Redistributive Equalization Payments in Federations[1]

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Abstract

We investigate the conditions under which an inequality averse and additively separable welfarist social planner would always choose to set up a redistributive equalization payment scheme in a federation. A redistributive equalization payment scheme is defined as a list of *per capita* net (possibly negative) subsidies - one such net subsidy for every jurisdiction - that are decreasing with respect to jurisdictions' *per capita* wealth. We examine the question in a setting in which the case for progressivity is *a priori* the strongest, namely: all households have the same utility function for the private good and the public good, households of a given jurisdiction are all identicals and are not able to move accross jurisdictions. We show that even in this favourable case, a redistributive equalization payment scheme is *not* necessarily a characteristic of a federation that a welfarist social planner would select. We show more specifically that a necessary and sufficient condition that the objective function of the social planner must satisfy to favour a redistributive equalization payment scheme for all distributions of wealth and households accross jurisdictions is to be additively separable between each jurisdiction's per capita wealth and number of households. When interpreted as a mean of order r social welfare function, this condition is shown to be equivalent to additive separability of the households' indirect utility function with respect to wealth and the price of the public good. Some implications of this restriction in the case where households' direct utility function are assumed to be additively separable are also provided.

1 Introduction

Many federal countries have developed *equalization payment schemes* by which a central government transfers money across jurisdictions. These equalization payments are entrenched in the Canadian constitution for instance. They also underlie the working of the European funds for structural development which are given to specific regions which suffer from economic backwardness. The alleged purpose of these schemes is, as their name suggests, to equalize the citizens' access to public services across jurisdictions. It is usually thought that these transfers should somehow correct for the unequal distribution of wealth across jurisdictions. More specifically, most equalization payment schemes that we are aware of are explicitly redistributive: they are designed in such a way that the (net) *per capita* subsidy received by a jurisdiction is decreasing with respect to its per capita wealth.

There are, of course, many reasons to question the soundness of this redistribution from a normative viewpoint. One such reason is cross-jurisdiction taste differences. Why should the inhabitants of a jurisdiction who have strong preferences for public goods and who must therefore contribute extensively to their public good spending be required to transfer money to people living in a slightly poorer jurisdiction whose inhabitants do not care at all about public goods and prefer spending their money on private consumption?

Another obvious source of skepticism with respect to redistributive equalization systems is within-jurisdiction heterogeneity. Suppose jurisdiction A has a slightly higher *per capita* wealth than jurisdiction B but that the distribution of wealth within A is much more unequal than B. Suppose in particular that a significant fraction of A's population is extremely poor while nobody experiments severe poverty in B. There is then no reason to expect A's citizens to transfer money to B's. As a matter of fact, standard inequality aversion considerations, such as those underlying the ranking of Lorenz curves, could very well recommend transfer from B to A in a case like this.

A third easy case that can be made against redistribution arises if mobility across jurisdictions is high. If citizens can easily move from one jurisdiction to the next, then redistributive equalization payments may not be sustainable because it may induce citizens from donator jurisdictions who make transfer payments (and who therefore receive less public good than the tax they pay) to move to recipient ones. In order to prevent such moves from occurring, the social planner may have to abandon part of its redistributive objective.

But suppose we abstract away these three reasons that mitigate the appeal of redistributive equalization payments in federations. Wouldn't redistribution become defensible then? The aim of this note is to provide a negative answer to this question. More specifically, we consider an arbitrary federation populated by a given number of individuals who have the same utility function for one private good and one public good. These individuals are partitioned into a given number of jurisdictions according to their wealth. All individuals within a jurisdiction have the same wealth, and individuals are not allowed to move from one jurisdiction to the next. In this stylized world, we examine the type of equalization payment systems that a welfarist "social planner" would adopt if its objective was the maximization of a symmetric, quasi-concave and additively separable function of the citizens' well-being. We show that, even in this *a priori* favorable case, the conditions for the optimal equalization payment scheme to be redistributive between jurisdictions for all distributions of wealth and population size are stringent.

More specifically, we show that a *necessary* and *sufficient* condition that the objective function of the social planner must satisfy in order to always choose a redistributive equalization payment scheme is to be *additively separable* with respect to a jurisdiction's per capita wealth and the jurisdiction's population size. The stringency of this condition can be appreciated better if the social welfare function used by the social planner to aggregate individual's utilities is specialized somewhat. A natural specialization to be considered in the redistributive context considered here is the *mean of order* r family of social welfare functions which contains as special cases many well-known and widely used social welfare functions such as Utilitarianism, Rawls, Nash-Bernouilli, etc. If such a specialization is adopted, we show that the additive separability of the social planner's objective function implies the additive separability of the individual's *indirect utility function* between wealth and the (Lindahl) price of the public good. This is a significant condition that restricts significantly the kind of preferences that citizens are allowed to have between public good and private consumption. For instance, if the *direct utility function* for the private and public good is assumed to be additively separable itself, the additive separability of the citizen's indirect utility implies the additive direct utility to be *logarithmic* with respect to public good.

That these severe restrictions are necessary and sufficient for a redistributive equalization payment system to be deemed optimal from a welfarist view point even in this stylized world are clear indications that redistribution is *not* a natural feature of an optimal equalization payment scheme. What the results presented in this paper show is that, when dealing with redistribution in multi-jurisdiction systems with public good provision, individual wealth is *not* the only variable of interest. Another one is the number of people living in a jurisdiction which contributes to reducing the jurisdiction *per capita* cost of providing a public good. Clearly, jurisdictions with a large population are able to afford a given amount of public good at a lower per capita tax cost.

The cost advantage of large population jurisdictions over small population ones must be accounted for by the social planner when performing cross-jurisdiction redistribution. Of course the specific nature of this account depends crucially upon the way by which the *social marginal value of wealth*, which the social planner seeks to equalize across jurisdictions, vary with the tax cost of the public good. If the social marginal value of wealth increases with the tax cost of the public good, then the social planner may transfer wealth from highly populated and relatively poor jurisdictions to richer but sparsely populated ones. Conversely, if the social marginal value of wealth decreases with respect to the tax cost, the social planner may transfer wealth from lowly populated and poor jurisdictions to richer and heavily populated ones. In these cases, the social planner may optionally depart from redistribution. It is only when the social marginal value of wealth is independent from the tax cost of the public good – which arises if the social objective function is additively separable between tax cost and wealth – that the social planner always find it optimal to redistribute.

The outline of the paper is as follows. The next section introduces the model and proves the main result. Section 3 interprets the result in the specific case where the social planner's objective is a mean of order r of the citizens' utility, and examines the implications of the result for the case where households' direct utility is assumed to be additively separable. Section 4 concludes.

2 The model and the main result

2.1 The model

We consider a country populated by $n \in \mathbb{N}_{++}$ households who live in $k \in \{1, ..., n\}$ *a priori* given jurisdictions. There are n_j households who live in jurisdiction j $(j = 1, ..., k)$ so that $\sum_{j=1}^{k} n_j = n$. A household living in jurisdiction j has a private wealth ω_j which it uses to pay taxes and to make private consumption. Jurisdictions are labeled in such a way that $\omega_1 \geq \omega_2 \geq ... \geq \omega_k$. Households derive utility from private consumption (whose quantity is denoted by c) and from a single public good (whose quantity is denoted by z). Specifically, all households in the country convert alternative bundles of private and public goods into well-being by the same strictly concave, monotonically increasing and twice continuously differentiable utility function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$.[1] We may at times assume, notably in section 3, that, in addition to the above properties, U is *additively separable* so that it can be written, for every bundle $(\overline{z}, \overline{c}) \in \mathbb{R}_+^2$, as

$$U(\overline{z}, \overline{c}) = f(\overline{z}) + h(\overline{c})$$

for some twice continuously differentiable, monotonically increasing and concave functions $f$and h from $\mathbb{R}_+$ to $\mathbb{R}$. For further use, we denote by V the indirect utility function defined by $V : \mathbb{R}_+^3 \rightarrow \mathbb{R}$

$$V(p_z, p_c, R) = \max_{z,c} U(z, c) \text{ subject to } p_z z + p_c c \leq R.$$

We also denote by $z^M(p_z, p_c, R)$ and $c^M(p_z, p_c, R)$ the (Marshallian) demands for public good and private consumption (respectively) when the prices for these two goods are p_z and p_c and when the wealth of the household is R. Given the assumptions on U, one can easily see that Marshallian demands and indirect utility are differentiable functions of prices and wealth. We denote by $\mathcal{U}$ the class of all direct utility functions that satisfy all these properties, and by $\mathcal{U}_A$ the subset of $\mathcal{U}$ consisting of those functions that are additively separable.

Public good production is financed by taxation and is allowed to differ across jurisdictions, that is, it is possible to exclude households of a given jurisdiction from consuming the public good of another jurisdiction. In this paper, we consider the case where households are not allowed to move across jurisdictions. Moreover, we allow the possibility for a control planner, the federal government,

[1] As a matter of terminology, a function $f : A \rightarrow B$ where $A \subseteq \mathbb{R}^l$ and $B \subseteq \mathbb{R}$ is:
(i) *concave* if $f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b)$ holds for every a, $b \in A$ and every $\lambda \in [0, 1]$
(ii) *strictly concave* if $f(\lambda a + (1 - \lambda)b) > \lambda f(a) + (1 - \lambda)f(b)$ holds for every a, $b \in A$ and every $\lambda \in]0, 1[$
(iii) *quasi-concave* if $f(\lambda a + (1 - \lambda)b) \geq y$ holds for every $y \in B$, $\lambda \in [0, 1]$ and a, $b \in A$ such that $f(a) \geq y$ and $f(b) \geq y$.
(iv) *symmetric* if $f(a) = f(b)$ holds for every a, $b \in A$ such that a is a permutation of b.

to redistribute purchasing power across jurisdictions in order to harmonize private and public goods consumption. In this setting, if $T_j = \omega_j - c_j$ denotes the tax paid by a household living in jurisdiction j, the country's feasibility constraint writes

$$\sum_{j=1}^{k} z_j \leq \sum_{j=1}^{k} n_j T_j, \tag{1}$$

where it is assumed that the price of the public good is one.

2.2 Conditions for the choice of a redistributive equalization payments scheme by a welfarist social planner

Assume that the federal government is *welfarist* in the sense that it ranks alternative packages of jurisdiction-specific taxes and public good levels according to the value taken by a monotonically increasing function of the citizens's well-being. This amounts to say that the federal government chooses jurisdiction taxes and public good levels that solve the following program.

$$\begin{array}{l} \max\limits_{z_1,T_1,\ldots,z_k,T_k} W(U(z_1,\omega_1 - T_1),...,U(z_k,\omega_k - T_k)) \\ \text{s. t. } \sum\limits_{j=1}^{k} z_j \leq \sum\limits_{j=1}^{k} n_j T_j,\ T_j \leq \omega_j \text{ and } z_j \geq 0 \text{ for all} \end{array} \tag{2}$$

for some continuous and monotonically increasing Bergson-Samuelson social welfare function $W : \mathbb{R}^n \to \mathbb{R}$. Consider a solution $(z_j^*, T_j^*)_{j=1}^k$ to this program (which exists by virtue of Weirstrass theorem). It is immediate to see that this solution satisfies (1) at equality.

Let $s_j^* = z_j^*/n_j - T_j^*$ denote the (possibly negative) *net per capita subsidy* received by a household jurisdiction j. This subsidy corresponds to the per capita amount needed above taxes to finance the public good. We want to know whether a welfarist social planner always finds it optimal to choose net per capita subsidies that are decreasing with respect to the jurisdiction per capita wealth. We call *redistributive* any such system of equalization payments.

We start by establishing a simple (but useful) lemma which says that the social planner's problem (2) can be thought of as being solved in two steps. In a *first* step, the social planner chooses per capita net subsidies $(s_1^*, ..., s_k^*)$ that maximize the composition of the social welfare function with the citizen's *indirect utilities.* In a *second* step, each jurisdiction j's household solves a fictitious standard consumer's problem of allocating optimally its private wealth and net per capita subsidy between private consumption (purchased at a price of 1) and public good spending (purchased at price $1/n_j$).

Lemma 1 *Let U be a utility function in $\mathcal{U}$. Then*

$$(z_j^M(1/n_j, 1, \omega_j + s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k)), \omega_j - c_j^M(1/n_j, 1, \omega_i + s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k))$$

for $j = 1, ..., k$ where

$$\begin{array}{rcl} \{s_j^*(n_1,...,n_k,\omega_1,...,\omega_k)\}_{j=1}^k & \in & \arg\max\limits_{s_1,\ldots,s_k} W(V(1/n_1,1,\omega_1+s_1),...,V(1/n_k,1,\omega_k+s_k)) \\ s.\ t. \sum\limits_{j=1}^{k} n_j s_j & \leq & 0 \text{ and } s_j \geq -\omega_j \text{ for all } j \end{array}$$

is a solution to (2).

Proof. Assume by contradiction that

$(z_j^M(1/n_j, 1, \omega_j + s_j^*(n_1..., n_k, \omega_1, ..., \omega_k)), \omega_j - c_j^M(1/n_j, 1, \omega_i + s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k))$

for $j = 1, ..., k$ does not solve (2), that is, assume that there exists $(\widehat{z}_1, \widehat{T}_1, ..., \widehat{z}_k, \widehat{T}_k)$ satisfying $\sum_{j=1}^{k} \widehat{z}_j \leq \sum_{j=1}^{k} n_j \widehat{T}_j$, $\widehat{T}_j \leq \omega_j$ and $\widehat{z}_j \geq 0$ for all j and such that

$$\begin{aligned} & W(U(\widehat{z}_1, \omega_1 - \widehat{T}_1), ..., U(\widehat{z}_k, \omega_k - \widehat{T}_k)) \\ > \quad & W(V(1/n_1, 1, \omega_1 + s_1^*(n_1, ..., n_k, \omega_1, ..., \omega_k)), ..., V(1/n_k, 1, \omega_k + s_k^*(n_1, ..., n_k, \omega_1, ..., \omega_k))). \end{aligned} \tag{3}$$

Without loss of generality, since (2) has a solution in which constraint (1) binds, one can assume that $\sum_{j=1}^{k} \widehat{z}_j = \sum_{j=1}^{k} n_j \widehat{T}_j$. Define, for every j, $\widehat{s}_j = \widehat{z}_j / n_j - \widehat{T}_j$ and $\widehat{c}_j = \omega_j - \widehat{T}_j$. Clearly, one has $\widehat{z}_j / n_j + \widehat{c}_j \leq \omega_j + \widehat{s}_j$. By definition of the indirect utility function, one has, for every j

$$V(1/n_j, 1, \omega_j + \widehat{s}_j) \geq U(\widehat{z}_j, \omega_j - \widehat{T}_j)$$

and, since W is monotonically increasing,

$$W(V(1/n_1, 1, \omega_1 + \widehat{s}_1), ..., V(1/n_k, 1, \omega_k + \widehat{s}_k)) \geq W(U(\widehat{z}_1, \omega_1 - \widehat{T}_1), ..., U(\widehat{z}_k, \omega_k - \widehat{T}_k)).$$

Since $\widehat{s}_j \geq -\omega_j$ for all j and $\sum_{j=1}^{k} \widehat{s}_j = 0$, this inequality, together with (3), contradicts the definition of $\{s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k)\}_{j=1}^{k}$ as the solution of the program

$$\begin{aligned} & \max_{s_1, \ldots, s_k} W(V(1/n_1, 1, \omega_1 + s_1), ..., V(1/n_k, 1, \omega_k + s_k)) \\ & \text{s. t. } \sum_{j=1}^{k} n_j s_j = 0 \text{ and } s_j \geq -\omega_j \text{ for all } j. \end{aligned}$$

QED ■

This lemma highlights the fact that, in a federation made of different homogeneous jurisdictions, jurisdiction's per capita wealth is *not* the only criterion used by the federal government to design equalization payment scheme. The federal government must also take into consideration the number of households living in a jurisdiction. The larger is this number, the lower is the per capita cost (or price) of providing one unit of public good in a jurisdiction. When redistributing wealth across jurisdictions, the federal government must take into consideration these cross-jurisdictions differences in the price of public good.

We can write the social planner's program as

$$\begin{aligned} & \max_{s_1, \ldots, s_k} W(V(1/n_1, 1, \omega_1 + s_1), ..., V(1/n_k, 1, \omega_k + s_k)) \\ & \text{s. t. } \sum_{j=1}^{k} n_j s_j \leq 0, \text{ and } s_j \geq -\omega_j \text{ for all } j. \end{aligned} \tag{4}$$

We want to know the conditions that the social planner's objective needs to satisfy in order for the solution to (4) to be redistributive for all distributions of households and wealth to the k jurisdictions.

To make this question somewhat interesting, it is natural to impose additional restrictions on the Bergson-Samuelson function used by the social planner. For redistribution is clearly not to be expected *a priori* for any arbitrary social welfare function which does not exhibit some inequality aversion. We therefore require W to be *quasi-concave* and *symmetric*. We also require W to be additively separable with respect to utilities. This latter assumption, which can be justified on normative grounds (see for instance Blackorby, Bossert and Donaldson [**?**]) is made for simplicity. These three assumptions amount to say that W can be defined by

$$W(u_1, ..., u_n) = F(\sum_{i=1}^{n} g(u_i)) \tag{5}$$

for some increasing function F and some monotonically increasing and continuous function $g : \mathbb{R} \to \mathbb{R}$ which needs to be concave if W is to be quasi-concave. A nice example of a Bergson-Samuelson social welfare function which fits in this setting is the mean-of-order r function where, for any $r \in]-\infty, 1]$, $W_r : \mathbb{R}^n_+ \to \mathbb{R}$ is defined by[2]

$$\begin{aligned} W_r(u_1, ..., u_n) &= [\sum_{j=1}^{n} u_j^r]^{\frac{1}{r}} \text{ if } r \neq 0 \text{ and} \\ W_r(u_1, ..., u_n) &= \sum_{j=1}^{n} \ln u_j \text{ otherwise} \end{aligned}$$

where the functions g^r referred to in (5) can be defined by

$$\begin{aligned} g^r(u) &= u^r \text{ if } r \in]0, 1] \\ g^0(u) &= \ln u \text{ if } r = 0 \\ g^r(u) &= -u^r \text{ if } r \in] - \infty, 0[. \end{aligned} \tag{6}$$

As is well-known (and can be easily seen), the case where $r = 1$ is that of a Utilitarian social planner, while the limiting case of $r = -\infty$ corresponds to an infinitely inequality averse Rawlsian one.

For a social welfare function satisfying (5), program (4) solved by the social planner writes

$$\max_{s_1,...,s_k} F(\sum_{j=1}^{k} n_j g(V(1/n_j, 1, \omega_j+s_j))) \text{ s. t. } \sum_{j=1}^{k} n_j s_j \leq 0, \text{ and } s_j \geq -\omega_j \text{ for all } j.$$

Defining the function $\Phi : \mathbb{R}^2_+ \to \mathbb{R}$ by

$$\Phi(p_z, R) = g(V(p_z, 1, R)),$$

we can more compactly write (4) as

$$\max_{s_1,...,s_k} F(\sum_{j=1}^{k} n_j \Phi(1/n_j, \omega_j + s_j)) \text{ s. t. } \sum_{j=1}^{k} n_j s_j \leq 0, \text{ and } s_j \geq -\omega_j \text{ for all } j.$$

[2] The definition of this social welfare function requires individual utilities to be measured in positive units. See Blackorby and Donaldson ([**?**]) for justifications and properties of this social welfare function.

$$(7)$$

This program has a unique solution s_j^* (for $j = 1, ..., k$) (due to the strict concavity of Φ with respect to its second argument). This solution is, thanks to Berge's maximum theorem, a continuous function $s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k)$ of the $2k$ parameters that define program (7), and is differentiable in those parameters if the objective function is twice differentiable. In particular, it follows that the first order conditions of (7) *characterize* any interior solution to this program.

These first order conditions write

$$\begin{aligned} \Phi_R^{j*}(\theta) &\equiv \Phi_R^{h*}(.)\ \forall\ h,\ j \in \{1, ..., k\} \text{ and} \\ \sum_{j=1}^{k} n_j s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k) &\equiv 0 \end{aligned}$$

where, for every jurisdiction j,

$$\Phi_R^{j*}(.) = \frac{\partial \Phi(1/n_j, \omega_j + s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k))}{\partial R}.$$

Differentiating these conditions with respect to ω_h, one obtains

$$\frac{\partial s_j^*(.)}{\partial \omega_h} \equiv \frac{\Phi_{RR}^{h*}(.)}{V_{RR}^{h*}(.)}(1 + \frac{\partial s_h^*(.)}{\partial \omega_h}) \text{ for all } h,\ j \in \{1, ..., k\} \tag{8}$$

and

$$\sum_{j=1}^{k} n_j \frac{\partial s_j^*(.)}{\partial \omega_h} \equiv 0. \tag{9}$$

Substituting (8) into (9) and rearranging yields

$$\frac{\partial s_h^*(.)}{\partial \omega_h} \equiv \frac{-\sum_{i \neq h} \frac{n_i}{\Phi_{RR}^{i*}(.)}}{\sum_{i=1}^{k} \frac{n_i}{\Phi_{RR}^{i*}(.)}} < 0 \tag{10}$$

Hence, thanks to the strict-concavity of Φ with respect to its second argument, an increase in the wealth of a household living in jurisdiction h always reduces the optimal subsidy received by this household. If one substitutes (10) back into (8) and rearranges the expression, one gets

$$\frac{\partial s_j^*(.)}{\partial \omega_h} \equiv \frac{n_h}{\Phi_{RR}^{j*}(.)(\sum_{i=1}^{k} \frac{n_i}{\Phi_{RR}^{i*}(.)})} > 0. \tag{11}$$

Here again, not surprisingly, the subsidy received a household living in a jurisdiction j is an increasing function of the wealth of any household living in another jurisdiction. More interestingly and relevant for Theorem 2 below, is the analogous comparative statics results that concern the relationship between a jurisdiction's optimal *per capita* net subsidy and its number of households. Using analogous comparative statics developments, one obtains

$$\frac{\partial s_h^*(.)}{\partial n_h} \equiv \frac{\frac{\Phi_{p_zR}^{h*}(.)}{n_h^2}[\sum_{i\neq h}\frac{n_i}{\Phi_{RR}^{i*}(.)}] - s_h^*(.)}{\Phi_{RR}^{h*}(.)[\sum_{i=1}^{k}\frac{n_i}{\Phi_{RR}^{i*}(.)}]} \tag{12}$$

and

$$\frac{\partial s_j^*(.)}{\partial n_h} = \frac{-[\frac{\Phi_{p_zR}^{h*}(.)}{n_h\Phi_{RR}^{h*}(.)} + s_h^*(.)]}{\Phi_{RR}^{j*}(.)[\sum_{i=1}^{k}\frac{n_i}{\Phi_{RR}^{i*}(.)}]}. \tag{13}$$

Note that the sign of each of these two expressions, given by the sign of its numerator, cannot be determined in general. This sign depends crucially upon that of $\Phi_{p_zR}^{h*}$ which measures how the social marginal utility of wealth varies with the price of the public good.

A way to understand how the optimal subsidy received by jurisdiction h depends upon the number of households living in h is to consider the situation in which the social planner finds optimal to give to h a 0 subsidy. Then, an exogenous increase in the number of households living in h will reduce the optimal subsidy received by h if and only if $\Phi_{p_zR}^{h}$ is positive. As it turns out, this simple fact is the basic ingredient of the proof of Theorem 2 below, which establishes that $\Phi_{p_zR}(p_z, R) = 0$ at every (p_z, R) is a necessary and sufficient condition that Φ must satisfy in order for redistributive net subsidies to be solution of (7) for all distributions of wealth and population across jurisdictions.

Theorem 2 *Let U be a utility function in $\mathcal{U}$. Then the solution $(s_1^*(.), ..., s_k^*(.))$ of (7) satisfies $s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k) \leq s_{j+1}^*(n_1, ..., n_k, \omega_1, ..., \omega_k)$ for $j = 1, ..., k-1$ for all $(n_1, ..., n_k, \omega_1, ..., \omega_k) \in \mathbb{N}_+^k \times \mathbb{R}^k$ if and only if $\Phi_{p_ZR}(p_z, R) = 0$ for every $(p_z, R) \in [0, 1] \times \mathbb{R}_+$ such that $p_z = 1/\widetilde{n}$ for some strictly positive integer $\widetilde{n}$.*

Proof. (1) Sufficiency. Assume that $\Phi_{p_ZR}(1/\widetilde{n}, R) = 0$ for every positive integer $\widetilde{n}$ and non-negative real number R. Then, restricting Φ to the domain $\{p_z \in [0, 1] : p_z = 1/\widetilde{n}$ for some strictly positive integer $\widetilde{n}\} \times \mathbb{R}^+$, one can write, for any such $(\widetilde{n}, R)$, $\Phi(1/\widetilde{n}, R) = \Upsilon(\widetilde{n}) + \Psi(R)$ for some functions $\Upsilon : \mathbb{N}_{++} \to \mathbb{R}$ monotonically increasing and $\Psi : \mathbb{R}_+ \to \mathbb{R}$ monotonically increasing and strictly concave (since V is concave with respect to wealth and g is concave). By contradiction, consider any $(n_1, ..., n_k, \omega_1, ..., \omega_k) \in \mathbb{N}_+^k \times \mathbb{R}^k$, and assume that the solution $(s_1^*(.), ..., s_k^*(.))$ of (7) involves $s_j^*(n_1, ..., n_k, \omega_1, ..., \omega_k) > s_{j+1}^*(n_1, ..., n_k, \omega_1, ..., \omega_k)$ for some $j \in \{1, ..., , k-1\}$. Consider then reducing the net *per capita* subsidy of jurisdiction j by $(\omega_j - \omega_{j+1} + s_j^*(.) - s_{j+1}^*(.))/2 > 0$ and increasing that of jurisdiction $j+1$ by the same amount. This change in *per capita* net subsidies obviously respects the constraints under which program (7) is defined. Notice in particular that

$$\begin{aligned} \omega_j + s_j^*(.) &> \omega_j + s_j^*(.) - \frac{\omega_j - \omega_{j+1} + s_j^*(.) - s_{j+1}^*(.)}{2} = \\ \frac{\omega_j + \omega_{j+1} + s_j^*(.) + s_{j+1}^*(.)}{2} &= \omega_{j+1} + s_{j+1}^*(.) + \frac{\omega_j - \omega_{j+1} + s_j^*(.) - s_{j+1}^*(.)}{2} \\ &> \omega_{j+1} + s_{j+1}^*(.) \end{aligned}$$

Now:

$$
\begin{aligned}
&\sum_{h\neq j,j+1} n_h\Phi(\frac{1}{n_h},\omega_h+s_h^*(.))+n_j\Phi(\frac{1}{n_j},\frac{\omega_j+\omega_{j+1}+s_j^*(.)+s_{j+1}^*(.)}{2})+\\
&n_{j+1}\Phi(\frac{1}{n_{j+1}},\frac{\omega_j+\omega_{j+1}+s_j^*(.)+s_{j+1}^*(.)}{2})\\
&+\sum_{j\neq k} n_j\Phi(\frac{1}{n_j},\omega_j+s_j^*)+n_h\Phi(\frac{1}{n_j},\omega_h+s_h^*)-\sum_{h=1}^{k} n_h\Phi(\frac{1}{n_h},\omega_h+s_h^*(.))\\
= \;& n_j\Phi(\frac{1}{n_j},\frac{\omega_j+\omega_{j+1}+s_j^*(.)+s_{j+1}^*(.)}{2})+n_{j+1}\Phi(\frac{1}{n_{j+1}},\frac{\omega_j+\omega_{j+1}+s_j^*(.)+s_{j+1}^*(.)}{2})\\
&-n_j\Phi(\frac{1}{n_j},\omega_j+s_j^*(.))+n_{j+1}\Phi(\frac{1}{n_{j+1}},\omega_{j+1}+s_{j+1}^*(.))\\
= \;& n_j\Psi(\frac{\omega_j+\omega_{j+1}+s_j^*(.)+s_{j+1}^*(.)}{2})+n_{j+1}\Psi(\frac{\omega_j+\omega_{j+1}+s_j^*(.)+s_{j+1}^*(.)}{2})\\
&-n_j\Psi(\omega_j+s_j^*(.))-n_{j+1}\Psi(\omega_{j+1}+s_{j+1}^*(.))\\
> \;& 0
\end{aligned}
$$

by the concavity of Ψ. This gives us the required contradiction that $(s_1^*(.),...,s_k^*(.))$ is a solution of (7).

(2) Necessity. Assume that $\Phi_{p_Z R}(\frac{1}{\widetilde{n}},R)\neq 0$ for some $(\widetilde{n},R)\in\mathbb{N}_{++}\times\mathbb{R}_+$. Consider the federation where $(n_1,\omega_1,...,n_k,\omega_k)=(\widetilde{n},R,\widetilde{n},R,....,\widetilde{n},R)$. The equalization payment system that solves (7) for this federation has clearly $s_j^*(\widetilde{n},R,\widetilde{n},R,....,\widetilde{n},R)=0$ for all j. Assume first that $\Phi_{p_Z R}(\frac{1}{\widetilde{n}},R)>0$, and consider increasing by a suitably small and strictly positive ε the number of inhabitants in some jurisdiction h. Using (13), one notices that the per capita subsidy received in jurisdictions $j\neq h$ increases and becomes positive while the subsidy received in j becomes negative. Let $\varepsilon^*=\min(1,\overline{\varepsilon})$ with $\overline{\varepsilon}$ defined by

$$\frac{\partial s_h^*(\widetilde{n},R,...,\widetilde{n},R,\widetilde{n}+\overline{\varepsilon},R,\widetilde{n},R,....,\widetilde{n},R)}{\partial n_h}=0.$$

Using (12) and (13), we have that $s_h^*(\widetilde{n},R,...,\widetilde{n},R,\widetilde{n}+\varepsilon^*,R,\widetilde{n},R,....,\widetilde{n},R)<0<s_j^*(\widetilde{n},R,...,\widetilde{n},R,\widetilde{n}+\varepsilon^*,R,\widetilde{n},R,....,\widetilde{n},R)$ for all $j\neq h$. Considering then a slight increase in the wealth of any individual j would give us, in view of (10) and the continuity of the optimal response functions s_j^*, the required violation of redistribution. The argument for the case where $\Phi_{p_Z R}(\frac{1}{\widetilde{n}},R)>0$ is analogous and therefore omitted.[3] ■

3 Interpretation for a mean-of-order r maximizer

We provide in this section some illustration of the restrictions on the *households*' utility function that are implied by the requirement of additive separability of Φ

[3] Obviously, the only slight problem that remains with the argument is that the ε^* may be considerably smaller than 1 so that it may not be possible to obtain the required violation of redistribution for a federation in which the number of households of any jurisdiction is required to be an integer.

characterized in Theorem 2. In the next theorem, we provide these restrictions for the case where the social planner aggregates households' utilities by a mean-of-order r social welfare function.

Theorem 3 *Let U be the utility function in $\mathcal{U}$ and let the social welfare function W used by the social planner be the mean-of-order r one defined as above where $r \in r \in]-\infty, 1]$. Then $\Phi_{p_Z R}(p_z, R) = 0$ for every $(p_z, R) \in \mathbb{R}^2_+$ if and only if each household's indirect utility function has the form*

$$V(p_z, p_c, R) = [\Upsilon_1^r(p_z, p_c) + \Upsilon_2^r(p_c, R)]^{\frac{1}{r}}$$

for $r \in]-\infty, 1] \backslash \{0\}$ and

$$V(p_z, p_c, R) = \Upsilon_1^0(p_z, p_c)\Upsilon_2^0(p_c, R)$$

otherwise.

Proof. Assume first that $r \in]-\infty, 1] \backslash \{0\}$. Using the definition of the mean-of-order r function provided by (6), we have

$$\begin{aligned} \Phi_{p_Z R}(p_z, R) &= 0 \\ &\Leftrightarrow \\ (r-1)V(p_z, 1, R)^{-1}\frac{\partial V(p_z, 1, R)}{\partial p_z}\frac{\partial V(p_z, 1, R)}{\partial R} + \frac{\partial^2 V(p_z, 1, R)}{\partial p_z \partial R} &= 0. \end{aligned}$$

For $r = 1$ (utilitarianism), this equality amounts to $\frac{\partial^2 V(p_z,1,R)}{\partial p_z \partial R} = 0$ which is equivalent to the additive separability of the indirect utility function with respect to p_z and R. If $r < 1$ (but $r \neq 0$), this equality can be written as

$$(1-r)\frac{\frac{\partial V(p_z,1,R)}{\partial p_z}}{V(p_z, 1, R)} = \frac{\frac{\partial^2 V(p_z,1,R)}{\partial p_z \partial R}}{\frac{\partial V(p_z,1,R)}{\partial R}}$$

or

$$\frac{(1-r)\,\partial(\ln V(p_z, 1, R))}{\partial p_z} = \frac{\partial(\ln \frac{\partial V(p_z,1,R)}{\partial R})}{\partial p_z}. \tag{14}$$

This is a first order partial differential equation assumed to hold for every $(p_z, R) \in \mathbb{R}^2_+$. Define $\widehat{V} : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $\widehat{V}(p_z, R) = V(p_z, 1, R)$. It can be checked easily that a solution to the partial differential equation (14) is given by

$$\widehat{V}(p_z, R) = (\widehat{\Upsilon}_1^r(p_z) + \widehat{\Upsilon}_2^r(R))^{\frac{1}{r}}$$

for some functions $\widehat{\Upsilon}_1^r : \mathbb{R}_+ \to \mathbb{R}_+$ and $\widehat{\Upsilon}_2^r : \mathbb{R}_+ \to \mathbb{R}_+$ (both depending upon r). By usual regularity arguments for partial differential equations, this solution is unique (up to irrelevant constant terms) (NOTE: THIS UNICITY ARGUMENT HAS TO BE TIGHTENED UP). For the case where $r = 0$, and using (6), we have that

$$\begin{aligned}\Phi_{p_Z R}(p_z,R) &= 0 \Leftrightarrow \frac{\partial^2(\ln \widehat{V}(p_z,R))}{\partial p_z \partial R}=0\\ &\Leftrightarrow \frac{\frac{\partial \widehat{V}(p_z,R)}{\partial p_z}}{\widehat{V}(p_z,R)}=\frac{\frac{\partial^2 \widehat{V}(p_z,R)}{\partial p_z \partial R}}{\frac{\partial \widehat{V}(p_z,R)}{\partial R}} \Leftrightarrow\\ \frac{\partial(\ln \widehat{V}(p_z,R))}{\partial p_z} &= \frac{\partial(\ln \frac{\partial \widehat{V}(p_z,R)}{\partial R})}{\partial p_z}.\end{aligned}$$

This is a partial differential equation whose (unique by standard argument) solution is

$$\widehat{V}(p_z,R)=\Upsilon_1^0(p_z)\Upsilon_2^0(R))$$

for some functions $\widehat{\Upsilon}_1^0:\mathbb{R}_+\to\mathbb{R}_+$ and $\widehat{\Upsilon}_2^0:\mathbb{R}_+\to\mathbb{R}_+$. We note finally that, since V is homogeneous of degree 0, we have $\widehat{V}(p_z,R)=V(\widehat{p}_z,\widehat{p}_c,\widehat{R})$ for every $(\widehat{p}_z,\widehat{p}_c,\widehat{R})\in\mathbb{R}^3_{++}$ such that $p_z=\frac{\widehat{p}_z}{\widehat{p}_c}$ and $R=\frac{\widehat{R}}{\widehat{p}_c}$. Hence the indirect utility function must have the form (for every $(p_z,p_c,R)\in\mathbb{R}^3_{++}$)

$$V(p_z,p_c,R)=[\Upsilon_1^r(p_z,p_c)+\Upsilon_2^r(p_c,R)]^{\frac{1}{r}}$$

for $r\in]-\infty,1]\backslash\{0\}$, and

$$V(p_z,p_c,R)=[\Upsilon_1^0(p_z,p_c)\Upsilon_2^0(p_c,R)]$$

otherwise. For all $r\in]-\infty,1]$, $\Upsilon_1^r:\mathbb{R}^2_{++}\to\mathbb{R}_+$ is a twice continuously differentiable function that is homogeneous of degree 0 and monotonically decreasing (resp. increasing) with respect to each of its two arguments if $r\geq 0$ (resp. <0), and $\Upsilon_2^r:\mathbb{R}^2_{++}\to\mathbb{R}_+$ is a twice continuously differentiable function that is homogeneous of degree 0, monotonically decreasing (resp. increasing) in its first argument and monotonically increasing (resp. decreasing) in its second argument if $r\geq 0$ (resp. <0). QED ■

This theorem illustrates the strength of the condition of additive separability of the social objective in terms of its implication on households' preferences for the public and the private goods. As Theorem 3 makes it clear, these indirect preferences must be additively separable with respect to the price of the public good and the household wealth. Furthermore, Theorem 3 indicates that the numerical representation of these indirect preferences that the social planner must use in order to achieve its objective must be the additive representation of these preferences raised at the power $1/r$. That this additive separability of the household's indirect preference is a significant restriction is now further emphasized by considering that households' direct preferences are additively separable and can therefore be represented by a direct utility function that belongs to $\mathcal{U}_A$.

Theorem 4 *Let a household's preference be represented by a direct utility function in $\mathcal{U}_A$. Then the conditions that*

$$V(p_z,p_c,R)=[\Upsilon_1^r(p_z,p_c)+\Upsilon_2^r(p_c,R)]^{\frac{1}{r}},$$

for $r \in]-\infty, 1]\backslash\{0\}$, *and*

$$V(p_z, p_c, R) = [\Upsilon_1^0(p_z, p_c)\Upsilon_2^0(p_c, R)],$$

otherwise, are satisfied if and only if the households' direct preferences can be represented by the utility function in $\mathcal{U}_A$ *having the specific form*

$$U(z, x) = a \cdot \ln z + h(c)$$

for all $(z, c) \in \mathbb{R}_+^2$.

Proof. The sufficiency part being straightforward, we only provide the proof of the necessity. We first note that if

$$V(p_z, p_c, R) = [\Upsilon_1^r(p_z, p_c) + \Upsilon_2^r(p_c, R)]^{\frac{1}{r}}$$

for $r \in]-\infty, 1]\backslash\{0\}$ and

$$V(p_z, p_c, R) = [\Upsilon_1^0(p_z, p_c)\Upsilon_2^0(p_c, R)]$$

otherwise, there exists a numerical representation $\widetilde{V}^r : \mathbb{R}_{++}^3 \to \mathbb{R}$ of the indirect preferences (for any admissible r) that writes

$$\widetilde{V}^r(p_z, p_c, R) = \widetilde{\Upsilon}_1^r(p_z, p_c) + \widetilde{\Upsilon}_2^r(p_c, R)$$

for any $(p_z, p_c, R) \in \mathbb{R}_{++}^3$ (the verification of this fact is immediate for $r \in [0, 1]$ and straightforward for $r \in]-\infty, 0[$ provided that the Υ_1^r and Υ_2^r functions are redefined appropriately). Taking the representation $\widetilde{V}^r$, we have, since U is additively separable that

$$\widetilde{\Upsilon}_1^r(p_z, p_c) + \widetilde{\Upsilon}_2^r(p_c, R) = \Psi(f(z^M(p_z, p_c, R)) + h(c^M(p_z, p_c, R)))$$

for some monotonically increasing and continuous function $\Psi : \mathbb{R}_{++} \to \mathbb{R}_+$. Considering

$$\Psi^{-1}(\widetilde{\Upsilon}_1^r(p_z, p_c) + \widetilde{\Upsilon}_2^r(p_c, R)) = f(z^M(p_z, p_c, R)) + h(c^M(p_z, p_c, R)),$$

one notices that, since $\partial z^M(p_z, p_c, R)/\partial p_z < 0$ thanks to the additive separability and strict concavity of U, this equality can only be satisfied if $c^M(p_z, p_c, R)$ is *independent* from p_z or, equivalently given the differentiability of c^M, if $\partial c^M(p_z, p_c, R)/\partial p_z = 0$ for every $(p_z, p_c, R) \in \mathbb{R}_{++}^3$. Yet the Marshallian demand function c^M is locally characterized by the first order condition of the standard consumer problem

$$-\frac{\partial f(R/p_z - p_c/p_z c^M(.))}{\partial z}\frac{p_c}{p_z} + \frac{\partial h(c^M(.))}{\partial c} \equiv 0.$$

Differentiating this (local) identity with respect to p_z and rearranging yields

$$\frac{\partial c^M(p_z, p_c, R)}{\partial p_z} \equiv \frac{\frac{-p_c}{p_z^2}(z^M(.)\frac{\partial^2 f(.)}{\partial^2 z} + \frac{\partial f(.)}{\partial z})}{(\frac{p_c}{p_z})^2\frac{\partial^2 f(.)}{\partial^2 z} + \frac{\partial^2 h(.)}{\partial^2 c}}. \tag{15}$$

Requiring $\partial c^M(p_z, p_c, R.)/\partial p_z = 0$ for every $(p_z, p_c, R) \in \mathbb{R}^3_{++}$ amounts therefore (since the denominator of (15) is strictly negative because of the strict concavity of U) to requiring

$$z\frac{\partial^2 f(z)}{\partial^2 z} + \frac{\partial f(z)}{\partial z} = 0 \tag{16}$$

to hold for every z for every $z \in \mathbb{R}_{++}$. The partial differential equation (16) can also be written as

$$\frac{\frac{\partial^2 f(z)}{\partial^2 z}}{\frac{\partial f(z)}{\partial z}} = \frac{-1}{z}$$

or

$$\begin{aligned} \frac{\partial(\ln \frac{\partial f(z)}{\partial z})}{\partial z} &= \frac{-1}{z} \Leftrightarrow \\ \ln \frac{\partial f(z)}{\partial z} &= -\ln z + b \end{aligned}$$

for some real number b. Taking the exponential on both side and rearranging yields

$$\frac{\partial f(z)}{\partial z} = \frac{e^b}{z}$$

or

$$f(z) = e^b \ln z$$

QED ■

4 Conclusion

To be provided.

References

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