf{k}[t_1]$ is inert in R (because $R = \mathbf{k}[t_1]^{[1]}$), it follows that $c t_1 + a_1 \in \mathbf{k}[t_1]$. Hence,

$$a_1, c \in \mathbf{k}.$$

Solving (15) for t_1 gives

$$t_1 = \frac{-a_1 t_2 + b}{c t_2 - a_2} \tag{15'}$$

so, by symmetry, the proof that $a_1, c \in \mathbf{k}$ shows that $-a_2, c \in \mathbf{k}$. Hence,

$$a_1, a_2, c \in \mathbf{k}. \tag{18}$$

Case $c = 0$. Since $a_1a_2 - bc \neq 0$, it follows that $a_1a_2 \neq 0$, so $a_1, a_2 \in \mathbf{k}^*$ by (18). Taking this into account, (15) gives

$$t_2 = \alpha t_1 + \beta, \quad \text{where } \alpha \in \mathbf{k}^* \text{ and } \beta \in \mathbf{k}[y]. \tag{19}$$

Note that assumption (i) can be written as $\mathbf{k}[t_1] \neq \mathbf{k}[\alpha t_1 + \beta]$, so $\beta \notin \mathbf{k}$. By (14) and (19) we have

$$\mathbf{k}[t_1, y] \subseteq B \subseteq \mathbf{k}(t_1)[y] \cap \mathbf{k}(\alpha t_1 + \beta)[y],$$

so 2.5.1 yields $B = \mathbf{k}[t_1, y]$. In particular, $(B, \mathbf{k})$ is a Danielewski surface.

Since $\mathbf{k}[t_1, y] = \mathbf{k}[t_2, y]$ by (19), we also have $B = \mathbf{k}[t_2, y]$. Now D_i is a derivation of $\mathbf{k}[t_i, y]$ with kernel $\mathbf{k}[t_i]$, so $D_i = (D_i y)\,\partial/\partial y$ and in particular $D_i(B) \subseteq (D_i y)B$. Now if (for each i) D_i is assumed to be irreducible, we have $D_i y \in \mathbf{k}^*$ and condition (2.5-1) holds.

Case $c \neq 0$. Then (18) implies that $c \in \mathbf{k}^*$ and $a_1, a_2 \in \mathbf{k}$.

Define $x_1 = ct_1 + a_1$ and $x_2 = ct_2 - a_2$ (so x_1, x_2 are not defined as in the statement). We now show that x_1, x_2 satisfy the following three conditions:

(2.5-2-a) $\ker D_1 = \mathbf{k}[x_1]$ and $\ker D_2 = \mathbf{k}[x_2]$

(2.5-2-b) The $\mathbf{k}$-homomorphism $\mathbf{k}[X_1, X_2, Y] \to B$ defined by $X_1 \mapsto x_1$, $X_2 \mapsto x_2$, $Y \mapsto y$ is surjective and has kernel $(\varphi - X_1X_2)\mathbf{k}[X_1, X_2, Y]$ for some nonconstant polynomial $\varphi \in \mathbf{k}[Y]$.

(2.5-2-c) If D_i is irreducible then $D_i(y) = \lambda_i x_i$ for some $\lambda_i \in \mathbf{k}^*$.

From the definition of x_1, x_2 together with $c \in \mathbf{k}^*$ and $a_1, a_2 \in \mathbf{k}$, we get $\mathbf{k}[x_i] = \mathbf{k}[t_i]$, so (2.5-2-a) holds. Clearly, we have $x_1, x_2 \notin \mathbf{k}$, so

$$x_1x_2 \notin \mathbf{k}, \tag{20}$$

for otherwise $x_1, x_2 \in B^* = (\ker D_1)^* = \mathbf{k}[x_1]^* = \mathbf{k}^*$, which is not the case. Using (15'), we get

$$\begin{aligned} x_1x_2 = (ct_1 + a_1)(ct_2 - a_2) &= \left[c\left(\frac{-a_1t_2 + b}{ct_2 - a_2}\right) + a_1\right](ct_2 - a_2) \\ &= c(-a_1t_2 + b) + a_1(ct_2 - a_2) = bc - a_1a_2, \end{aligned}$$

so $x_1x_2 \in \mathbf{k}[y]$; thus $x_1x_2 \in \mathbf{k}[y] \setminus \mathbf{k}$ by (20) and consequently:

(21) For some nonconstant polynomial $\varphi \in \mathbf{k}[Y]$, we have $\varphi(y) = x_1x_2$.

Let $\pi : \mathbf{k}[X_1, X_2, Y] \to B$ be the $\mathbf{k}$-homomorphism defined by $\pi(X_1) = x_1$, $\pi(X_2) = x_2$ and $\pi(Y) = y$. The image of π is the affine $\mathbf{k}$-domain $\mathbf{k}[x_1, x_2, y]$, whose transcendence degree over $\mathbf{k}$ is 2; consequently $\ker \pi$ is a height one prime ideal of $\mathbf{k}[X_1, X_2, Y]$; since $(\varphi - X_1X_2)\mathbf{k}[X_1, X_2, Y]$ is a prime ideal and, by (21), is contained in $\ker \pi$, we have:

$$\ker \pi = (\varphi - X_1X_2)\mathbf{k}[X_1, X_2, Y]. \tag{22}$$

Since we have

$$\mathbf{k}[x_1, x_2, y] \subseteq B \subseteq \mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y]$$

by (14), and since

$$\mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y] = \mathbf{k}[x_1, x_2, y]$$

by 2.2, we obtain:

$$B = \mathbf{k}[x_1, x_2, y]. \tag{23}$$

Thus $\pi : \mathbf{k}[X_1, X_2, Y] \to B$ is surjective. Together with (22), this implies that (2.5-2-b) holds.

Hence, $(B, \mathbf{k})$ is a Danielewski surface and $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$.

Assume that D_1 and D_2 are irreducible. Write $\gamma = (x_1, x_2, y)$ and consider the D_i^γ of 2.3. For each $i \in \{1, 2\}$, applying 1.9.3 to (D_i, D_i^γ) gives $D_i = \lambda_i D_i^\gamma$ for some $\lambda_i \in \mathbf{k}^*$. Thus $D_i(y) = \lambda_i D_i^\gamma(y) = \lambda_i x_i$, so (2.5-2-c) holds and the condition (2.5-2) of the theorem is satisfied.

In the two cases ($c = 0$ or $c \neq 0$), we proved that $(B, \mathbf{k})$ is a Danielewski surface. Assuming that D_1 and D_2 are irreducible, we also proved the two implications $c = 0 \Rightarrow$ (2.5-1) and $c \neq 0 \Rightarrow$ (2.5-2); so exactly one of (2.5-1), (2.5-2) is true and the proof of 2.5 is complete. □

In the special case where B is factorial, we have another characterization of Danielewski surfaces (see also 2.6.2):

2.6. **Theorem.** *Let B be a factorial $\mathbf{k}$-domain of transcendence degree 2. If B admits a $\mathbf{k}$-simple derivation, then $(B, \mathbf{k})$ is a Danielewski surface.*

2.6.1. **Example.** Let $B = \mathbf{k}[x, y, y^2/x, y^3/x^2]$ where x, y are indeterminates over $\mathbf{k}$. Then $D = x\,\partial/\partial y : B \to B$ is a $\mathbf{k}$-simple derivation but $(B, \mathbf{k})$ is not a Danielewski surface. Note that B is normal but not factorial. (We leave it to the reader to verify that $\ker D = \mathbf{k}[x]$, that D is irreducible and that B is not a Danielewski surface.)

Proof of 2.6. Consider a $\mathbf{k}$-simple derivation $D_1 : B \to B$, i.e., an irreducible $D_1 \in \mathrm{LND}(B)$ satisfying $\mathbf{k}[D_1(y)] = \ker D_1$ for some $y \in B$. Let $x_1 = D_1(y)$, then

$$\ker D_1 = \mathbf{k}[x_1] = \mathbf{k}^{[1]}.$$

In particular, x_1 is a prime element of $\ker D_1$; since B is factorial and $\ker D_1$ is inert in B,

$$x_1 \text{ is a prime element of } B. \tag{24}$$

Observe that $D_1(y) = x_1 \in \ker(D_1) \setminus \{0\}$ implies that $B_{x_1} = (\ker D_1)_{x_1}[y] = \mathbf{k}[x_1, 1/x_1, y]$, so

$$B \subseteq \mathbf{k}[x_1, 1/x_1, y]. \tag{25}$$

It follows from (24) that $\mathfrak{m} = \mathbf{k}[y] \cap x_1 B$ is a prime ideal of $\mathbf{k}[y]$. We claim that $\mathfrak{m}$ is nonzero. To see this, choose $\beta \in B$ such that $D_1(\beta) \notin x_1 B$ (this is possible because

D_1 is irreducible). It is clear that D_1 maps $\mathbf{k}[x_1, y]$ in x_1B, so $\beta \notin \mathbf{k}[x_1, y]$. In view of (25), we may write

$$\beta = \frac{F(x_1, y)}{x_1^n}, \quad \text{for some } F \in \mathbf{k}[X_1, Y] \text{ and } n \geq 0.$$

Note that we must have $n > 0$, because $\beta \notin \mathbf{k}[x_1, y]$. Assume that n is minimal, i.e., $F(0, Y) \neq 0$. Then $F(x_1, y) = x_1^n \beta \in x_1B$, so $F(0, y) \in x_1B$ and consequently $F(0, y) \in \mathfrak{m}$. Since $F(0, Y) \neq 0$, we have $F(0, y) \neq 0$ (because, by (25), y is transcendental over $\mathbf{k}$), so $\mathfrak{m} \neq \{0\}$. Thus $\mathbf{k}[y] \cap x_1B$ is a maximal ideal of $\mathbf{k}[y]$ and

(26) $\qquad \mathbf{k}[y] \cap x_1B = \varphi(y)\mathbf{k}[y] \quad$ for some irreducible element φ of $\mathbf{k}[Y]$.

Let $x_2 = \varphi(y)/x_1 \in B$. Let $\pi : \mathbf{k}[X_1, X_2, Y] \to B$ be the homomorphism of $\mathbf{k}$-algebras defined by $\pi(X_1) = x_1$, $\pi(X_2) = x_2$ and $\pi(Y) = y$. Since $\operatorname{im}(\pi) = \mathbf{k}[x_1, x_2, y]$ contains $\mathbf{k}[x_1, y]$, which is birational to B by (25), $\operatorname{im}(\pi)$ has transcendence degree 2 over $\mathbf{k}$. It follows that $\ker \pi$ is a height one prime ideal of $\mathbf{k}[X_1, X_2, Y]$. It is clear that $\Phi = \varphi - X_1X_2$ is an irreducible element of $\mathbf{k}[X_1, X_2, Y]$ and that $\Phi \in \ker \pi$, so

$$\ker \pi = \Phi\mathbf{k}[X_1, X_2, Y].$$

Let us observe that

Any two elements of $\{x_1, x_2, y\}$ are algebraically independent over $\mathbf{k}$.

In fact, (25) implies that x_1, y are algebraically independent over $\mathbf{k}$ and, from $x_1x_2 = \varphi(y)$, one easily deduces that each pair, x_2, y and x_1, x_2, is algebraically independent.

Let $K = \mathbf{k}(x_2)$ and $B_K = K \otimes_{\mathbf{k}[x_2]} B$. Note that $x_1 \in K[y]$, since $x_2 \in K^*$ and $x_1x_2 \in \mathbf{k}[y] \subseteq K[y]$. We claim:

(27) $\qquad x_1$ is a prime element of B_K and also of $K[y]$.

Begin with the observation that $D_1(x_2) = \varphi'(y) \notin \varphi(y)\mathbf{k}[y] = \mathbf{k}[y] \cap x_1B$; since $D_1(x_2) \in \mathbf{k}[y]$, we get $D_1(x_2) \notin x_1B$. So, if $\overline{D}_1 : B/x_1B \to B/x_1B$ denotes $D_1 \pmod{x_1B}$, we have $x_2 + x_1B \notin \ker(\overline{D}_1)$, so $x_2 + x_1B$ is transcendental over $\mathbf{k}$ and consequently $\mathbf{k}[x_2] \setminus \{0\} \cap x_1B = \varnothing$. This implies that

(28) $\qquad x_1 \notin B_K^*$ and $x_1 \notin K[y]^*$.

Since x_1 is prime in B and $x_1 \notin B_K^*$, x_1 is a prime element of B_K.

On the other hand, $\varphi(y)$ is prime in $\mathbf{k}[y] \implies \varphi(y)$ is prime in $\mathbf{k}[x_2, y] = \mathbf{k}[y]^{[1]}$ $\implies \varphi(y)$ is either prime or a unit in $\mathbf{k}(x_2)[y] = K[y] \implies x_1$ is either prime or a unit in $K[y]$ (because x_1 and $\varphi(y)$ are associates in $K[y]$). By (28), x_1 is prime in $K[y]$ and (27) is proved.

Next, we show that

(29) $$B_K = K[y].$$

In fact, (25) implies that $B_K \subseteq K[y]_{x_1}$, so

(30) $$K[y] \subseteq B_K \subseteq K[y]_{x_1}.$$

By (27), $K[y] \cap x_1 B_K$ is a prime ideal of $K[y]$ and $x_1 K[y]$ is a maximal ideal of $K[y]$; since $x_1 K[y] \subseteq K[y] \cap x_1 B_K$, we have $K[y] \cap x_1 B_K = x_1 K[y]$ and by induction on n we deduce:

$$\forall\, n \in \mathbb{N}, \quad K[y] \cap x_1^n B_K = x_1^n K[y]. \tag{31}$$

Then (29) follows from (30) and (31).

In particular, (29) implies that $B \subseteq K[y] = \mathbf{k}(x_2)[y]$, so (25) gives

$$\mathbf{k}[x_1, x_2, y] \subseteq B \subseteq \mathbf{k}(x_1)[y] \cap \mathbf{k}(x_2)[y]$$

and we obtain $B = \mathbf{k}[x_1, x_2, y]$ by 2.2, i.e., π is surjective. We showed that $(B, \mathbf{k})$ is a Danielewski surface, which completes the proof of 2.6. □

Note the following reformulation of 2.6:

2.6.2. **Corollary.** *Let B be a factorial $\mathbf{k}$-domain and suppose that $D \in \text{LND}(B)$ and $y \in B$ satisfy:*

$$\ker D = \mathbf{k}[Dy] = \mathbf{k}^{[1]}.$$

Then $(B, \mathbf{k})$ is a Danielewski surface and the following hold:

1. *If D is irreducible then there exists $x \in B$ such that $(Dy, x, y) \in \Gamma_{\mathbf{k}}(B)$.*
2. *If D is not irreducible then $B = \mathbf{k}[y, Dy] = \mathbf{k}^{[2]}$.*

Proof. The hypotheses imply that $\operatorname{trdeg}_{\mathbf{k}} B = 2$. If D is irreducible then it is $\mathbf{k}$-simple, so the hypothesis of 2.6 is satisfied; then the proof of 2.6 actually shows that $(Dy, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ for some $x_2 \in B$, so assertion (1) is true. If D is not irreducible then $D = D(y)D_0$ for some $D_0 \in \text{LND}(B)$, because $D(y)$ is a prime element of B; thus $D_0(y) = 1$ and assertion (2) follows from part (3) of 1.6. □

The Transitivity Theorem and some consequences

2.7. Assume that $(B, \mathbf{k})$ is a Danielewski surface, fix an element $\gamma = (x_1, x_2, y)$ of $\Gamma_{\mathbf{k}}(B)$ and let φ be the unique element of $\mathbf{k}[Y] \setminus \mathbf{k}$ satisfying $\varphi(y) = x_1 x_2$. Thus

$$B \cong \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1 X_2).$$

2.7.1. **Notations** ([1]-2.2).

- Define $\tau \in \operatorname{Aut}_{\mathbf{k}}(B)$ by $\tau(x_1) = x_2$, $\tau(x_2) = x_1$ and $\tau(y) = y$.
- For each $f \in \mathbf{k}[x_1]$, define $\Delta_f \in \operatorname{Aut}_{\mathbf{k}}(B)$ by $\Delta_f(x_1) = x_1$ and $\Delta_f(y) = y + x_1 f$. (Then $\Delta_f(x_2) = x_1^{-1}\varphi(y + x_1 f)$.)
- Let G_γ be the subgroup of $\operatorname{Aut}_{\mathbf{k}}(B)$ generated by $\{\tau\} \cup \{\, \Delta_f \mid f \in \mathbf{k}[x_1] \,\}$.
- Given $f \in \mathbf{k}[x_1]$, also define $\delta_f = \Delta_f \circ \tau \in G$. Note that $\delta_0 = \tau$ and that G_γ is generated by the set $\{\, \delta_f \mid f \in \mathbf{k}[x_1] \,\}$.

The assignment $(\alpha, A) \longmapsto \alpha(A)$, where $\alpha \in \operatorname{Aut}_{\mathbf{k}}(B)$ and $A \in \text{KLND}(B)$, is a left-action of the group $\operatorname{Aut}_{\mathbf{k}}(B)$ on the set $\text{KLND}(B)$. We restrict this action to the subgroup G_γ of $\operatorname{Aut}_{\mathbf{k}}(B)$ defined in 2.7.1. Then the main result of [1] is:

2.7.2. **Transitivity Theorem.** *The action of G_γ on $\text{KLND}(B)$ is transitive.*

Results 2.8, 2.9 and 2.10 are consequences of the Transitivity Theorem.

2.8. **Lemma.** *Suppose that $(B, \mathbf{k})$ is a Danielewski surface and consider an irreducible $D \in \text{LND}(B)$. Then D is $\mathbf{k}$-simple, i.e., $\exists\,(x,y) \in B \times B$ such that $Dy = x$ and* $\ker D = \mathbf{k}[x]$. *Moreover, for each such pair (x,y) we have $(x, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ for some $x_2 \in B$.*

Proof. **Case 1:** $B = \mathbf{k}^{[2]}$. Rentschler's Theorem 1.8 gives a pair (x', s) such that $B = \mathbf{k}[x', s]$, $\ker D = \mathbf{k}[x']$ and $D = u\,\partial/\partial s$ for some $u \in \mathbf{k}[x']$. Since D is irreducible, we have $u \in \mathbf{k}^*$ and in fact we may choose s in such a way that $Ds = 1$. Then $y' = x's$ satisfies $D(y') = x'$, showing that D is $\mathbf{k}$-simple.

Now consider any $x, y \in B$ such that $Dy = x$ and $\ker D = \mathbf{k}[x]$. Since $B = \mathbf{k}[x', s]$ and $\mathbf{k}[x] = \mathbf{k}[x']$, $B = \mathbf{k}[x, s]$ (where $Ds = 1$, as before). Then $D(y - xs) = 0$, so we may write $y - xs = a + xf(x)$ for some $a \in \mathbf{k}$ and $f(x) \in \mathbf{k}[x]$. Define a $\mathbf{k}$-homomorphism $\pi : \mathbf{k}[X_1, X_2, Y] \to B$ by $\pi(X_1) = x$, $\pi(X_2) = s + f(x)$ and $\pi(Y) = y$. Then π is surjective and $Y - a - X_1X_2$ belongs to $\ker \pi$, so $(x, s + f(x), y) \in \Gamma_{\mathbf{k}}(B)$.

Case 2: $B \neq \mathbf{k}^{[2]}$. Pick any $(x_1^{(1)}, x_2^{(1)}, y^{(1)}) \in \Gamma_{\mathbf{k}}(B)$.

Given $(x_1^{(j)}, x_2^{(j)}, y^{(j)}) \in \Gamma_{\mathbf{k}}(B)$, let $D_1^{(j)} \in \text{LND}(B)$ be the $\mathbf{k}$-simple derivation given by 2.3, i.e., $\ker D_1^{(j)} = \mathbf{k}[x_1^{(j)}]$ and $D_1^{(j)}(y^{(j)}) = x_1^{(j)}$.

By the Transitivity Theorem, there exists $\theta_1 \in \text{Aut}_{\mathbf{k}}(B)$ such that $\theta_1\big(\mathbf{k}[x_1^{(1)}]\big) = \ker D$. Let $(x_1^{(2)}, x_2^{(2)}, y^{(2)}) = \big(\theta_1(x_1^{(1)}), \theta_1(x_2^{(1)}), \theta_1(y^{(1)})\big) \in \Gamma_{\mathbf{k}}(B)$, then $\ker D = \mathbf{k}[x_1^{(2)}] = \ker D_1^{(2)}$. By 1.9.3 applied to the pair $D_1^{(2)}, D$,

$$D \text{ is } \mathbf{k}\text{-simple and } D = \lambda_2 D_1^{(2)} \text{ for some } \lambda_2 \in \mathbf{k}^*.$$

For the second assertion, consider $x, y \in B$ such that $D(y) = x$ and $\ker D = \mathbf{k}[x]$. Then $\mathbf{k}[x] = \mathbf{k}[x_1^{(2)}]$ and consequently $x = \lambda' x_1^{(2)} + \mu$ for some $\lambda' \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$. Define $\theta_2 \in \text{Aut}_{\mathbf{k}}(B)$ by

$$\theta_2 : \qquad x_1^{(2)} \longmapsto \lambda' x_1^{(2)}, \qquad x_2^{(2)} \longmapsto (\lambda')^{-1} x_2^{(2)} \quad \text{and} \quad y^{(2)} \longmapsto y^{(2)}$$

and define $(x_1^{(3)}, x_2^{(3)}, y^{(3)}) = \big(\theta_2(x_1^{(2)}), \theta_2(x_2^{(2)}), \theta_2(y^{(2)})\big) \in \Gamma_{\mathbf{k}}(B)$. Then $x = x_1^{(3)} + \mu$ and $D = \lambda D_1^{(3)}$ for some $\lambda \in \mathbf{k}^*$. Let $s = \lambda y - y^{(3)}$, then $D_1^{(3)}(s) = (\lambda D_1^{(3)})(y) - D_1^{(3)} y^{(3)} = x - x_1^{(3)} = \mu$.

We must have $\mu = 0$ for otherwise $D_1^{(3)}(s) \in \mathbf{k}^*$ would imply $B = \mathbf{k}[x, s] = \mathbf{k}^{[2]}$, which is not the case. Thus $x = x_1^{(3)}$ and, since $D_1^{(3)}(s) = 0$, $\lambda y - y^{(3)} = a + xf(x)$ for some $a \in \mathbf{k}$ and $f(x) \in \mathbf{k}[x]$. As we know, there is an automorphism $\Delta \in \text{Aut}_{\mathbf{k}}(B)$ satisfying

$$\Delta : \qquad x_1^{(3)} \longmapsto x_1^{(3)} \quad \text{and} \quad y^{(3)} \longmapsto y^{(3)} + x_1^{(3)} f(x_1^{(3)}).$$

Let $(x_1^{(4)}, x_2^{(4)}, y^{(4)}) = \big(\Delta(x_1^{(3)}), \Delta(x_2^{(3)}), \Delta(y^{(3)})\big) \in \Gamma_{\mathbf{k}}(B)$, then

$$x_1^{(4)} = x \quad \text{and} \quad y^{(4)} = \lambda y - a.$$

For each $j \in \{1, 2, 3, 4\}$, let $\pi_j : \mathbf{k}[X_1, X_2, Y] \to B$ be the $\mathbf{k}$-homomorphism defined by $\pi_j(X_1) = x_1^{(j)}$, $\pi_j(X_2) = x_2^{(j)}$ and $\pi_j(Y) = y^{(j)}$.

Finally, consider $\Psi \in \mathrm{Aut}_{\mathbf{k}}\big(\mathbf{k}[X_1, X_2, Y]\big)$ defined by

$$\Psi: \qquad X_1 \longmapsto X_1, \qquad X_2 \longmapsto X_2 \quad \text{and} \quad Y \longmapsto \lambda Y - a$$

and define $\pi_5 = \pi_4 \circ \Psi^{-1} : \mathbf{k}[X_1, X_2, Y] \to B$, i.e., we have constructed a commutative diagram (where we write $R = \mathbf{k}[X_1, X_2, Y]$):

$$\begin{array}{ccccccccc} R & \xrightarrow{\mathrm{id}} & R & \xrightarrow{\mathrm{id}} & R & \xrightarrow{\mathrm{id}} & R & \xrightarrow{\Psi} & R \\ {\scriptstyle\pi_1}\big\downarrow & & {\scriptstyle\pi_2}\big\downarrow & & {\scriptstyle\pi_3}\big\downarrow & & {\scriptstyle\pi_4}\big\downarrow & & {\scriptstyle\pi_5}\big\downarrow \\ B & \xrightarrow[\theta_1]{} & B & \xrightarrow[\theta_2]{} & B & \xrightarrow[\Delta]{} & B & \xrightarrow[\mathrm{id}]{} & B \end{array} \tag{32}$$

Then π_5 is surjective and $\ker \pi_5 = \Psi\big(\ker \pi_4\big)$ is of the required form, i.e., if we define

$$(x_1^{(5)}, x_2^{(5)}, y^{(5)}) = \big(\pi_5(X_1), \pi_5(X_2), \pi_5(Y)\big)$$

then $(x_1^{(5)}, x_2^{(5)}, y^{(5)}) \in \Gamma_{\mathbf{k}}(B)$. Since $\pi_5(X_1) = x$ and $\pi_5(Y) = y$, we are done. $\square$

2.9. **Lemma.** *Suppose that $(B, \mathbf{k})$ is a Danielewski surface. If $(x_1, x_2, y), (x_1', x_2', y') \in \Gamma_{\mathbf{k}}(B)$ then there exists $\theta \in \mathrm{Aut}_{\mathbf{k}}(B)$ satisfying:*

$$\theta(x_1') = x_1, \qquad \theta(x_2') = cx_2, \text{ for some } c \in \mathbf{k}^*, \qquad \text{and} \qquad \theta\big(\mathbf{k}[y']\big) = \mathbf{k}[y].$$

Proof. If $B = \mathbf{k}^{[2]}$ then an element of $\Gamma_{\mathbf{k}}(B)$ is a triple $(x_1, x_2, \alpha x_1 x_2 + \beta)$ such that $B = \mathbf{k}[x_1, x_2]$, $\alpha \in \mathbf{k}^*$ and $\beta \in \mathbf{k}$. In this case, the assertion is trivial and we may even arrange $c = 1$.

The case $B \neq \mathbf{k}^{[2]}$ is in fact a corollary of the proof of 2.8. We know that there exists an irreducible $D \in \mathrm{LND}(B)$ such that $Dy = x_1$ and $\ker D = \mathbf{k}[x_1]$ (D is the "D_1^{γ}" of 2.3, where $\gamma = (x_1, x_2, y)$); so the pair (x_1, y) satisfies the hypothesis of 2.8. Start the proof of 2.8 with $(x_1^{(1)}, x_2^{(1)}, y^{(1)}) = (x_1', x_2', y')$ instead of picking an arbitrary $(x_1^{(1)}, x_2^{(1)}, y^{(1)}) \in \Gamma_{\mathbf{k}}(B)$. Going through the proof, we obtain the commutative diagram (32).

Now let $\theta = \Delta \circ \theta_2 \circ \theta_1$; then θ maps $x_i' = x_i^{(1)}$ on $x_i^{(4)}$ (for $i = 1, 2$) and $y' = y^{(1)}$ on $y^{(4)}$. Recall that the x in the proof of 2.8 corresponds to x_1 here, so

$$\theta(x_1') = x_1^{(4)} = x = x_1. \tag{33}$$

We also have $\theta(y') = y^{(4)} = \lambda y - a$ (where $\lambda \in \mathbf{k}^*$ and $a \in \mathbf{k}$), so

$$\theta\big(\mathbf{k}[y']\big) = \mathbf{k}[y]. \tag{34}$$

By 2.4, $x_1' x_2'$ generates the ideal $\mathbf{k}[y'] \cap x_1' B$ of $\mathbf{k}[y']$; applying θ and taking (33) and (34) into account, we obtain:

$$x_1 \theta(x_2') \text{ generates the ideal } \mathbf{k}[y] \cap x_1 B \text{ of } \mathbf{k}[y].$$

But 2.4 implies that $x_1 x_2$ is another generator of the same ideal of $\mathbf{k}[y]$. Thus $x_1\theta(x_2')$ and $x_1 x_2$ are associates in B, so $\theta(x_2') = cx_2$ for some $c \in \mathbf{k}^*$. $\square$

2.10. **Lemma.** *Suppose that $(B, \mathbf{k})$ is a Danielewski surface. Then the polynomial $\varphi \in \mathbf{k}[Y]$ in a representation*

$$B \cong \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2)$$

is uniquely determined by B, up to a $\mathbf{k}$-automorphism of $\mathbf{k}[Y]$ and multiplication by a unit. In particular, the degree of φ is uniquely determined by B.

Proof. Consider $(x_1, x_2, y), (x_1', x_2', y') \in \Gamma_{\mathbf{k}}(B)$ and the corresponding $\varphi, \psi \in \mathbf{k}[Y]$ satisfying $x_1x_2 = \varphi(y)$ and $x_1'x_2' = \psi(y')$. By 2.9, there exists $\theta \in \mathrm{Aut}_{\mathbf{k}}(B)$ such that $\theta(x_1') = x_1$, $\theta(x_2') = cx_2$ and $\theta(y') = \lambda y - a$, for some $c, \lambda \in \mathbf{k}^*$ and $a \in \mathbf{k}$. Thus

$$c\varphi(y) = x_1(cx_2) = \theta(x_1'x_2') = \theta(\psi(y')) = \psi(\lambda y - a),$$

so $c\varphi = \psi(\lambda Y - a)$ and in particular $\deg_Y \varphi = \deg_Y \psi$, as claimed. □

3. Definition of $\underline{\text{KLND}}\,(B)$ and $\underline{\mathfrak{R}}(B)$

Given an arbitrary integral domain B (of characteristic zero), the graphs $\underline{\text{KLND}}\,(B)$ and $\underline{\mathfrak{R}}(B)$ are defined in 3.3 and 3.8 respectively. These graphs are invariants of the ring B and the group of automorphisms of B acts on each one of them.

See 1.1 for the notations A_R, R_R, etc.

3.1. **Terminology of graphs.** By a *graph*, we mean an undirected graph such that no edge connects a vertex to itself and at most one edge joins any given pair of vertices. In such a graph, the edge joining vertices u and v is represented by the set $\{u, v\}$. Two vertices are called *neighbors* if they are joined by an edge. If u is a vertex in a graph $\mathcal{G}$, the set of neighbors of u in $\mathcal{G}$ is denoted $\mathcal{N}(u)$ or $\mathcal{N}_{\mathcal{G}}(u)$. A *path* in $\mathcal{G}$ is a sequence $P = (u_0, \dots, u_k)$ of vertices satisfying $k \geq 0$ and:

If $k \geq 1$ then $\quad e_1 = \{u_0, u_1\},\ e_2 = \{u_1, u_2\},\ \dots,\ e_k = \{u_{k-1}, u_k\}\quad$ are edges in $\mathcal{G}$.

If the edges $e_1, \dots, e_k$ of P are distinct, we call P a *simple* path; if P satisfies the weaker condition:

$$e_i \neq e_{i+1} \quad \text{for } 1 \leq i < k \qquad \text{(or equivalently } \ u_{i-1} \neq u_{i+1} \ \text{ for } 1 \leq i < k\text{)},$$

we say that P is *locally simple.*

A *spanning tree* of a graph $\mathcal{G}$ is a subgraph of $\mathcal{G}$ which is a tree and whose vertex set is equal to that of $\mathcal{G}$.

Let $\mathcal{G}$ and $\mathcal{H}$ be graphs with vertex sets G and H respectively. By a *homomorphism of graphs* $f : \mathcal{G} \to \mathcal{H}$ we mean a set map $f : G \to H$ satisfying:

$$\text{for every edge } \{u, v\} \text{ of } \mathcal{G},\ \{f(u), f(v)\} \text{ is an edge of } \mathcal{H}$$

(note that this condition implies, in particular, that $f(u) \neq f(v)$).

3.2. **Definitions.** Let $R \subset B$ be domains such that $\mathrm{trdeg}_R(B) = 2$.

3.2.1. If $A \in \mathrm{KLND}_R(B)$, define

$$\Omega_R(A) = \big\{ \, y \in B \mid \exists \text{ an irreducible } D \in \mathrm{LND}_A(B) \text{ such that } A_R = R_R[Dy] \, \big\}.$$

Remarks.

(1) If $A \in \mathrm{KLND}(B)$, then $\mathrm{LND}_A(B)$ is the set of locally nilpotent derivations of B with kernel *equal* to A.
[This is because A is algebraically closed in B and $\mathrm{trdeg}_A(B) = 1$, by 1.7.]

(2) If $A \in \mathrm{KLND}_R(B)$ and $\Omega_R(A) \neq \varnothing$, then $A_R = (R_R)^{[1]}$.
[Indeed, $A_R = R_R[Dy]$ for some D and y, and Dy must be transcendental over R_R since $\mathrm{trdeg}_R(B) = 2$ and (by 1.7) $\mathrm{trdeg}_A(B) = 1$.]

3.2.2. Let $\underline{\mathrm{KLND}}_R(B)$ be the graph with vertex set $\mathrm{KLND}_R(B)$ and whose edges are defined as follows: Given distinct $A, A' \in \mathrm{KLND}_R(B)$,

$$\{A, A'\} \textit{ is an edge of } \underline{\mathrm{KLND}}_R(B) \quad \iff \quad \Omega_R(A) \cap \Omega_R(A') \neq \varnothing.$$

3.3. **Definition.** Given an integral domain B, let $\underline{\mathrm{KLND}}(B)$ be the graph with vertex set $\mathrm{KLND}(B)$ and where distinct $A, A' \in \mathrm{KLND}(B)$ are neighbors if:

$$\{A, A'\} \text{ is an edge of } \underline{\mathrm{KLND}}_R(B), \text{ for some subring } R \text{ of } B \text{ with } \mathrm{trdeg}_R(B) = 2.$$

We also define:

$$\underline{\mathrm{KLND}}_*(B) = \text{the subgraph of } \underline{\mathrm{KLND}}(B) \text{ obtained by deleting all isolated vertices.}$$

3.4. **Lemma.** *Let $R \subset B$ be domains such that $\mathrm{trdeg}_R(B) = 2$ and suppose that A and A' are distinct elements of $\mathrm{KLND}_R(B)$. If R is inert in B, then $R = A \cap A'$.*

Proof. Note that one of the inclusions in

$$R \subseteq A \cap A' \subseteq A \tag{35}$$

must be an algebraic extension of rings, because $\mathrm{trdeg}_R(A) = 1$; since each of R, A, A', $A \cap A'$ is an inert subring of B, and hence is algebraically closed in B, one of the inclusions in (35) must actually be an equality. Now $A \neq A'$ and $\mathrm{trdeg}_A(B) = 1 = \mathrm{trdeg}_{A'}(B)$ imply that $A \neq A \cap A'$, so $R = A \cap A'$. □

3.5. **Lemma.** *Let B be a domain and let $\{A, A'\}$ be an edge of $\underline{\mathrm{KLND}}(B)$. Then there exists a unique* inert *subring R of B satisfying*

$$\mathrm{trdeg}_R(B) = 2 \textit{ and } \{A, A'\} \textit{ is an edge of } \underline{\mathrm{KLND}}_R(B).$$

Moreover, $R = A \cap A'$ and (B_R, R_R) is a Danielewski surface.

Proof. The assumption implies that the set

$$\Sigma = \big\{ \, R \mid R \text{ is a subring of } B, \ \mathrm{trdeg}_R(B) = 2 \text{ and } \{A, A'\} \text{ is an edge of } \underline{\mathrm{KLND}}_R(B) \, \big\}$$

is nonempty. Consider any $R_1 \in \Sigma$ and define $R = B \cap \mathrm{Frac}(R_1)$. Then $R_1 \subseteq R$ and $\mathrm{Frac}(R_1) = \mathrm{Frac}(R)$; it follows that $\Omega_{R_1}(A) = \Omega_R(A)$ and $\Omega_{R_1}(A') = \Omega_R(A')$, so $\Omega_R(A) \cap \Omega_R(A') \neq \varnothing$ and $R \in \Sigma$. Since A is inert in B, A_R is inert in B_R and consequently $(B_R)^* = (A_R)^*$. On the other hand, the fact that $\Omega_R(A) \neq \varnothing$ implies that $A_R = (R_R)^{[1]}$, so $(B_R)^* = (A_R)^* = (R_R)^*$. Thus the first part of

$$(B_R)^* = (R_R)^* \quad \text{and} \quad B \cap \operatorname{Frac}(R) = R$$

holds, and so does the second part by definition of R. By 1.4, it follows that R is an inert subring of B. This proves that at least one element of Σ is an inert subring of B. Then 3.4 gives:

$$\big\{\, R \in \Sigma \mid R \text{ is an inert subring of } B \,\big\} = \big\{ A \cap A' \big\}.$$

To complete the proof, we show that if R is any element of Σ then (B_R, R_R) is a Danielewski surface. Pick $y \in \Omega_R(A) \cap \Omega_R(A')$. Then there exist $D, D' \in \mathrm{LND}(B)$ satisfying $\ker D = A$, $\ker D' = A'$, $A_R = R_R[Dy]$ and $A'_R = R_R[D'y]$. Then the localized derivations $D_R, D'_R \in \mathrm{LND}(B_R)$ satisfy $\ker(D_R) = A_R = (R_R)^{[1]}$ and $\ker(D'_R) = A'_R = (R_R)^{[1]}$. Since D_R and D'_R are extensions of D and D' respectively, we have $A = B \cap \ker(D_R)$ and $A' = B \cap \ker(D'_R)$; so D_R and D'_R have distinct kernels. Since y is a preslice of both D_R and D'_R, 2.5 applied to the triple (y, D_R, D'_R) gives that (B_R, R_R) is a Danielewski surface. $\square$

3.6. **Definition.** Given an integral domain B, let $\mathcal{R}(B)$ denote the following set of subrings of B:

$$\mathcal{R}(B) = \big\{\, R \mid R \text{ is an inert subring of } B \text{ and } (B_R, R_R) \text{ is a Danielewski surface} \,\big\}.$$

Note that if $R \in \mathcal{R}(B)$ then $\operatorname{trdeg}_R(B) = 2$.

The following result will be improved later (see 5.1). In part (a) of 3.7, "$\subseteq$" means *"is a subgraph of"*.

3.7. **Corollary.** *If B is an integral domain then:*

(a) $\underline{\mathrm{KLND}}_*(B) \subseteq \bigcup_{R \in \mathcal{R}(B)} \underline{\mathrm{KLND}}_R(B) \subseteq \underline{\mathrm{KLND}}(B)$

(b) *If R_1, R_2 are distinct elements of $\mathcal{R}(B)$, the graphs $\underline{\mathrm{KLND}}_{R_1}(B)$ and $\underline{\mathrm{KLND}}_{R_2}(B)$ have at most one vertex in common.*

Proof. Assertion (a) follows from 3.5 and (b) from 3.4. $\square$

Result 3.7 suggests a natural way to turn $\mathcal{R}(B)$ into a graph:

3.8. **Definition.** Given an integral domain B, let $\underline{\mathcal{R}}(B)$ be the graph with vertex set $\mathcal{R}(B)$ and where distinct $R_1, R_2 \in \mathcal{R}(B)$ are neighbors if and only if $\mathrm{KLND}_{R_1}(B) \cap \mathrm{KLND}_{R_2}(B) \neq \varnothing$.

Equivalently, $R_1, R_2 \in \mathcal{R}(B)$ are neighbors in $\underline{\mathcal{R}}(B)$ if and only if there exists a nonzero locally nilpotent derivation $D : B \to B$ satisfying $D(R_1 \cup R_2) = \{0\}$.

The structures of the graphs $\underline{\text{KLND}}(B)$ and $\underline{\mathcal{R}}(B)$ are closely related and (as can be inferred from 5.1, below) this is particularly true when B is factorial and affine over some field. However, we will not elaborate on this point. Let us simply say that the graphs $\underline{\text{KLND}}(B)$ and $\underline{\mathcal{R}}(B)$ are two invariants of the ring B, and that $\underline{\mathcal{R}}(B)$ should be thought of as a simplified version of $\underline{\text{KLND}}(B)$.

3.9. **Actions of $\mathbf{Aut}(B)$.** Let B be an integral domain and θ an automorphism of B. Then the following claims are trivial.

(1) If $D \in \text{LND}(B)$ and $D' = \theta \circ D \circ \theta^{-1}$, then $D' \in \text{LND}(B)$ and $\ker D' = \theta(\ker D)$; if D is irreducible then so is D'.

(2) If $R \in \mathcal{R}(B)$ and $A \in \text{KLND}_R(B)$ then:

$$\theta(R) \in \mathcal{R}(B), \quad \theta(A) \in \text{KLND}_{\theta(R)}(B) \quad \text{and} \quad \theta\big(\Omega_R(A)\big) = \Omega_{\theta(R)}\big(\theta(A)\big).$$

(3) If $R \in \mathcal{R}(B)$ and A_1, A_2 are distinct elements of $\text{KLND}_R(B)$, then:

$$\{A_1, A_2\} \text{ is an edge of } \underline{\text{KLND}}_R(B) \Leftrightarrow \{\theta(A_1), \theta(A_2)\} \text{ is an edge of } \underline{\text{KLND}}_{\theta(R)}(B).$$

Consequently,

3.9.1. *Let* $\text{Aut}(B)$ *denote the group of ring automorphisms of* B.

- *There is a left-action of* $\text{Aut}(B)$ *on the graph* $\underline{\text{KLND}}(B)$, *given by*
$$(\theta, A) \mapsto \theta A = \theta(A).$$
- *There is a left-action of* $\text{Aut}(B)$ *on the graph* $\underline{\mathcal{R}}(B)$, *given by*
$$(\theta, R) \mapsto \theta R = \theta(R).$$

3.10. **The one-dimensional case.** Suppose that B is a domain containing a field over which B has transcendence degree one or less.

Then it is well-known that if $0 \neq D : B \to B$ is a locally nilpotent derivation then B is a polynomial ring in one variable over some field, and this field is in fact the kernel of D. This simple fact can be phrased as follows:

- $\underline{\text{KLND}}(B)$ *is either the empty graph or the graph with one vertex (and no edge).*
- $\underline{\text{KLND}}(B)$ *is nonempty if and only if* $B = \mathbf{k}^{[1]}$ *for some field* $\mathbf{k}$, *in which case* $\text{KLND}(B) = \{\mathbf{k}\}$.
- $\underline{\mathcal{R}}(B)$ *is the empty graph* [this is because $R \in \mathcal{R}(B)$ implies $\text{trdeg}_R(B) = 2$ and $B^* = R^*$].

4. Description of the graph $\underline{\text{KLND}}_{\mathbf{k}}(B)$ in the two-dimensional case

The beginning of this section considers the problem of describing the graph $\underline{\text{KLND}}(B)$ where B is an integral domain which has transcendence degree two over some field (of characteristic zero). However 4.3 shows that this problem reduces to the following: *Describe the graph* $\underline{\text{KLND}}_{\mathbf{k}}(B)$ *where* $\mathbf{k}$ *is a field,* B *is an integral domain containing* $\mathbf{k}$ *as a subring and* B *has transcendence degree* 2 *over* $\mathbf{k}$. Solving this reformulated problem then becomes the aim of this section (this viewpoint is adopted in 4.4).

In 4.6 (but see also 4.3) we show that $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is non-discrete (i.e., has at least one edge) if and only if $(B, \mathbf{k})$ is a Danielewski surface. From 4.7 to the end of the section, we restrict our attention to the case where $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is non-discrete and give a quite satisfactory description of that graph. In particular, we show that it is connected, we identify in which cases it is a tree and, in all cases, we describe a spanning tree of $\underline{\text{KLND}}_{\mathbf{k}}(B)$.

Result 5.1, below, is the motivation for giving such a detailed description of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ in the non-discrete case.

The case where $\underline{\text{KLND}}\,(B)$ is a discrete graph deserves to be investigated, but this is not done in this paper. In particular, one would like to know which two-dimensional rings B are such that $\underline{\text{KLND}}\,(B)$ has many vertices but no edges (see 6.2 for an interesting example).

We begin by showing that the graph $\underline{\mathcal{R}}(B)$ has at most one vertex in the two-dimensional case:

4.1. **Proposition.** *Let B be an integral domain which has transcendence degree 2 over some field. Then $\mathcal{R}(B)$ is the set of fields $\mathbf{k}$ contained in B and satisfying: $(B, \mathbf{k})$ is a Danielewski surface. In particular, $\mathcal{R}(B)$ has at most one element.*

Proof. Consider an arbitrary element $\mathbf{k}$ of $\mathcal{R}(B)$ (a priori, $\mathbf{k}$ is not necessarely a field). Note that $B^* = \mathbf{k}^*$, since $\mathbf{k}$ is an inert subring of B. By assumption, there exists a field $k \subset B$ such that $\operatorname{trdeg}_k(B) = 2$. Then $k^* \subseteq B^* = \mathbf{k}^*$, so $k \subseteq \mathbf{k}$. Since $\operatorname{trdeg}_k(B) = 2 = \operatorname{trdeg}_{\mathbf{k}}(B)$, $\mathbf{k}$ is integral over k, so $\mathbf{k}$ is a field. It follows that $\mathbf{k} = \{0\} \cup \mathbf{k}^* = \{0\} \cup B^*$ is uniquely determined by B, so $\mathcal{R}(B) = \{\mathbf{k}\}$. Obviously, $\mathbf{k} \in \mathcal{R}(B)$ implies that $(B, \mathbf{k})$ is a Danielewski surface.

Conversely, suppose that $\mathbf{k} \subset B$ is a field such that $(B, \mathbf{k})$ is a Danielewski surface. We have $B^* = \mathbf{k}^*$ by 2.3, so $\mathbf{k}$ is an inert subring of B and $\mathbf{k} \in \mathcal{R}(B)$. □

Next we point out that there are edges in the graph $\underline{\text{KLND}}\,(B)$ of a Danielewski surface:

4.2. **Example.** Suppose that $(B, \mathbf{k})$ is a Danielewski surface and consider $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$. Let $A_i = \mathbf{k}[x_i] \in \text{KLND}(B)$ $(i = 1, 2)$. Then

$$\{A_1, A_2\} \text{ is an edge in } \underline{\text{KLND}}_{\mathbf{k}}(B).$$

Proof. For each $i \in \{1, 2\}$, consider the derivation $D_i^\gamma : B \to B$ of 2.3, where $\gamma = (x_1, x_2, y)$. Then D_i^γ is an irreducible derivation, belongs to $\text{LND}_{A_i}(B)$, and satisfies $\ker D_i^\gamma = \mathbf{k}[D_i^\gamma(y)]$. So

$$y \in \Omega_{\mathbf{k}}(A_1) \cap \Omega_{\mathbf{k}}(A_2)$$

and $\{A_1, A_2\}$ is an edge in $\underline{\text{KLND}}_{\mathbf{k}}(B)$. □

4.3. **Corollary.** *Let B be an integral domain which has transcendence degree 2 over some field. Then the following three conditions are equivalent:*

(1) $\mathcal{R}(B) \neq \varnothing$
(2) *There exists a field $\mathbf{k} \subset B$ such that $(B, \mathbf{k})$ is a Danielewski surface.*
(3) $\underline{\text{KLND}}\,(B)$ *has at least one edge.*

Moreover, we have

(∗) $\underline{\text{KLND}}\,(B) = \underline{\text{KLND}}_{\mathbf{k}}(B)$

for some field $\mathbf{k} \subset B$ *satisfying* $\operatorname{trdeg}_{\mathbf{k}}(B) = 2$. *More precisely:*

(4) *If conditions* (1–3) *hold then the unique element* $\mathbf{k}$ *of* $\mathcal{R}(B)$ *satisfies* (∗).

(5) *If conditions* (1–3) *do not hold then* (∗) *holds for any field* $\mathbf{k} \subset B$ *satisfying* $\operatorname{trdeg}_{\mathbf{k}}(B) = 2$.

Proof. We have (1) $\overset{4.1}{\Longleftrightarrow}$ (2) $\overset{4.2}{\Longrightarrow}$ (3) $\overset{3.7}{\Longrightarrow}$ (1).

To prove (4), assume that (1–3) hold and consider the unique element $\mathbf{k}$ of $\mathcal{R}(B)$. Since $\mathbf{k}$ is a field contained in B, we have $\text{KLND}(B) = \text{KLND}_{\mathbf{k}}(B)$ by 1.6. So 3.7 and $\mathcal{R}(B) = \{\mathbf{k}\}$ give $\underline{\text{KLND}}\,(B) = \underline{\text{KLND}}_{\mathbf{k}}(B)$.

To prove (5), assume that (1–3) do not hold and consider any field $\mathbf{k} \subset B$ satisfying $\operatorname{trdeg}_{\mathbf{k}}(B) = 2$. Again, we have $\text{KLND}(B) = \text{KLND}_{\mathbf{k}}(B)$ by 1.6. This immediately implies that $\underline{\text{KLND}}\,(B) = \underline{\text{KLND}}_{\mathbf{k}}(B)$, since $\underline{\text{KLND}}\,(B)$ has no edges. □

Result 4.3 implies, in particular, that the study of $\underline{\text{KLND}}\,(B)$ reduces to that of $\underline{\text{KLND}}_{\mathbf{k}}(B)$. Until the end of this section, our aim is to describe the graph $\underline{\text{KLND}}_{\mathbf{k}}(B)$ where $(B, \mathbf{k})$ is a pair satisfying:

4.4. **Global assumptions.** $\mathbf{k}$ is a field, B is an integral domain containing $\mathbf{k}$ as a subring and B has transcendence degree 2 over $\mathbf{k}$.

4.5. Let $(B, \mathbf{k})$ be a pair satisfying 4.4. Recall the following facts from 3.2:

(1) For each $A \in \text{KLND}(B)$ we define

$$\Omega_{\mathbf{k}}(A) = \big\{\, y \in B \mid \exists \text{ an irreducible } D \in \text{LND}_A(B) \text{ such that } \mathbf{k}[\,Dy\,] = A \,\big\}.$$

Regarding the set $\Omega_{\mathbf{k}}(A)$, note the following.

(i) If $y \in \Omega_{\mathbf{k}}(A)$ then $\mathbf{k}[\,Dy\,] = A$ holds for *every* irreducible $D \in \text{LND}_A(B)$, by 1.9.3.

(ii) If $\Omega_{\mathbf{k}}(A) \neq \varnothing$ then $A = \mathbf{k}^{[1]}$.

(iii) If $\Omega_{\mathbf{k}}(A) \neq \varnothing$ for some A, then $B^* = \mathbf{k}^*$ [because $B^* = A^*$ and $A = \mathbf{k}^{[1]}$].

(2) $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is the graph with vertex set $\text{KLND}_{\mathbf{k}}(B) = \text{KLND}(B)$ and whose edges are defined as follows: Given distinct vertices $A, A' \in \text{KLND}_{\mathbf{k}}(B)$,

$$\{A, A'\} \textit{ is an edge if and only if } \Omega_{\mathbf{k}}(A) \cap \Omega_{\mathbf{k}}(A') \neq \varnothing.$$

Recall that a graph is *non-discrete* if it has at least one edge.

4.6. **Corollary.** *Let* $(B, \mathbf{k})$ *be a pair satisfying* 4.4. *Then* $\underline{\text{KLND}}_{\mathbf{k}}(B)$ *is non-discrete if and only if* $(B, \mathbf{k})$ *is a Danielewski surface.*

Moreover, if $\{A_1, A_2\}$ *is any edge of* $\underline{\text{KLND}}_{\mathbf{k}}(B)$ *then there exists* $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ *satisfying* $A_1 = \mathbf{k}[x_1]$ *and* $A_2 = \mathbf{k}[x_2]$.

Proof. By 4.2, if $(B, \mathbf{k})$ is a Danielewski surface then $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is non-discrete.

Conversely, suppose that $\{A_1, A_2\}$ is an edge of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ (where A_1, A_2 are distinct elements of $\text{KLND}(B)$). Then $\Omega_{\mathbf{k}}(A_1) \cap \Omega_{\mathbf{k}}(A_2) \neq \varnothing$, so we may pick an element y of that intersection. For each $i \in \{1,2\}$ we have $y \in \Omega_{\mathbf{k}}(A_i)$ and consequently there exists an irreducible $D_i \in \text{LND}_{A_i}(B)$ satisfying $A_i = \mathbf{k}[D_i(y)]$. Let $x_i = D_i(y)$, then

$$\ker D_i = \mathbf{k}[x_i] = \mathbf{k}^{[1]} \qquad (\text{for each } i \in \{1,2\}).$$

Thus (y, D_1, D_2) satisfies the hypothesis of 2.5. Since it is clear that (2.5-1) is false, (2.5-2) must hold. So $(B, \mathbf{k})$ is a Danielewski surface and $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$. □

The graph of a Danielewski surface of degree n

In view of 2.10, the following is well-defined:

4.7. **Terminology.** Let n be a positive integer. The phrase "$(B, \mathbf{k})$ is a Danielewski surface of degree n" means that $(B, \mathbf{k})$ is a Danielewski surface and that the polynomial $\varphi \in \mathbf{k}[Y]$ satisfying $B \cong \mathbf{k}[X_1, X_2, Y]/(\varphi - X_1X_2)$ has degree n.

Until the end of this section, we consider a Danielewski surface $(B, \mathbf{k})$ of degree n and our aim is to describe $\underline{\text{KLND}}_{\mathbf{k}}(B)$. This is an important problem because of 5.1, below.

4.8. **Theorem.** *If $(B, \mathbf{k})$ is a Danielewski surface then the graph $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is connected.*

Proof. Choose $\gamma = (x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ and write $A_1 = \mathbf{k}[x_1]$ and $A_2 = \mathbf{k}[x_2]$. By 4.2, A_1 and A_2 belong to the same connected component $\mathcal{C}$ of $\underline{\text{KLND}}_{\mathbf{k}}(B)$.

Consider the subgroup $G = G_\gamma$ of $\text{Aut}_{\mathbf{k}}(B)$ generated by the set $E = \{\tau\} \cup \big\{ \Delta_f \mid f \in \mathbf{k}[x_1] \big\}$ (see 2.7.1). If $g \in G$ then, by 3.9.1, $g\mathcal{C}$ is a connected component of $\underline{\text{KLND}}_{\mathbf{k}}(B)$. It is immediate that if $g \in E$ then $gA_1 \in \{A_1, A_2\}$, so $gA_1 \in \mathcal{C}$, so $g\mathcal{C} = \mathcal{C}$; it follows that $g\mathcal{C} = \mathcal{C}$ for all $g \in G$. Since G acts transitively on the set $\text{KLND}(B)$ (by 2.7.2), we conclude that $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is connected. □

The main result of this subsection is 4.10.4, but we also point-out:

4.9. **Theorem.** *Suppose that $(B, \mathbf{k})$ is a Danielewski surface of degree n. Then $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is a tree if and only if $n > 2$.*

The proof of 4.9 consists of 4.9.1, 4.9.2 and part (4) of 4.10.4.

We begin by showing (in 4.9.1 and 4.9.2) that if $(B, \mathbf{k})$ is a Danielewski surface of degree 1 or 2, then $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is very far from being a tree: *Each vertex belongs to a subgraph of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ isomorphic to the complete graph on the set $\mathbf{k}$.*

4.9.1. **Example.** Suppose that $(B, \mathbf{k})$ is a Danielewski surface of degree 1 (which is equivalent to $B = \mathbf{k}^{[2]}$ by 2.3). We first note that:

(36) *The edge set of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is $\big\{\{\mathbf{k}[u], \mathbf{k}[v]\} \mid B = \mathbf{k}[u, v]\big\}$.*

Indeed, if $u, v \in B$ are such that $B = \mathbf{k}[u, v]$, then it is immediate that $(u, v, uv) \in \Gamma_{\mathbf{k}}(B)$; so 4.2 implies that $\big\{\mathbf{k}[u], \mathbf{k}[v]\big\}$ is an edge. Conversely, suppose that $\{A_1, A_2\}$ is an edge. Then, by 4.6, there exists $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ such that $A_1 = \mathbf{k}[x_1]$ and

$A_2 = \mathbf{k}[x_2]$. Since $\varphi(y) = x_1x_2$ for some $\varphi \in \mathbf{k}[Y]$ of degree one, we have $y \in \mathbf{k}[x_1, x_2]$, so $B = \mathbf{k}[x_1, x_2]$. This proves (36).

Let $A \in \text{KLND}(B)$. By Rentschler's Theorem 1.8 we may choose x_1, x_2 such that $B = \mathbf{k}[x_1, x_2]$ and $A = \mathbf{k}[x_1]$. For each $\lambda = (\lambda_1 : \lambda_2) \in \mathbb{P}^1_{\mathbf{k}}$, let $A_\lambda = \mathbf{k}[\lambda_1 x_1 + \lambda_2 x_2]$. Then $U = \{ A_\lambda \mid \lambda \in \mathbb{P}^1_{\mathbf{k}} \}$ is a subset of $\text{KLND}(B)$ of cardinality $|\mathbf{k}|$, $A \in U$ and, by (36), $\{A_\lambda, A_{\lambda'}\}$ is an edge of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ whenever λ, λ' are distinct elements of $\mathbb{P}^1_{\mathbf{k}}$. In other words, the complete graph on the set U is a subgraph of $\underline{\text{KLND}}_{\mathbf{k}}(B)$. □

4.9.2. **Example.** Suppose that $(B, \mathbf{k})$ is a Danielewski surface of degree 2. Let $A \in \text{KLND}(B)$. By the Transitivity Theorem (or by 2.8), there exists $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ such that $A = \mathbf{k}[x_1]$. Consider the polynomial $\varphi \in \mathbf{k}[Y]$ which satisfies $x_1x_2 = \varphi(y)$. Then φ has degree two and depends on our choice of (x_1, x_2, y). In fact we may choose (x_1, x_2, y) in $\Gamma_{\mathbf{k}}(B)$ in such a way that $A = \mathbf{k}[x_1]$ and:

$$\varphi = Y^2 + c \text{ for some } c \in \mathbf{k}. \tag{37}$$

[To see this, it suffices to observe that if $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$, $\nu \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$, then $(x_1, \nu x_2, y+\mu) \in \Gamma_{\mathbf{k}}(B)$.] For each $\lambda = (\lambda_1 : \lambda_2) \in \mathbb{P}^1_{\mathbf{k}}$, let $A_\lambda = \mathbf{k}[\lambda_1^2 x_1 + 2\lambda_1\lambda_2 y + \lambda_2^2 x_2]$. Observe that $A = A_{(1:0)}$. We claim:

$$\{A_\lambda, A_{\lambda'}\} \text{ is an edge of } \underline{\text{KLND}}_{\mathbf{k}}(B) \text{ whenever } \lambda, \lambda' \text{ are distinct elements of } \mathbb{P}^1_{\mathbf{k}}. \tag{38}$$

Clearly, if this is true then A belongs to a subgraph of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ isomorphic to the complete graph on the set $\mathbf{k}$. To prove (38), let $\alpha, \beta \in \mathbf{k}$ and consider the element $\theta = \Delta_\alpha \circ \tau \circ \Delta_\beta$ of $\text{Aut}_{\mathbf{k}}(B)$ (see 2.7.1). Note that, given $t \in \mathbf{k}$, $\Delta_t(x_1) = x_1$, $\Delta_t(y) = y + tx_1$ and (taking (37) into account) $\Delta_t(x_2) = t^2x_1 + 2ty + x_2$. It follows that $\theta(x_1) = \alpha^2 x_1 + 2\alpha y + x_2$, so

$$\theta(\mathbf{k}[x_1]) = A_{(\alpha:1)}.$$

Also, $\theta(x_2) = (1 + \alpha\beta)^2 x_1 + 2(1 + \alpha\beta)\beta y + \beta^2 x_2$, so

$$\theta(\mathbf{k}[x_2]) = A_{(1+\alpha\beta:\beta)}.$$

Since $\{\mathbf{k}[x_1], \mathbf{k}[x_2]\}$ is an edge in $\underline{\text{KLND}}_{\mathbf{k}}(B)$ by 4.2, so is $\{A_{(\alpha:1)}, A_{(1+\alpha\beta:\beta)}\}$ by 3.9.1. The claim (38) follows from this. □

4.10. **Statement of the main result.**

Suppose that $(B, \mathbf{k})$ is a Danielewski surface of degree n and fix an element $\gamma = (x_1, x_2, y)$ of $\Gamma_{\mathbf{k}}(B)$. Consider the subgroup $G = G_\gamma$ of $\text{Aut}_{\mathbf{k}}(B)$ and its generating set $\{ \delta_f \mid f \in \mathbf{k}[x_1] \}$, as in 2.7.1. Let $A_1 = \mathbf{k}[x_1] \in \text{KLND}(B)$.

We now define a tree $\underline{\mathcal{F}}_\gamma$, a subtree $\underline{\mathcal{F}}^\circ_\gamma$ of $\underline{\mathcal{F}}_\gamma$ and homomorphisms of graphs $\mathcal{P}_\gamma : \underline{\mathcal{F}}_\gamma \to \underline{\text{KLND}}_{\mathbf{k}}(B)$ and $\mathcal{P}^\circ_\gamma : \underline{\mathcal{F}}^\circ_\gamma \to \underline{\text{KLND}}_{\mathbf{k}}(B)$.

4.10.1. **Definition.** $\mathcal{E}_\gamma = \begin{cases} \mathbf{k}[x_1], & \text{if } n > 1 \\ x_1\mathbf{k}[x_1], & \text{if } n = 1. \end{cases}$

4.10.2. **Definition.** Let $\mathcal{F}_\gamma$ be the set of finite sequences $(f_1, \dots, f_k)$ of elements of $\mathcal{E}_\gamma$ satisfying :

$$f_i \neq 0 \text{ for all } i \neq 1.$$

Let $\mathcal{F}_\gamma^\circ$ be the subset of $\mathcal{F}_\gamma$ whose elements are the finite sequences $(f_1, \dots, f_k)$ in $\mathcal{E}_\gamma$ satisfying:

$$\deg_{x_1}(f_i) \geq 3 - n \text{ for all } i \neq 1.$$

Note that the empty sequence $\varnothing$ is an element of both $\mathcal{F}_\gamma$ and $\mathcal{F}_\gamma^\circ$.

Let $\underline{\mathcal{F}}_\gamma$ (resp. $\underline{\mathcal{F}}_\gamma^\circ$) be the tree with vertex-set $\mathcal{F}_\gamma$ (resp. $\mathcal{F}_\gamma^\circ$) and where the edges are the pairs of the form

$$\Big\{ (f_1, \dots, f_k),\ (f_1, \dots, f_k, f_{k+1}) \Big\}.$$

It is clear that $\underline{\mathcal{F}}_\gamma$ is a tree, that $\underline{\mathcal{F}}_\gamma^\circ$ is a subtree of $\underline{\mathcal{F}}_\gamma$ and that $\underline{\mathcal{F}}_\gamma^\circ = \underline{\mathcal{F}}_\gamma$ whenever $n \geq 3$.

4.10.3. **Definition.** Define a map $\mathcal{P}_\gamma : \mathcal{F}_\gamma \to \mathrm{KLND}_{\mathbf{k}}(B)$ by declaring that the element $(f_1, \dots, f_k)$ of $\mathcal{F}_\gamma$ is mapped to the element $(\delta_{f_1} \circ \cdots \circ \delta_{f_k})(A_1)$ of $\mathrm{KLND}_{\mathbf{k}}(B)$.

Let $\mathcal{P}_\gamma^\circ : \mathcal{F}_\gamma^\circ \to \mathrm{KLND}_{\mathbf{k}}(B)$ be the restriction of $\mathcal{P}_\gamma$ to $\mathcal{F}_\gamma^\circ$.

4.10.4. **Theorem.** *The maps $\mathcal{P}_\gamma$ and $\mathcal{P}_\gamma^\circ$ have the following properties:*

(1) $\mathcal{P}_\gamma : \underline{\mathcal{F}}_\gamma \to \underline{\mathrm{KLND}}_{\mathbf{k}}(B)$ *and* $\mathcal{P}_\gamma^\circ : \underline{\mathcal{F}}_\gamma^\circ \to \underline{\mathrm{KLND}}_{\mathbf{k}}(B)$ *are homomorphisms of graphs (see* 3.1 *for definition).*

(2) $\mathcal{P}_\gamma$ *is surjective, both as a map of vertices and as a map of edges.*

(3) $\mathcal{P}_\gamma^\circ$ *is bijective, as a map of vertices. Consequently,* $\mathcal{P}_\gamma^\circ$ *defines an isomorphism of trees from* $\underline{\mathcal{F}}_\gamma^\circ$ *to some spanning tree of* $\underline{\mathrm{KLND}}_{\mathbf{k}}(B)$.

(4) *If* $n > 2$ *then* $\mathcal{P}_\gamma : \underline{\mathcal{F}}_\gamma \to \underline{\mathrm{KLND}}_{\mathbf{k}}(B)$ *is an isomorphism and consequently* $\underline{\mathrm{KLND}}_{\mathbf{k}}(B)$ *is a tree.*

4.11. **Preliminaries to the proof of 4.10.4.**

Throughout 4.11, we suppose that $(B, \mathbf{k})$ is a Danielewski surface of degree n and we fix an element $\gamma = (x_1, x_2, y)$ of $\Gamma_{\mathbf{k}}(B)$. Let φ be the unique element of $\mathbf{k}[Y] \setminus \mathbf{k}$ such that $x_1 x_2 = \varphi(y)$. Consider the subgroup $G = G_\gamma$ of $\mathrm{Aut}_{\mathbf{k}}(B)$ and its elements τ, Δ_f and δ_f (where $f \in \mathbf{k}[x_1]$), as in 2.7.1. Let $A_i = \mathbf{k}[x_i] \in \mathrm{KLND}(B)$ for $i \in \{1, 2\}$.

Bidegree. Some of the material on bidegree is reproduced from [1], but there are also some additions.

Since $\gamma = (x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ is fixed, we may embed B in $\mathbf{k}[x_1, x_1^{-1}, y]$. Each element g of $\mathbf{k}[x_1, x_1^{-1}, y]$ is a sum

$$g = \sum_{(i,j) \in \mathbb{Z} \times \mathbb{N}} g_{ij}\, x_1^i y^j$$

where $g_{ij} \in \mathbf{k}$ for all (i, j) and where the set $\mathrm{supp}_\gamma(g) = \big\{ (i, j) \in \mathbb{Z} \times \mathbb{N} \mid g_{ij} \neq 0 \big\}$ is finite. As in [1]-2.7, we define the bidegree map determined by γ

$$\begin{aligned} \mathrm{bideg}_\gamma \ : \ \mathbf{k}[x_1, x_1^{-1}, y] &\longrightarrow \mathbb{N} \times \mathbb{N} \\ g &\longmapsto (u, v) \end{aligned}$$

by declaring that u, v are the following integers:

$$u = \max\big[\{0\} \cup \{\, i \in \mathbb{N} \mid (i,0) \in \operatorname{supp}_\gamma(g) \,\}\big]$$
$$v = \max\big[\{0\} \cup \{\, j \in \mathbb{N} \mid j(-1,n) \in \operatorname{supp}_\gamma(g) \,\}\big].$$

Since γ is fixed throughout 4.11, we may simply write $\operatorname{supp} g$ and $\operatorname{bideg} g$.

4.11.1. ([1]-2.7.2). *Let $g \in \mathbf{k}[x_1, 1/x_1, y]$ and $(a,b) = \operatorname{bideg} g$. Then:*

$$a > 0 \implies (a,0) \in \operatorname{supp} g \qquad \text{and} \qquad b > 0 \implies (-b, bn) \in \operatorname{supp} g.$$

Given $g \in \mathbf{k}[x_1, 1/x_1, y]$, $(a,b) = \operatorname{bideg} g$, let $C(g)$ be the unique subset of $\mathbb{R}^2$ which is closed, convex and has boundary $H_a \cup E \cup H_b$, where E is the line segment joining $(-b, bn)$ to $(a,0)$, $H_a = \{\, (s,0) \mid s \le a \,\}$ and $H_b = \{\, (s, bn) \mid s \le -b \,\}$.

4.11.2. ([1]-2.7.3). *Given $A \in \text{KLND}(B)$ and $g \in A$,* $\quad \operatorname{supp}(g) \subset C(g)$.

4.11.3. ([1]-2.7.6). *Given $A \in \text{KLND}(B)$ and $g \in A$,*

$$\operatorname{bideg} g = (a,b) \implies \operatorname{bideg} \tau(g) = (b,a).$$

As in [1]-3.6.6, let $\mathbb{N} \times \mathbb{N}$ be endowed with the reverse lexicographic order:

$$(a,b) < (a',b') \iff b < b' \text{ or } (b = b' \text{ and } a < a')$$

and define for each $A \in \text{KLND}(B)$

$$\operatorname{bideg}_\gamma(A) = \min\{\, \operatorname{bideg}_\gamma f \mid f \in A \setminus \mathbf{k} \,\} \in \mathbb{N} \times \mathbb{N}$$

[which makes sense because $\mathbb{N} \times \mathbb{N}$ is well-ordered]. So we have a well-defined map

$$\operatorname{bideg}_\gamma : \text{KLND}(B) \longrightarrow \mathbb{N} \times \mathbb{N}.$$

Recall that $A = \mathbf{k}^{[1]}$; it is a straightforward exercise to prove:

4.11.4. *Given $A \in \text{KLND}(B)$ and $f \in A$,* $\quad \operatorname{bideg} f = \operatorname{bideg}(A) \Leftrightarrow A = \mathbf{k}[f]$.

So applying 4.11.3 (resp. [1]-3.6.4) to a generator of A yields 4.11.5 (resp. 4.11.6):

4.11.5. *Given $A \in \text{KLND}(B)$,* $\quad \operatorname{bideg}(A) = (a,b) \implies \operatorname{bideg}(\tau A) = (b,a)$.

4.11.6. *Let $A \in \text{KLND}(B)$ and $(a,b) = \operatorname{bideg} A$. Then*

$$a = 0 \iff A = \mathbf{k}[x_2] \qquad \text{and} \qquad b = 0 \iff A = \mathbf{k}[x_1].$$

Finally we quote:

4.11.7. ([1]-3.9). *Let $A \in \text{KLND}(B) \setminus \{\mathbf{k}[x_1]\}$, let $(a,b) = \operatorname{bideg}(A)$ and suppose that $a \ge b$. Then there exists $(\lambda, s) \in \mathbf{k}^* \times \mathbb{N}$ such that if we set $u = \lambda x_1^s$ then the ring $A' = \Delta_u(A)$ satisfies $\operatorname{bideg}(A') = (a', b)$ and $a' < a$. Moreover, $s = (a+b)/\gcd(nb, a+b) - 1$.*

Remark. The last assertion of 4.11.7 implies, in particular, that $u \in \mathcal{E}_\gamma$. To see this, we may assume that $n = 1$ (otherwise $\mathcal{E}_\gamma = \mathbf{k}[x_1]$); then $s = (a+b)/\gcd(b, a+b) - 1 = (a+b)/\gcd(a,b) - 1$, and if this is not positive then $a = 0$ or $b = 0$. However, $A \neq A_1$ and 4.11.6 give $b \neq 0$, and $a \neq 0$ follows from $a \ge b$; so $s > 0$ and $u \in x_1\mathbf{k}[x_1] = \mathcal{E}_\gamma$.

We continue to prepare for the proof of 4.10.4. See the beginning of 4.11 for the notation.

4.11.8. **Lemma.** *Suppose that $n > 1$ (resp. $n = 1$). Then for $f \in \mathbf{k}[x_1]$ we have:*

$$\Delta_f(A_2) = A_2 \iff f = 0 \quad (\textit{resp. } f \in \mathbf{k}).$$

Proof. Let $F \in \mathbf{k}[X_1]$ be such that $F(x_1) = f$, let $\varphi^{(k)} \in \mathbf{k}[Y]$ be the k-th derivative of φ, define

$$G = X_2 + \sum_{k=1}^{n} \frac{\varphi^{(k)}}{k!} X_1^{k-1} F(X_1)^k \in \mathbf{k}[X_1, X_2, Y]$$

and note that $\Delta_f(x_2) = G(x_1, x_2, y)$. Then we have

$$\begin{aligned} \Delta_f(A_2) = A_2 &\iff \exists_{\substack{\lambda\in\mathbf{k}^* \\ \mu\in\mathbf{k}}} \ \Delta_f(x_2) = \lambda x_2 + \mu \\ &\iff \exists_{\substack{\lambda\in\mathbf{k}^* \\ \mu\in\mathbf{k}}} \ G(x_1, x_2, y) = \lambda x_2 + \mu \\ &\iff \exists_{\substack{\lambda\in\mathbf{k}^* \\ \mu\in\mathbf{k}}} \ G = \lambda X_2 + \mu \ \ (\text{equality in } \mathbf{k}[X_1, X_2, Y]), \end{aligned}$$

where the last equivalence is a consequence of 2.2 and

$$\deg_Y G = \left\{ \begin{array}{ll} 0, & \text{if } f = 0 \\ n-1, & \text{if } f \neq 0 \end{array} \right\} < n \qquad \text{and} \qquad \deg_Y(\lambda X_2 + \mu) < n.$$

The desired result follows. $\square$

In the next result, $\mathcal{N}(A_1)$ denotes the set of neighbors of the vertex A_1 in the graph $\underline{\text{KLND}}\,(B)$.

4.11.9. **Lemma.** *Let $A_1 = \mathbf{k}[x_1] \in \text{KLND}(B)$. Then*

$$\begin{array}{rcl} \mathcal{E}_\gamma & \to & \mathcal{N}(A_1) \\ f & \mapsto & \delta_f(A_1) \end{array}$$

is a well-defined bijection.

Proof. Since A_1 is a neighbor of $A_2 = \mathbf{k}[x_2]$, it follows from 3.9.1 that $\delta_f(A_1)$ is a neighbor of $\delta_f(A_2) = A_1$ for every $f \in \mathbf{k}[x_1]$. Thus $\eta : \mathbf{k}[x_1] \to \mathcal{N}(A_1)$, $\eta(f) = \delta_f(A_1)$, is a well-defined map.

We show that η is surjective.

Case $n = 1$. Let $A \in \mathcal{N}(A_1)$. Define $a \in \mathbf{k}^*$ by the condition $\varphi = aY + b$ (for some $b \in \mathbf{k}$); note that $\Delta_f(x_2) = x_2 + af$, for every $f \in \mathbf{k}[x_1]$.

By (36), $A = \mathbf{k}[v]$ for some v satisfying $B = \mathbf{k}[x_1, v]$. Then $\mathbf{k}[x_1, x_2] = \mathbf{k}[x_1, v]$, so $\lambda v = x_2 + af$ for some $\lambda \in \mathbf{k}^*$ and $f \in \mathbf{k}[x_1]$. Now $\Delta_f(x_2) = x_2 + af = \lambda v$, so $\eta(f) = \delta_f(A_1) = \Delta_f(A_2) = \mathbf{k}[v] = A$.

Case $n > 1$ (i.e., $B \neq \mathbf{k}^{[2]}$)**.** Let $A \in \mathcal{N}(A_1)$. By 4.6, there exists $\gamma' = (x_1', x_2', y') \in \Gamma_{\mathbf{k}}(B)$ such that $A_1 = \mathbf{k}[x_1']$ and $A = \mathbf{k}[x_2']$. Applying 4.2 to γ' gives $y' \in \Omega_{\mathbf{k}}(A_1)$; thus $A_1 = \mathbf{k}[D_1(y')]$, where $D_1 = D_1^\gamma \in \text{LND}_{A_1}(B)$ is such that $D_1(y) = x_1$ (see 2.3 and remark (i) in part (1) of 4.5). So $\mathbf{k}[D_1(y')] = \mathbf{k}[x_1]$, which implies that $D_1(y') = \lambda x_1 + \mu$ for some $\lambda \in \mathbf{k}^*$ and $\mu \in \mathbf{k}$. Since $(x_1', x_2', \lambda^{-1} y') \in \Gamma_{\mathbf{k}}(B)$, we may in fact arrange that

$D_1(y') = x_1 + \mu$ for some $\mu \in \mathbf{k}$. Since $D_1(y' - y) = \mu$ and $B \neq \mathbf{k}^{[2]}$, $\mu = 0$; so $D_1(y') = x_1 = D_1(y)$.

Note that there is an irreducible $D_1' \in \text{LND}_{A_1}(B)$ such that $D_1'(y') = x_1'$ (namely, $D_1' = D_1^{\gamma'}$). By 1.9.3, we have $D_1 = \lambda D_1'$ (some $\lambda \in \mathbf{k}^*$), so $x_1 = D_1(y') = \lambda D_1'(y') = \lambda x_1'$ and consequently $(x_1, x_2', y') = (\lambda x_1', x_2', y') \in \Gamma_{\mathbf{k}}(B)$. To summarize,

$$(x_1, x_2', y') \in \Gamma_{\mathbf{k}}(B), \quad A = \mathbf{k}[x_2'] \quad \text{and} \quad D_1(y') = x_1 = D_1(y).$$

Since $D_1(y') = D_1(y)$, $y' - y \in \mathbf{k}[x_1]$. Noting that $(x_1, x_2', y' + c) \in \Gamma_{\mathbf{k}}(B)$ for every $c \in \mathbf{k}$, we may also arrange that $y' - y \in x_1\mathbf{k}[x_1]$. Then for some $f \in \mathbf{k}[x_1]$ we have $y' = y + x_1 f = \Delta_f(y)$.

Since $(x_1, x_2, y) \in \Gamma_{\mathbf{k}}(B)$, it follows that

$$(x_1, \Delta_f(x_2), y') = (\Delta_f(x_1), \Delta_f(x_2), \Delta_f(y)) \in \Gamma_{\mathbf{k}}(B).$$

Hence, both (x_1, x_2', y') and $(x_1, \Delta_f(x_2), y')$ belong to $\Gamma_{\mathbf{k}}(B)$. By 2.4, each of x_1x_2' and $x_1\Delta_f(x_2)$ generates the ideal $\mathbf{k}[y'] \cap x_1 B$ of $\mathbf{k}[y']$. It follows that x_2' and $\Delta_f(x_2)$ are associates, so

$$\eta(f) = \delta_f(A_1) = \Delta_f(A_2) = \mathbf{k}[\Delta_f(x_2)] = \mathbf{k}[x_2'] = A.$$

So $\eta : \mathbf{k}[x_1] \to \mathcal{N}(A_1)$ is a surjective map.

Consider again $A \in \mathcal{N}(A_1)$ and pick $f_0 \in \mathbf{k}[x_1]$ such that $\eta(f_0) = A$. Then, for $g \in \mathbf{k}[x_1]$ we have

$$\eta(g) = A \iff \eta(g) = \eta(f_0) \iff \Delta_g(A_2) = \Delta_{f_0}(A_2) \iff \Delta_{g-f_0}(A_2) = A_2$$

and, in view of 4.11.8, the last condition is equivalent to $g = f_0$ (resp. $g - f_0 \in \mathbf{k}$) if $n > 1$ (resp. if $n = 1$). Thus η is bijective if $n > 1$; and if $n = 1$ then exactly one element g of $x_1\mathbf{k}[x_1]$ satisfies $\eta(g) = A$. □

4.11.10. **Proposition.** *Let $h \in B \setminus \mathbf{k}$, let $f \in \mathcal{E}_\gamma \setminus \{0\}$ and assume the following:*

(i) *$h \in A$ for some $A \in \text{KLND}(B)$*

(ii) *$a > b$, where $(a, b) = \operatorname{bideg} h$.*

Then $\operatorname{bideg} \delta_f(h) = (ea, a)$ *where $e = n\big[1 + \deg_{x_1}(f)\big] - 1$. Moreover,*

$$e \geq 1 \quad \textit{and} \quad \big[\ e > 1 \Leftrightarrow n + \deg_{x_1}(f) \geq 3\ \big].$$

Proof. Let $d = 1 + \deg_{x_1}(f) \geq 1$, then $e = nd - 1 \geq 0$. If $e = 0$, then $nd = 1$, so $n = 1$ and $f \in \mathbf{k}^*$, which contradicts the assumption that $f \in \mathcal{E}_\gamma$. Hence, $e \geq 1$. The equivalence $e > 1 \Leftrightarrow n + \deg_{x_1}(f) \geq 3$ is trivial if $n \geq 3$, and is easily verified for each $n \in \{1, 2\}$.

Note that $a \geq 1$, because $(a, b) \in \mathbb{N} \times \mathbb{N}$ and $a > b$; also, $h \in A$ and 4.11.3 imply

$$\operatorname{bideg}(\tau h) = (b, a). \tag{39}$$

We have $B \subset R$, where $R = \mathbf{k}[x_1, x_1^{-1}, y]$, and observe that $\Delta_f \in \operatorname{Aut}_{\mathbf{k}}(B)$ extends to $\Delta_f \in \operatorname{Aut}_{R_0}(R)$, where $R_0 = \mathbf{k}[x_1, x_1^{-1}]$. Given $(i, j) \in \mathbb{Z} \times \mathbb{N}$, consider

$$S_{ij} = \operatorname{supp}\big[\Delta_f(x_1^i y^j)\big] = \operatorname{supp}\big[x_1^i (y + x_1 f)^j\big].$$

Direct calculation shows that $(i,j), (i+jd, 0) \in S_{ij} \subset T_{ij}$, where $T_{ij} \subset \mathbb{R}^2$ denotes the triangular region with vertices (i,j), $(i+j,0)$ and $(i+jd,0)$.

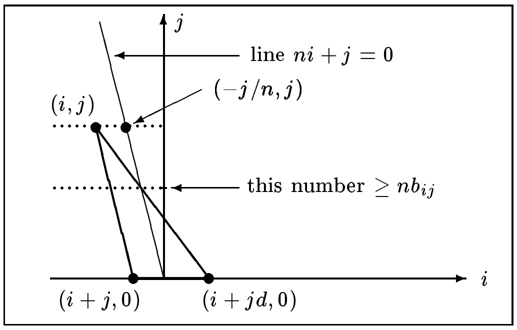

Figure 1

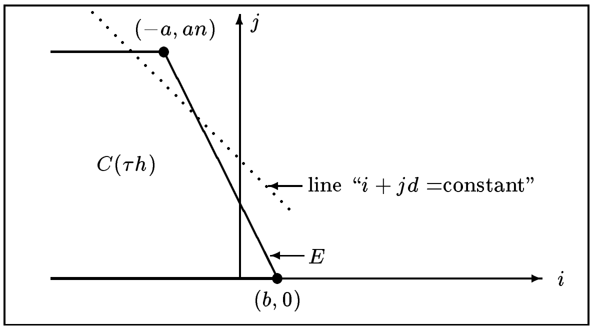

Figure 2

Thus bideg $\left[\Delta_f(x_1^i y^j)\right] = (i+jd, b_{ij})$, for some b_{ij}. By definition of bidegree, $b_{ij}(-1,n) \in \operatorname{supp}[\Delta_f(x_1^i y^j)] \subset T_{ij}$, so $nb_{ij} \leq j$ [because any point (i',j') of T_{ij} satisfies $j' \leq j$]; we record:

$$b_{ij} \leq j/n, \tag{40}$$

where equality holds if and only if $ni + j = 0$ or $j = 0$ (see Figure 1).

Suppose now that $(i,j) \in \operatorname{supp}(\tau h)$. Since $\tau(h) \in \tau A \in \text{KLND}(B)$, we may apply 4.11.2 to $\tau(h)$ and conclude that $\operatorname{supp}(\tau h) \subset C(\tau h)$; since bideg$(\tau h) = (b,a)$ by (39), we have[1]

$$(b,0), (-a, an) \in \operatorname{supp}(\tau h) \subset C(\tau h).$$

In particular, $(i,j) \in C(\tau h)$ implies that $j \leq an$, so

$$j/n \leq a. \tag{41}$$

By (40) and (41) we have $b_{ij} \leq a$ for all $(i,j) \in \operatorname{supp}(\tau h)$. If $(i,j) \in \operatorname{supp}(\tau h)$ satisfies $b_{ij} = a$, then equality must hold in both (40) and (41), so $(i,j) = (-a, na)$. Note that $(-a, na)$ does belong to $\operatorname{supp}(\tau h)$ and $b_{-a,na} = a$ [because if we regard $\Delta_f(x_1^{-a} y^{na}) = x_1^{-a}(y + x_1 f)^{na}$ as a polynomial in y with coefficients in $\mathbf{k}[x_1]$, then the leading term is $x_1^{-a} y^{na}$, which shows that $b_{-a,na} = a$]. So the second component of bideg $\left[\Delta_f(\tau h)\right]$ is a, i.e.,

$$\text{The second component of bideg } \left[\delta_f(h)\right] \text{ is } a.$$

Clearly, the slope of a line "$i + jd$ =constant" is equal to $-1/d$, and the slope of the line segment E joining $(b,0)$ to $(-a, an)$ is $-na/(a+b)$; thus

$$\text{(slope of line "}i + jd \text{ =constant")} - \text{(slope of } E) =$$

$$\frac{na}{a+b} - \frac{1}{d} = -\frac{nda - a - b}{d(a+b)} = \frac{ea - b}{d(a+b)} > 0$$

[1]In our case, $C(\tau h)$ is the closed and convex subset of $\mathbb{R}^2$ with boundary $H \cup E \cup H'$, where E is the line segment joining $(-a, an)$ to $(b,0)$, $H = \{\, (s,0) \mid s \leq b \,\}$ and $H' = \{\, (s, an) \mid s \leq -a \,\}$.

because $e \geq 1$ and $a > b$. Consequently,

$$0 > \text{slope of line "}i + jd = \text{constant"} > \text{slope of } E.$$

Hence, the maximum value of $i + jd$ on $\operatorname{supp}(\tau h)$ is reached at the point $(-a, an)$ and at no other point (see Figure 2). Since $\operatorname{bideg}\big[\Delta_f(x_1^i y^j)\big] = (i + jd, b_{ij})$, it follows that the first component of $\operatorname{bideg}\big[\Delta_f(\tau h)\big]$ is $-a + and = (-1 + nd)a = ea$. So

$$\operatorname{bideg}\big[\delta_f(h)\big] = (ea, a),$$

as desired. $\square$

4.11.11. **Proposition.** *For each $A \in \mathrm{KLND}(B) \setminus \{A_1\}$, there exists a unique $f \in \mathcal{E}_\gamma$ satisfying the following condition:*

> *If we define $(a, b) = \operatorname{bideg}(A)$, $A' = \delta_f^{-1}(A)$ and $(a', b') = \operatorname{bideg}(A')$, then $(a', b') < (a, b)$ and $a' > b'$.*

Moreover, we have $\deg_{x_1}(f) \geq 3 - n \iff a > b$.

Proof. We prove the existence of f by induction on $\operatorname{bideg}(A)$. Note that $\operatorname{bideg}(A) \geq (0, 1)$, by 4.11.6.

If $a < b$ then $f = 0 \in \mathcal{E}_\gamma$ satisfies the desired condition, by (4.11.5). In particular, this proves the case $\operatorname{bideg}(A) = (0, 1)$, i.e., the base case of induction.

Assume that $a \geq b$; by 4.11.7 and the remark following it, there exists $u \in \mathcal{E}_\gamma$ such that, if we write $R = \Delta_u(A)$, then $\operatorname{bideg}(R) = (a_1, b)$ with $a_1 < a$, so $\operatorname{bideg}(R) < \operatorname{bideg}(A)$. Observe that if $R = A_1$ then $A = \Delta_{-u}(A_1) = A_1$, a contradiction; hence $R \neq A_1$.

Since $R \neq A_1$ and $\operatorname{bideg}(R) < \operatorname{bideg}(A)$, we may assume by induction that there exists $g \in \mathcal{E}_\gamma$ such that, if we set $A' = \delta_g^{-1}(R)$, then $\operatorname{bideg}(A') < \operatorname{bideg}(R)$ and $a' > b'$ where $(a', b') = \operatorname{bideg}(A')$. Then

$$A' = \delta_g^{-1}(R) = \delta_g^{-1}\Delta_u(A) = \tau\Delta_{-g}\Delta_u(A) = \tau\Delta_{u-g}(A) = \delta_{g-u}^{-1}(A).$$

Note that $\mathcal{E}_\gamma$ is closed under addition, so $g - u \in \mathcal{E}_\gamma$. Thus $f = g - u$ satisfies the desired condition, which proves the existence of f.

We now prove uniqueness of f. Suppose that $f, g \in \mathcal{E}_\gamma$ satisfy the conditions $a' > b'$ and $a'' > b''$, where:

$$A' = \delta_f^{-1}(A), \quad (a', b') = \operatorname{bideg}(A'), \qquad A'' = \delta_g^{-1}(A), \quad (a'', b'') = \operatorname{bideg}(A'').$$

Since $\delta_f(A') = \delta_g(A'')$, it follows that $\Delta_f\tau(A') = \Delta_g\tau(A'')$, so $\Delta_{f-g}\tau(A') = \tau(A'')$, i.e.,

$$\delta_{f-g}(A') = \tau A''. \tag{42}$$

By (4.11.5), $\tau A''$ has bidegree (b'', a''); since $b'' < a''$, the bidegree of the ring $\delta_{f-g}(A') = \tau A''$ cannot be of the form (ea', a'), where e is a positive integer.

If $f - g \neq 0$ then $f - g \in \mathcal{E}_\gamma \setminus \{0\}$ (and $a' > b'$), so 4.11.10 implies that $\operatorname{bideg}[\Delta_{f-g}\tau(A')] = (ea', a')$ for some positive integer e. This contradicts the preceding paragraph, so $f - g = 0$, i.e., f is unique.

Finally, we prove the last assertion of 4.11.11. Let $e = n\big[1 + \deg_{x_1}(f)\big] - 1$.

Suppose that $a > b$. If $f = 0$ then $A' = \delta_0^{-1}(A) = \tau(A)$, so $(a', b') = \operatorname{bideg}(A') = (b, a)$ by (4.11.5), and since $a' > b'$ we get $a < b$, a contradiction. Hence, $f \in \mathcal{E}_\gamma \setminus \{0\}$. Since we also have $a' > b'$, 4.11.10 implies that $\operatorname{bideg}\big[\delta_f(A')\big] = (ea', a')$, where $e \geq 1$. So $(a, b) = (ea', a')$; now the assumption $a > b$ implies that $e > 1$, which gives $\deg_{x_1}(f) \geq 3 - n$.

Conversely, suppose that $\deg_{x_1}(f) \geq 3 - n$. Then $f \neq 0$, so $f \in \mathcal{E}_\gamma \setminus \{0\}$; together with $a' > b'$ and 4.11.10, this implies that $(a, b) = (ea', a')$ where $e \geq 1$. But in fact the condition $\deg_{x_1}(f) \geq 3 - n$ implies that $e > 1$, so $a > b$. $\square$

4.11.12. **Lemma.** *Let $(f_1, \dots, f_k) \in \mathcal{F}_\gamma^\circ$ and define*

$$R_i = (\delta_{f_{i+1}} \cdots \delta_{f_k})(A_1) \quad \textit{and} \quad (a_i, b_i) = \operatorname{bideg}(R_i) \qquad (0 \leq i \leq k).$$

Then $(1, 0) = (a_k, b_k) < \cdots < (a_0, b_0)$ and, for each $i > 0$, $a_i > b_i$. Moreover,

$$a_0 > b_0 \iff \deg_{x_1}(f_1) \geq 3 - n.$$

Proof. Since $R_k = A_1$, $(1, 0) = (a_k, b_k)$ and $a_k > b_k$ are clear. Suppose that for some $j \in \{1, \dots, k\}$ we have

$$(1, 0) = (a_k, b_k) < \cdots < (a_j, b_j) \quad \text{and, for each } i \in \{j, \dots, k\}, \quad a_i > b_i.$$

Proceding by descending induction, it suffices to prove:

$$(a_j, b_j) < (a_{j-1}, b_{j-1}) \text{ and } [j = 1 \text{ or } a_{j-1} > b_{j-1}]. \tag{43}$$

We consider two cases. If $j > 1$ then the definition of $\mathcal{F}_\gamma^\circ$ gives $f_j \in \mathcal{E}_\gamma$ and $\deg_{x_1}(f_j) \geq 3-n$; together with $a_j > b_j$ and 4.11.10, this implies that $\operatorname{bideg}\big[\delta_{f_j}(R_j)\big] = (ea_j, a_j)$ for some $e > 1$. Since $\delta_{f_j}(R_j) = R_{j-1}$, this gives $(a_{j-1}, b_{j-1}) = (ea_j, a_j)$, so $(a_j, b_j) < (a_{j-1}, b_{j-1})$ and $a_{j-1} > b_{j-1}$, i.e., (43) holds.

If $j = 1$ then we still have $f_1 \in \mathcal{E}_\gamma$ and $a_1 > b_1$. If $f_1 = 0$ then $R_0 = \delta_0 R_1 = \tau R_1$ has bidegree (b_1, a_1) by (4.11.5), so $(a_1, b_1) < (a_0, b_0)$; if $f_1 \neq 0$ then 4.11.10 implies that $R_0 = \delta_{f_1} R_1$ has bidegree (ea_1, a_1) for some $e \geq 1$, so again $(a_1, b_1) < (a_0, b_0)$. Hence, (43) holds in all cases.

To prove that $a_0 > b_0 \Leftrightarrow \deg_{x_1}(f_1) \geq 3 - n$, observe that the conditions

$$f_1 \in \mathcal{E}_\gamma, \quad R_1 = \delta_{f_1}^{-1}(R_0), \quad (a_1, b_1) < (a_0, b_0) \quad \text{and} \quad a_1 > b_1$$

show that f_1 is the unique element of $\mathcal{E}_\gamma$ determined by $R_0 \in \text{KLND}(B) \setminus \{A_1\}$ (see 4.11.11); then the last assertion of 4.11.11 is the desired result. $\square$

Proof of 4.10.4. Consider an edge $\{\mathfrak{f}, \mathfrak{f}'\}$ of $\underline{\mathcal{F}}_\gamma$, where

$$\mathfrak{f} = (f_1, \dots, f_k) \quad \text{and} \quad \mathfrak{f}' = (f_1, \dots, f_k, f_{k+1});$$

write $\delta = \delta_{f_1} \circ \cdots \circ \delta_{f_k}$, $A = \mathcal{P}_\gamma(\mathfrak{f}) = \delta(A_1)$ and $A' = \mathcal{P}_\gamma(\mathfrak{f}') = \delta \circ \delta_{f_{k+1}}(A_1)$. Since $\delta^{-1}(A') = \delta_{f_{k+1}}(A_1)$ is a neighbor of $\delta^{-1}(A) = A_1$ by 4.11.9, it follows that A' is a neighbor of A. This proves (1).

Observe that the connectedness of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ (4.8) implies that every vertex of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ is an endpoint of some edge; so, in order to prove (2), it suffices to prove surjectivity on the edges. Now, again by connectedness of $\underline{\text{KLND}}_{\mathbf{k}}(B)$, if e is any edge of $\underline{\text{KLND}}_{\mathbf{k}}(B)$ then there exists a simple path P with initial point A_1 and which traverses e. So it suffices to prove:

(44) *Suppose that $P = (R_0, \dots, R_k)$ is a locally simple path in $\underline{\text{KLND}}_{\mathbf{k}}(B)$ such that $R_0 = A_1$. Then there exists $\mathfrak{f} = (f_1, \dots, f_k) \in \mathcal{F}_\gamma$ such that $\big\{(\delta_{f_1} \cdots \delta_{f_i})(A_1)\big\}_{i=0}^{k} = P$.*

If $k = 0$ then $\mathfrak{f} = \varnothing$ (empty sequence) satisfies (44). Assume that $k > 0$ and that $(f_1, \dots, f_{k-1}) \in \mathcal{F}_\gamma$ is such that

$$\big\{(\delta_{f_1} \cdots \delta_{f_i})(A_1)\big\}_{i=0}^{k-1} = (R_0, \dots, R_{k-1}).$$

Write $\delta = \delta_{f_1} \circ \cdots \circ \delta_{f_{k-1}}$. Then $\delta^{-1}(R_k)$ is a neighbor of $\delta^{-1}(R_{k-1}) = A_1$ so, by 4.11.9, there is a unique $f_k \in \mathcal{E}_\gamma$ such that $\delta_{f_k}(A_1) = \delta^{-1}(R_k)$; this implies that $(\delta_{f_1} \cdots \delta_{f_k})(A_1) = R_k$ so there remains only to check that $(f_1, \dots, f_k) \in \mathcal{F}_\gamma$.

Assume that $(f_1, \dots, f_k) \notin \mathcal{F}_\gamma$, then we must have $k > 1$ and $f_k = 0$; writing $\delta' = \delta_{f_1} \circ \cdots \circ \delta_{f_{k-2}}$, we have

$$R_k = \delta' \circ \delta_{f_{k-1}} \circ \delta_0(A_1) = \delta' \circ \Delta_{f_{k-1}} \circ \tau^2(A_1) = \delta'(A_1) = R_{k-2},$$

contradicting the hypothesis that P is locally simple. So (44) is proved and so is assertion (2).

Let $A \in \text{KLND}(B)$. By induction on $(a, b) = \text{bideg}(A)$, we show that A is in the image of $\mathcal{P}_\gamma^\circ : \mathcal{F}_\gamma^\circ \to \text{KLND}_{\mathbf{k}}(B)$. If $(a, b) = (1, 0)$ then $A = A_1$ by 4.11.6, so $\mathcal{P}_\gamma^\circ(\varnothing) = A$.

Suppose that $(a, b) > (1, 0)$; then $A \in \text{KLND}(B) \setminus \{A_1\}$. By 4.11.11, there exists $f \in \mathcal{E}_\gamma$ such that, if we define

$$A' = \delta_f^{-1}(A) \qquad \text{and} \qquad (a', b') = \text{bideg}(A'),$$

then $a' > b'$ and $(a', b') < (a, b)$. By induction, we may assume that $A' = \mathcal{P}_\gamma^\circ(\mathfrak{f}')$ for some vertex $\mathfrak{f}' = (f_1, \dots, f_k)$ of $\underline{\mathcal{F}}_\gamma^\circ$. We claim that

(45) $$\mathfrak{f} = (f, f_1, \dots, f_k) \text{ is a vertex of } \underline{\mathcal{F}}_\gamma^\circ.$$

To see this, it suffices to show that, if $\mathfrak{f}' \neq \varnothing$, then $\deg_{x_1}(f_1) \geq 3 - n$. Assume that $\mathfrak{f}' \neq \varnothing$ and apply 4.11.11 to $(f_1, \dots, f_k)$; then the last assertion of 4.11.11 reads $\deg_{x_1}(f_1) \geq 3 - n \Leftrightarrow a' > b'$. Since $a' > b'$ does hold, (45) follows. Clearly, $\mathcal{P}_\gamma^\circ(\mathfrak{f}) = \delta_f(A') = A$. Thus $\mathcal{P}_\gamma^\circ$ is surjective on vertices.

Notice the following consequence of 4.11.12: If $\mathfrak{f}$ is a vertex of $\underline{\mathcal{F}}_\gamma^\circ$ other than $\varnothing$, then $\mathcal{P}_\gamma^\circ(\mathfrak{f})$ has bidegree strictly greater than $(1, 0)$; in other words, the only element of ${\mathcal{P}_\gamma^\circ}^{-1}(A_1)$ is the empty sequence.

Suppose that $\mathcal{P}_\gamma^\circ$ is not injective (on vertices). Then we may choose distinct vertices $\mathfrak{f}, \mathfrak{f}'$ of $\underline{\mathcal{F}}_\gamma^\circ$ such that $\mathcal{P}_\gamma^\circ(\mathfrak{f}) = \mathcal{P}_\gamma^\circ(\mathfrak{f}')$. Write $\mathfrak{f} = (f_1, \dots, f_k)$ and $\mathfrak{f}' = (g_1, \dots, g_m)$ and assume that we have chosen $\mathfrak{f}, \mathfrak{f}'$ such that $k + m$ is minimal. Write $A = \mathcal{P}_\gamma^\circ(\mathfrak{f}) = \mathcal{P}_\gamma^\circ(\mathfrak{f}')$.

Since $\mathfrak{f} \neq \mathfrak{f}'$, at least one of $\mathfrak{f}, \mathfrak{f}'$ is nonempty, so $A \neq A_1$ by the preceding paragraph, so both $\mathfrak{f}$ and $\mathfrak{f}'$ are nonempty.

Result 4.11.12 implies that each element f of $\{f_1, g_1\}$ satisfies:

> *If we define* $(a,b) = \operatorname{bideg}(A)$, $A' = \delta_f^{-1}(A)$ *and* $(a',b') = \operatorname{bideg}(A')$, *then* $(a',b') < (a,b)$ *and* $a' > b'$.

So the uniqueness part of 4.11.11 implies that $f_1 = g_1$.

Notice that $\mathfrak{f}_* = (f_2, \dots, f_k)$ and $\mathfrak{f}'_* = (g_2, \dots, g_m)$ belong to $\mathcal{F}_\gamma^\circ$. Since $f_1 = g_1$, $\mathcal{P}_\gamma^\circ(\mathfrak{f}_*) = \mathcal{P}_\gamma^\circ(\mathfrak{f}'_*)$; by minimality of $k+m$ we obtain $\mathfrak{f}_* = \mathfrak{f}'_*$, which implies that $\mathfrak{f} = \mathfrak{f}'$, a contradiction. This proves assertion (3).

If $n > 2$ then $\mathcal{F}_\gamma = \mathcal{F}_\gamma^\circ$ and $\mathcal{P}_\gamma = \mathcal{P}_\gamma^\circ$. By (1–3), $\mathcal{P}_\gamma$ is a homomorphism of graphs which is bijective on vertices and surjective on edges; it follows that it is an isomorphism, so (4) is true.

This completes the proof of 4.10.4. $\square$

5. Factorial affine domains

By a *factorial affine domain*, we mean a UFD which is affine over some field (of characteristic zero, as always in this paper). The main result of this section is 5.1, which improves 3.7.

5.1. Theorem. *If B is a factorial affine domain then:*

1. $\underline{\text{KLND}}_*(B) = \bigcup_{R \in \mathcal{R}(B)} \underline{\text{KLND}}_R(B)$
2. *For each $R \in \mathcal{R}(B)$, $\underline{\text{KLND}}_R(B)$ is isomorphic to $\underline{\text{KLND}}_{R_R}(B_R)$ and (B_R, R_R) is a Danielewski surface. In particular, $\underline{\text{KLND}}_R(B)$ is infinite and connected.*
3. *If R, R' are distinct elements of $\mathcal{R}(B)$, the graphs $\underline{\text{KLND}}_R(B)$ and $\underline{\text{KLND}}_{R'}(B)$ have at most one vertex in common.*

Remarks.

- Assertion 5.1(3) simply repeats 3.7(b).
- In 5.1(2), the fact that (B_R, R_R) is a Danielewski surface follows from the definition of $\mathcal{R}(B)$, and we know by section 4 that the graph of a Danielewski surface is infinite and connected.

The proof of 5.1 requires some preparation. First, we define a set $\mathcal{R}^{\text{in}}(B)$ of subrings of B which is larger than $\mathcal{R}(B)$:

5.2. Definition. Given an integral domain B,

$$\mathcal{R}^{\text{in}}(B) = \big\{\, R \mid R \text{ is an inert subring of } B \text{ and } \operatorname{trdeg}_R(B) = 2 \,\big\}.$$

5.3. Lemma. *Let B be a factorial affine domain and $R \in \mathcal{R}^{\text{in}}(B)$. Then:*

(a) *The map $\Lambda : \text{KLND}_R(B) \to \text{KLND}_{R_R}(B_R)$, $A \mapsto A_R$, is well-defined and bijective. Its inverse is given by $\mathcal{A} \mapsto \mathcal{A} \cap B$.*

(b) *The bijection Λ is an isomorphism of graphs, $\underline{\text{KLND}}_R(B) \to \underline{\text{KLND}}_{R_R}(B_R)$.*

Proof. It is clear that B_R has transcendence degree two over R_R, so it makes sense to consider the graph $\underline{\text{KLND}}_{R_R}(B_R)$. Note that $\text{KLND}_{R_R}(B_R) = \text{KLND}(B_R)$, by 1.6 and the fact that R_R is a field contained in B_R.

We prove (a) now, and (b) will be proved after 5.3.3, below.

The fact that $\Lambda : \text{KLND}_R(B) \to \text{KLND}_{R_R}(B_R)$ is well-defined and injective is a consequence of part (2) of 1.6.

Before proving that Λ is surjective, we first note that B is affine over R. Indeed, we have $B^* = R^*$ because R is an inert subring of B. Let $\mathbf{k} \subseteq B$ be a field over which B is affine. Then $\mathbf{k}^* \subseteq B^* = R^*$, so $\mathbf{k} \subseteq R \subset B$ and it follows that B is affine over R.

To show that Λ is surjective, consider $\mathcal{A} \in \text{KLND}_{R_R}(B_R)$. Choose $\mathcal{D} \in \text{LND}_{R_R}(B_R)$ such that $\ker \mathcal{D} = \mathcal{A}$. Since B is affine over R, we may consider $b_1, \dots, b_n$ such that $B = R[b_1, \dots, b_n]$. For each $i \in \{1, \dots, n\}$, we have $\mathcal{D}(b_i) \in B_R$; so there exists $r \in R \setminus \{0\}$ satisfying

$$\forall_i \quad r\mathcal{D}(b_i) \in B.$$

Since the derivation $r\mathcal{D} : B_R \to B_R$ maps R to 0 and maps each b_i in B, it maps B into itself; also, $r\mathcal{D}$ is locally nilpotent, since $r \in \ker \mathcal{D}$. Let $D : B \to B$ be the restriction of $r\mathcal{D}$, then $D \in \text{LND}_R(B)$ and $\ker D = A$, where we define $A = B \cap \mathcal{A}$. Since D has a *unique* extension to a derivation of B_R, we have $D_R = r\mathcal{D}$; by 1.6, the kernel of D_R is A_R, so we obtain $\mathcal{A} = A_R = \Lambda(A)$. So Λ is surjective and (a) is proved. ☐

The next three facts are needed for the proof of 5.3(b). The first one is well-known and easy to prove.

5.3.1. *Let B be a UFD and $A \in \text{KLND}(B)$. Then:*

(1) *There exists an irreducible $D \in \text{LND}_A(B)$.*
(2) *If $D_1, D_2 \in \text{LND}_A(B)$ are irreducible, then $D_2 = \lambda D_1$ for some $\lambda \in B^*$.*

5.3.2. **Lemma.** *Let B be a UFD, R an inert subring of B and $D : B \to B$ an irreducible R-derivation. Then $D_R : B_R \to B_R$ is irreducible.*

Proof. Assume the contrary; then there exists $b \in B_R \setminus {B_R}^*$ such that $D_R(B_R) \subseteq bB_R$. In fact, such an element b may be chosen in B. Then some prime factor $p \in B$ of b satisfies $p \notin {B_R}^*$.

Since D is irreducible and $p \notin B^*$, we may choose $x \in B$ such that $Dx \notin pB$. Since $D(x) = D_R(x) \in pB_R$, there exists $r \in R \setminus \{0\}$ such that $p \mid rD(x)$ in B. Then $p \mid r$ in B; since $r \in R \setminus \{0\}$ and R is an inert subring of B, $p \in R \setminus \{0\}$. Thus $p \in {B_R}^*$, a contradiction. ☐

5.3.3. **Lemma.** *Let B be a UFD, $R \in \mathcal{R}^{\text{in}}(B)$ and $K = R_R$. Then, for each $A \in \text{KLND}_R(B)$,*

$$\Omega_R(A) = B \cap \Omega_K(A_R).$$

Remark. Since K is a field contained in A_R, we have $(A_R)_K = A_R$. So the definition of $\Omega_K(A_R)$ reads:

$$\Omega_K(A_R) = \big\{\, \zeta \in B_R \mid \exists \text{ an irreducible } \Delta \in \text{LND}_{A_R}(B_R) \text{ such that } A_R = K[\Delta\zeta] \,\big\}.$$

Proof of 5.3.3. Let $y \in \Omega_R(A)$. Then there exists an irreducible $D \in \mathrm{LND}_A(B)$ such that $A_R = K[Dy]$. By 1.6, $D_R : B_R \to B_R$ belongs to $\mathrm{LND}_{A_R}(B_R)$; moreover, D_R is irreducible by 5.3.2. Since $A_R = K[Dy] = K[D_R(y)]$, we have $y \in \Omega_K(A_R)$. This proves that $\Omega_R(A) \subseteq B \cap \Omega_K(A_R)$.

Conversely, suppose that $y \in B \cap \Omega_K(A_R)$. Then there exists an irreducible $\Delta \in \mathrm{LND}_{A_R}(B_R)$ such that $A_R = K[\Delta y]$. On the other hand, 5.3.1 allows us to consider an irreducible $D \in \mathrm{LND}_A(B)$ and, by 5.3.2, D_R is irreducible. Thus D_R and Δ are two irreducible derivations belonging to $\mathrm{LND}_{A_R}(B_R)$; using 5.3.1 again, we get $D_R = \lambda\Delta$ for some $\lambda \in {B_R}^*$. Since R is inert in B, K is inert in B_R, so ${B_R}^* = K^*$ and $\lambda \in K^*$. So

$$K[Dy] = K[D_R(y)] = K[\lambda\Delta(y)] = K[\Delta(y)] = A_R,$$

showing that $y \in \Omega_R(A)$. This proves that $B \cap \Omega_K(A_R) \subseteq \Omega_R(A)$. □

Proof of 5.3(b). Write $K = R_R$. We have to verify that, given distinct $A, A' \in \mathrm{KLND}_R(B)$,

$$\Omega_R(A) \cap \Omega_R(A') \neq \varnothing \iff \Omega_K(A_R) \cap \Omega_K(A'_R) \neq \varnothing. \tag{46}$$

By 5.3.3, we have in particular $\Omega_R(A) \subseteq \Omega_K(A_R)$ and $\Omega_R(A') \subseteq \Omega_K(A'_R)$, so "$\implies$" holds in (46).

Conversely, suppose that $\omega \in \Omega_K(A_R) \cap \Omega_K(A'_R)$. For any $\lambda \in K^*$, we have $\lambda\omega \in \Omega_K(A_R) \cap \Omega_K(A'_R)$; choose $\lambda \in R \setminus \{0\}$ such that $\lambda\omega \in B$, then 5.3.3 gives

$$\lambda\omega \in B \cap \Omega_K(A_R) \cap \Omega_K(A'_R) = \Omega_R(A) \cap \Omega_R(A'),$$

so "$\impliedby$" holds in (46). This proves 5.3(b). □

Proof of 5.1. Assertion (3) (of 5.1) is given in 3.7, so only (1) and (2) need proof.

If $R \in \mathcal{R}(B)$ then (by definition) (B_R, R_R) is a Danielewski surface; so $\underline{\mathrm{KLND}}_{R_R}(B_R)$ is connected by 4.8, and contains infinitely many vertices by (say) 4.11.9. Now $R \in \mathcal{R}(B)$ also implies that $R \in \mathcal{R}^{\mathrm{in}}(B)$, so $\underline{\mathrm{KLND}}_R(B) \cong \underline{\mathrm{KLND}}_{R_R}(B_R)$ by 5.3; this proves assertion (2).

For each $R \in \mathcal{R}(B)$, assertion (2) implies that $\underline{\mathrm{KLND}}_R(B)$ has no isolated vertex; thus $\bigcup_{R\in\mathcal{R}(B)} \underline{\mathrm{KLND}}_R(B) \subseteq \underline{\mathrm{KLND}}_*(B)$. This and 3.7 imply assertion (1).

This completes the proof of 5.1. □

6. Some philosophical remarks

Given any integral domain B (of characteristic zero) we have defined three graphs, $\underline{\mathrm{KLND}}(B)$, $\underline{\mathrm{KLND}}_*(B)$ and $\underline{\mathcal{R}}(B)$, which are invariants of B up to isomorphism. Moreover, the structures of $\underline{\mathrm{KLND}}_*(B)$ and $\underline{\mathcal{R}}(B)$ are closely related and $\underline{\mathcal{R}}(B)$ should be thought of as a "simplified version" of $\underline{\mathrm{KLND}}_*(B)$: If B is factorial and affine, $\underline{\mathcal{R}}(B)$ is isomorphic to the graph obtained from $\underline{\mathrm{KLND}}_*(B)$ by shrinking each connected subgraph $\underline{\mathrm{KLND}}_R(B)$ (where $R \in \mathcal{R}(B)$) to a single vertex.

To illustrate the claim that $\underline{\text{KLND}}_*(B)$ and $\underline{\mathcal{R}}(B)$ have closely related structures we mention the following easy consequence of 5.1:

If B is a factorial affine domain then $\underline{\text{KLND}}_*(B)$ and $\underline{\mathcal{R}}(B)$ have the same number of connected components. In particular,

$$\underline{\text{KLND}}_*(B) \text{ is connected} \iff \underline{\mathcal{R}}(B) \text{ is connected.}$$

Consider the problem of describing $\underline{\text{KLND}}(B)$. In view of 5.1 and of the fact that the graphs $\underline{\text{KLND}}_R(B) \cong \underline{\text{KLND}}_{R_R}(B_R)$ are described in section 4, we are justified to state the following:

6.1. **Aphorism.** *Let B be a factorial affine domain. To achieve a satisfactory description of $\underline{\text{KLND}}(B)$, it suffices to solve the following problems:*

(1) *Describe the kernels $A \in \text{KLND}(B)$ which are isolated vertices of $\underline{\text{KLND}}(B)$.*
(2) *Describe the graph $\underline{\mathcal{R}}(B)$.*

A particularly interesting factorial affine domain is $B = \mathbf{k}[X, Y, Z] = \mathbf{k}^{[3]}$. For this ring, the above problems (1) and (2) are still open but there are some partial results that we intend to give in a subsequent paper. Let us mention that a crucial rôle is played by the polynomials $f \in \mathbf{k}[X, Y, Z]$ whose generic fiber is a Danielewski surface, i.e.,

$$\textit{the pair } \big(\mathbf{k}(f)[X, Y, Z],\, \mathbf{k}(f)\big) \textit{ is a Danielewski surface.}$$

In fact, it is not too difficult to show that $\mathcal{R}(B)$ is precisely the set of rings $\mathbf{k}[f]$ such that $f \in B$ is a polynomial whose generic fiber is a Danielewski surface.

It seems to this author that, in order to understand the locally nilpotent derivations (and the automorphisms) of $\mathbf{k}^{[3]}$, it will be necessary to better understand the polynomials whose generic fiber is a Danielewski surface. It may be a good idea to think of those polynomials as generalized variables.

Isolated vertices

This paper made some progress in the understanding of $\underline{\text{KLND}}_*(B)$, but essentially nothing has been said about isolated vertices of $\underline{\text{KLND}}(B)$. In particular, it would be interesting to classify two-dimensional rings B such that $\underline{\text{KLND}}(B)$ is a discrete graph with many vertices. The smooth surfaces $X_{m,j}$ which are studied in [4] give examples of such rings:

6.2. **Example.** Fix two integers $0 < j < m$ such that $\gcd(j, m) = 1$. Consider the Danielewski surface $B = \mathbb{C}[x_1, x_2, y]$ defined by $x_1 x_2 = y^m - 1$. Let $\zeta \in \mathbb{C}$ be a primitive m-th root of unity and define $\theta \in \text{Aut}_{\mathbb{C}}(B)$ by $\theta(x_1) = \zeta x_1$, $\theta(x_2) = \zeta^{-1} x_1$ and $\theta(y) = \zeta^j y$. Finally, let $B_{m,j} = \big\{ b \in B \mid \theta(b) = b \big\}$. Then Theorem 2.9 of [4] shows, among other things, that the smooth surface $X_{m,j} = \text{Spec}(B_{m,j})$ is a $\mathbb{Q}$-homology plane with $|\operatorname{Pic}(X_{m,j})| = m$. We claim:

(47) $\underline{\text{KLND}}(B_{m,j})$ *is a discrete graph whose vertex set has the cardinality of* $\mathbb{C}$.

Proof of (47). Assume that $\underline{\text{KLND}}(B_{m,j})$ is non-discrete. Then, by 4.3, there exists a field $\mathbf{k} \subset B_{m,j}$ such that $(B_{m,j}, \mathbf{k})$ is a Danielewski surface; since $\mathbf{k}$ must satisfy

$B^*_{m,j} = \mathbf{k}^*$, and since it is clear that $B^*_{m,j} = \mathbb{C}^*$ (because $\mathbb{C} \subset B_{m,j} \subseteq B$ and $B^* = \mathbb{C}^*$), we must then have $\mathbf{k} = \mathbb{C}$. However, it is known (see 2.8 of [4]) that any smooth Danielewski surface (over $\mathbb{C}$) of degree n has a Picard group isomorphic to $\mathbb{Z}^{n-1}$; since $X_{m,j}$ is smooth and has a Picard group of order m, it cannot be a Danielewski surface over $\mathbb{C}$. This contradiction shows that $\underline{\mathrm{KLND}}(B_{m,j})$ is discrete.

Theorem 2.9 of [4] also implies that $\mathrm{KLND}(B_{m,j})$ has at least two elements. In view of 1.10, it follows that $|\,\mathrm{KLND}(B_{m,j})| = |\mathbb{C}|$. $\square$

Local slice construction

As mentioned in the introduction, the present work started as an attempt to understand [3]. In that paper, Freudenburg presents a method for modifying a given kernel $A \in \mathrm{KLND}(B)$, where $B = \mathbf{k}^{[3]}$, so as to obtain another one, say $A' \in \mathrm{KLND}(B)$; in that case he says that A' is obtained from A *by local slice construction.*

To conclude this paper, we show that the graph $\underline{\mathrm{KLND}}(B)$ can be interpreted as method for modifying kernels, in the same spirit as [3]. This works best when B is a factorial affine domain:

6.3. **Proposition.** *Let B be a factorial affine domain and consider a triple (R, A, y) where $R \in \mathcal{R}(B)$, $A \in \mathrm{KLND}_R(B)$ and $y \in \Omega_R(A)$. Then there exists exactly one $A' \in \mathrm{KLND}_R(B)$ such that*

$$y \in \Omega_R(A') \quad \text{and} \quad A' \neq A.$$

6.3.1. **Definition.** In the situation of 6.3, we say:

A' is obtained from (R, A, y) by local slice construction.

Remark. There is a method for computing A' from (R, A, y), similar to the method described in [3], but we leave this aspect to the reader.

The first step in the proof of 6.3 is:

6.3.2. **Lemma.** *Let $(B, \mathbf{k})$ be a Danielewski surface and $y \in B$. Then the set*

$$E = \{\, A \in \mathrm{KLND}(B) \mid y \in \Omega_{\mathbf{k}}(A) \,\}$$

has cardinality zero or two.

Proof. Suppose that $A \in E$. Then $y \in \Omega_{\mathbf{k}}(A)$, so there exists an irreducible $D \in \mathrm{LND}_A(B)$ satisfying $A = \mathbf{k}[Dy]$. Write $x = Dy$, then 2.8 implies that $(x, x_2, y) \in \Gamma_{\mathbf{k}}(B)$ for some $x_2 \in B$. Write $A' = \mathbf{k}[x_2]$, then $A' \in \mathrm{KLND}(B)$, $A' \neq A$ and 4.2 gives $y \in \Omega_{\mathbf{k}}(A) \cap \Omega_{\mathbf{k}}(A')$; so $A' \in E$. This shows that if $E \neq \varnothing$ then $|E| \geq 2$.

To finish the proof of 6.3.2, it suffices to show that if $A_1, A_2, A_3 \in E$ satisfy

$$A_1 \neq A_2 \quad \text{and} \quad A_1 \neq A_3, \tag{48}$$

then $A_2 = A_3$.

Suppose that (48) holds. For each $i \in \{1, 2, 3\}$ we have $y \in \Omega_{\mathbf{k}}(A_i)$ and consequently there exists an irreducible $D_i \in \mathrm{LND}_{A_i}(B)$ satisfying $A_i = \mathbf{k}[D_i(y)]$. Let $x_i = D_i(y)$, then $\ker D_i = A_i = \mathbf{k}[x_i] = \mathbf{k}^{[1]}$ (for each $i \in \{1, 2, 3\}$).

Let $j \in \{2,3\}$. Since $A_1 \neq A_j$, (y, D_1, D_j) satisfies the hypothesis of 2.5; since (2.5-1) is false, (2.5-2) must hold, so $(x_1, x_j, y) \in \Gamma_{\mathbf{k}}(B)$. This and 2.4 imply that $\mathbf{k}[y] \cap x_1 B$ is the principal ideal of $\mathbf{k}[y]$ generated by $x_1 x_j$.

So $x_1 x_2$ and $x_1 x_3$ are associates in $\mathbf{k}[y]$ and consequently $x_3 = \lambda x_2$ for some $\lambda \in \mathbf{k}^*$. So $A_2 = A_3$ and 6.3.2 is proved. $\square$

Proof of 6.3. Let B be a factorial affine domain, let $R \in \mathcal{R}(B)$ and let $y \in B$. We have to show that the set

$$E_{(B,R,y)} = \left\{ A \in \text{KLND}(B) \mid y \in \Omega_R(A) \right\}$$

has cardinality zero or two. Since $R \in \mathcal{R}(B)$, the pair (B_R, R_R) is a Danielewski surface so the set

$$E = \left\{ \mathcal{A} \in \text{KLND}(B_R) \mid y \in \Omega_{R_R}(\mathcal{A}) \right\}$$

has cardinality zero or two by 6.3.2. By 5.3,

$$\begin{array}{rcl} \Lambda : \text{KLND}_R(B) & \longrightarrow & \text{KLND}(B_R) \\ A & \longmapsto & A_R \end{array}$$

is a well-defined bijection and, in view of 5.3.3, for each $A \in \text{KLND}_R(B)$ we have

$$A \in E_{(B,R,y)} \Leftrightarrow y \in \Omega_R(A) \Leftrightarrow y \in B \cap \Omega_{R_R}(A_R) \Leftrightarrow y \in \Omega_{R_R}(A_R) \Leftrightarrow A_R \in E,$$

i.e., $E_{(B,R,y)} = \Lambda^{-1}(E)$. So $E_{(B,R,y)}$ has cardinality zero or two and 6.3 is proved. $\square$

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