Coordinatization of MV algebras
and some fun things about effect algebras

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MV algebras

Definition

An MV algebra is a set $M$ with elements 0, 1, binary operation $\oplus$, and unary operation $\neg$, satisfying

1. **Associativity**: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
2. **Commutativity**: $a \oplus b = b \oplus a$.
3. **Zero law**: $a \oplus 0 = a$.
4. **Involution**: $\neg \neg a = a$.
5. **Absorption law**: $a \oplus 1 = 1$.
6. **Łukasiewicz axiom**: $\neg (\neg a \oplus b) \oplus b = \neg (\neg b \oplus a) \oplus a$.

Examples

1. Every boolean algebra is an MV algebra, with $a \oplus b = a \lor b$. Łukasiewicz axiom becomes $\neg (\neg a \lor b) \lor b = \neg (\neg b \lor a) \lor a$.
2. The unit interval $[0, 1]$ with $x \oplus y = \min(1, x + y)$, and $\neg x = 1 - x$.
3. The Łukasiewicz chains, $\mathcal{L}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \subseteq [0, 1]$.

Theorem (Chang's completeness theorem)

An MV equation holds in an arbitrary MV algebra if and only if it holds in $[0, 1]$. 

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Effect algebras

**Definition**

An *effect algebra* is a set $E$ with elements 0, 1, partial binary operation $\tilde{\oplus}$, and unary operation $(-)^\perp$, satisfying

1. **Commutativity**: if $a \tilde{\oplus} b \downarrow$, then $b \tilde{\oplus} a \downarrow$, and $a \tilde{\oplus} b = b \tilde{\oplus} a$.

2. **Associativity**: if $b \tilde{\oplus} c \downarrow$ and $a \tilde{\oplus} (b \tilde{\oplus} c) \downarrow$, then $a \tilde{\oplus} b \downarrow$, $(a \tilde{\oplus} b) \tilde{\oplus} c \downarrow$, and $a \tilde{\oplus} (b \tilde{\oplus} c) = (a \tilde{\oplus} b) \tilde{\oplus} c$.

3. **Zero law**: $a \tilde{\oplus} 0 \downarrow$ and $a \tilde{\oplus} 0 = a$.

4. **Orthocomplement law**: $a^\perp$ is the unique element satisfying $a \tilde{\oplus} a^\perp = 1$.

5. **Zero-one law**: if $a \tilde{\oplus} 1 \downarrow$, then $a = 0$. 
Effect algebras

Examples

1. Every boolean algebra is an effect algebra, with \( p \sim \oplus q \downarrow \text{iff} \ p \land q = \bot \), in which case \( p \sim \oplus q \) is \( p \lor q \).
2. The unit interval \([0, 1]\) with \( x \sim \oplus y \downarrow \text{iff} \ x + y \leq 1 \), in which case \( \sim \oplus \) is \( + \).
3. Denote \( SA_n \) the \( n \times n \) self-adjoint complex matrices. Define an order on \( SA_n \)
   \[
   M \leq N \iff \langle x, Mx \rangle \leq \langle x, Nx \rangle \text{ for all } x \in \mathbb{C}^n.
   \]
   Then, \( \textit{effects} \) are the self-adjoint matrices
   \[
   \mathcal{E}_n = \{ M \in SA_n \mid 0 \leq A \leq I_n \},
   \]
   with \( \sim \oplus \) given by matrix addition and defined iff the sum is in \( \mathcal{E}_n \).
Let $A$ be an MV algebra. Define the \textit{algebraic order} $x \leq y$ iff there exists $z \in A$ s.t. $x \oplus z = y$.

(Similarly for an effect algebra $E$: $x \leq y$ iff there exists $z \in E$ s.t. $x \tilde{\oplus} z = y$.)

Fact: MV algebras (w.r.t. $\leq$) are always a lattice. \textit{Not} true for effect algebras.

\textbf{Definition (Riesz decomposition property)}

An effect algebra $E$ has the \textit{Riesz decomposition property} (RDP) if:

For all $a, b_1, b_2 \in E$, if $a \leq b_1 \tilde{\oplus} b_2$, then there exist $a_1, a_2 \in E$ such that $a = a_1 \tilde{\oplus} a_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$. 
Theorem

There is a natural one-to-one correspondence between lattice-ordered effect algebras with RDP and MV algebras.

Proof.

(Sketch): If $(A, \oplus)$ is an MV algebra, we can get an effect algebra $(A, \tilde{\oplus})$ by defining

$$a \tilde{\oplus} b = \begin{cases} a \oplus b & \text{if } a \leq \neg b, \\ \uparrow & \text{otherwise.} \end{cases}$$

This effect algebra is lattice ordered and has RDP.

If $(E, \bar{\oplus})$ is an effect algebra with lattice order and RDP, we can get an MV algebra $(E, \oplus)$ by defining

$$a \oplus b = a \bar{\oplus} (a^\perp \land b).$$

Definition (MV-effect algebra)

An effect algebra that has lattice order has RDP is called an MV-effect algebra.

Remark: references to and proofs of the above theorem in the present literature is an absolute mess.
But what about the morphisms?

Definition (Homomorphism (of MV algebras))

A function \( f : A \rightarrow B \) is an \emph{MV algebra homomorphism} if for all \( a, b \in A \):

1. \( f(0_A) = 0_B \).
2. \( f(a \oplus b) = f(a) \oplus f(b) \).
3. \( f(\neg a) = \neg f(a) \).

Definition (Homomorphism (of effect algebras))

A function \( f : E \rightarrow F \) an \emph{effect algebra homomorphism} if:

1. \( f(1_E) = 1_F \).
2. If \( a, b \in E \) and \( a \bowtie b \downarrow \), then \( f(a) \bowtie f(b) \downarrow \) and \( f(a \bowtie b) = f(a) \bowtie f(b) \).
Let $\mathbf{EA}$ and $\mathbf{MV}$ be categories of effect algs and MV algs, respectively.

**Proposition**

There are a continuum of effect algebra maps from $[0,1]^2 \rightarrow [0,1]$. There are only two MV algebra maps from $[0,1]^2 \rightarrow [0,1]$.

Let $\mathbf{MVEA}$ be the (nonfull) subcategory of $\mathbf{EA}$ of MV-effect algebras and effect algebra homomorphisms which **preserve the lattice operations** $\land$, $\lor$.

**Theorem**

There is an isomorphism of categories $\mathbf{MV} \cong \mathbf{MVEA}$.
MV algebras are an equational class. Limits and colimits are easy and obvious.

For effect algebras... products and equalizers exactly as for sets. Coproducts also easy; take coproduct of sets and identify all 0s and 1s.

Coequalizers of effect algebras are kinda hard, though.

Problem: congruences on partial structures aren’t nice. Can’t simply say “Let R be the effect algebra cong. generated by \( \{ f(a), g(a) \mid a \in E \} \)”.
Definition (BCM, BCM homomorphism)

A \textit{barred commutative monoid} (BCM) is a commutative monoid $M$ with an element $u$ called the \textit{bar} of $M$, s.t. for all $a, b, c \in M$:

1. (Positivity): If $a + b = 0$, then $a = b = 0$.
2. (Cancellation under the bar): If $a + b = a + c = u$, then $b = c$.

A BCM hom $f : M \to N$ is a monoid hom which preserves the bar.

Define $\mathcal{T}o : \text{EA} \to \text{BCM}$ as follows. For effect algebra $(E, \widetilde{\oplus})$, let $\mathcal{M}(E)$ be free commutative monoid on $E$. Then, $\mathcal{T}o(E) = (\mathcal{M}(E)/\sim, 1(1_E))$, where $\sim$ is modding out by obvious stuff.

Define $\mathcal{P}a : \text{BCM} \to \text{EA}$ by $\mathcal{P}a(M, u) = [0, u] = \{x \in M \mid x \leq u\}$, with $x \sim \oplus y \iff x + y \leq u$, in which case $x \oplus y = x + y$.

Theorem (B. Jacobs, 2012)

The functors $\mathcal{T}o : \text{EA} \rightleftarrows \text{BCM} : \mathcal{P}a$ form an adjunction with $\mathcal{T}o$ full and faithful. So $\text{EA} \hookrightarrow \text{BCM}$ is a coreflection.
Coequalizers

**Proposition**

Let $f, g: M \to N$ be BCM homomorphisms. The coequalizer of $f, g$ is $\pi: N \to N/\approx$, where $\approx$ is the smallest BCM congruence on $N$ containing $\{(f(x), g(x)) \mid x \in M\}$ and $\pi$ is the natural projection onto the quotient.

**Corollary**

Let $f, g: E \to F$ be effect algebra homomorphisms. Let $\pi: \mathcal{T}o(F) \to \mathcal{T}o(F)/\approx$ be the coequalizer in BCM of $\mathcal{T}o(f), \mathcal{T}o(g): \mathcal{T}o(E) \to \mathcal{T}o(F)$.

\[
\begin{array}{c}
\mathcal{T}o(E) \xrightarrow{\mathcal{T}o(f)} \mathcal{T}o(F) \\
\searrow \mathcal{T}o(g) \quad \nearrow \pi \\
\end{array}
\]

Then, the coequalizer of $f, g$ in EA is $\mathcal{P}a(\pi) \circ \eta_F: F \to \mathcal{P}a(\mathcal{T}o(F)/\approx)$.

\[
\begin{array}{c}
E \xrightarrow{f} F \xrightarrow{\eta_F} \mathcal{P}a\mathcal{T}o(F) \\
\mathcal{P}a(\pi) \quad \mathcal{P}a(\pi) \\
\end{array}
\]
### Monomorphisms

**Non-definition (“Monomorphism”)**

An effect algebra homomorphism $f$ is a “monomorphism” (definition given in the present literature) if $f(a) \tilde{\oplus} f(b) \downarrow$ implies $a \tilde{\oplus} b \downarrow$.

**Lemma**

*The monomorphisms of $\textbf{EA}$ are precisely the injective effect algebra homomorphisms.*

**Example**

Consider the inclusion $i : \{0, \frac{1}{4}, \frac{3}{4}, 1\} \to [0,1]$. This is injective, and so a monomorphism. But it’s not a “monomorphism”, as $i(\frac{1}{4}) \tilde{\oplus} i(\frac{1}{4}) \downarrow$ but $\frac{1}{4} \tilde{\oplus} \frac{1}{4} \uparrow$.

**Theorem**

*The “monomorphisms” are actually the regular monomorphisms (equalizers).*
Definition (Inverse semigroup)

An inverse semigroup is a pair \((S, \ast)\) consisting of a set \(S\) and a binary operation \(\ast: S \times S \to S\) satisfying (writing simply \(xy\) for \(x \ast y\)):

1. **Associativity**: for all \(x, y, z \in S\), \((xy)z = x(yz)\).

2. **Pseudoinverse**: for all \(x \in S\), there exists a unique \(x^{-1} \in S\) such that \(xx^{-1}x = x\) and \(x^{-1}xx^{-1} = x^{-1}\).

For an inverse semigroup \(S\), write \(E(S) = \{x \in S \mid x\) is idempotent\}\.

For \(x, y \in S\), define \(x \leq y\) to mean there exists \(e \in E(S)\) such that \(x = ye\). This is a partial order.
Example

Let $X$ be a set. Then the \textit{partial bijections} on $X$ (that is, partially defined functions $X \rightarrow X$ which are injective), $\mathcal{I}(X)$, is an inverse semigroup called the \textit{symmetric inverse monoid} on $X$.

Given $f, g \in \mathcal{I}(X)$, we have $gf = g \circ f$, where
\[
\text{dom}(g \circ f) = f^{-1}(\text{dom } g \cap \text{im } f),
\]
and when $x \in \text{dom}(g \circ f)$, then
\[
(g \circ f)(x) = g(f(x)).
\]

When $X$ is a finite set of $n$ elements, we write $\mathcal{I}(X)$ as $\mathcal{I}_n$.

Facts: The idempotents of $\mathcal{I}(X)$ are the partial identity maps. For $f, g \in \mathcal{I}(X)$, $f \leq g$ means $g$ is an extension of $f$.

Theorem (Wagner-Preston theorem)

For every inverse semigroup $S$, there exists a set $X$ and in injective homomorphism $i: S \hookrightarrow \mathcal{I}(X)$ such that, for $a, b \in S$,
\[
a \leq b \iff i(a) \leq i(b).
\]
Definition (Boolean inverse monoid)

An inverse semigroup is a **Boolean inverse monoid** if it has an identity 1 (note this does not mean $xx^{-1} = 1$), absorbing element 0, and

1. $E(S)$ is a Boolean algebra.
2. For all $a, b \in S$, if $ab^{-1} = 0 = a^{-1}b$, then $a \vee b$ exists.
3. Multiplication distributes over binary joins.

Definition (Factorizable)

An inverse monoid is **factorizable** if every element is beneath an element in the group of units i.e.

$$\forall x \in S, \exists y \in S \text{ s.t. } yy^{-1} = 1 = y^{-1}y \text{ and } x \leq y.$$  

A factorizable boolean inverse monoid is called a **Foulis monoid**. Symmetric inverse monoids $\mathcal{I}(X)$ are Foulis monoids iff $X$ is finite.
Green’s $\mathcal{D}$ and $\mathcal{J}$ relations

For an inverse semigroup $S$, $a, b \in S$, define

$$ a \mathcal{D} b \iff \exists c \in S \text{ such that } a^{-1}a = c^{-1}c \text{ and } cc^{-1} = b^{-1}b. $$

For $f, g \in \mathcal{I}_n$, $f \mathcal{D} g$ iff $|\text{dom } f| = |\text{dom } g|$.

**Proposition**

$\mathcal{D}$ is an equivalence relation. Consider a distributive inverse semigroup $S$ with zero and define

$$ a \hat{\oplus} b = \begin{cases} a \lor b, & \text{if } ab^{-1} = 0 = a^{-1}b, \\ \uparrow, & \text{otherwise}. \end{cases} $$

- $S$ with $\hat{\oplus}$ is a partial commutative monoid, and so is $S/\mathcal{D}$.
- $S$ a Foulis monoid $\Rightarrow S/\mathcal{D}$ an effect algebra with RDP.
- $S$ a Foulis monoid with lattice order $\Rightarrow S/\mathcal{D}$ an MV-effect algebra.

Define $a \mathcal{J} b \iff SaS = SbS$.

Fact: In a Foulis monoid, $\mathcal{D} = \mathcal{J}$. 
Coordinatization Theorem

**Definition (AF inverse monoid)**

Let

\[ S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots \]

be a sequence of finite products of finite symmetric inverse monoids and injective maps. The directed colimit of such a sequence is called an *AF inverse monoid*.

**Definition (Coordinatizable)**

An MV algebra \( A \) is said to be *coordinatizable* if there is a Foulis monoid \( S \) such that \( S/\mathcal{D} \cong A \).

**Theorem (Lawson & Scott)**

*Every countable MV algebra \( A \) can be coordinatized by some Foulis monoid \( S \). Moreover, \( S \) can be taken to be an AF inverse monoid.*

**Theorem (Wehrung)**

*Every MV algebra can be coordinatized. A direct generalization of AF inverse monoids applies at cardinality \( \aleph_1 \), but not at \( \aleph_2 \) and beyond.*
The Łukasiewicz chains, $\mathcal{L}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \subseteq [0, 1]$ are coordinatized by $\mathcal{I}_n$. Recall that a $\mathcal{D}$-class of $\mathcal{I}_n$ consists of all the partial bijections with the same size domain. The $\mathcal{D}$-class of partial bijections on $m$ elements corresponds to the element $\frac{m}{n}$.

**Theorem (Lawson & Scott)**

The dyadic rationals in $[0, 1]$ (those with denominator $2^k$) are coordinatized by the dyadic inverse monoid, which turns out to be isomorphic to the directed colimit of

$$
\mathcal{I}_1 \rightarrow \mathcal{I}_2 \rightarrow \mathcal{I}_4 \rightarrow \mathcal{I}_8 \rightarrow \mathcal{I}_{16} \rightarrow \ldots.
$$

How to generalize to get all the rationals in $[0, 1]$?
First idea...

\[ I_1 \xrightarrow{\tau_1} I_2 \xrightarrow{\tau_2} I_3 \xrightarrow{\tau_3} I_4 \xrightarrow{\tau_4} \ldots \]

Problem: maps don’t make sense. Successive maps are “inclusions”. For \( \tau : I_n \rightarrow I_m \) to make sense, \( n \) must divide \( m \).

Idea: if \( nq = m \), then each element of the underlying set of \( X_n = \{x_1, \ldots, x_n\} \) is identified with a subset of \( q \) elements in \( X_m = \{x_1, \ldots x_{nq}\} \) in an obvious way; e.g. identify \( x_1 \in X_n \) with \( \{x_1, \ldots, x_q\} \subseteq X_m \). Extend this to identification to functions.
## Definition (Omnidivisional sequence)

A sequence $D = \{n_i\}_{i=1}^{\infty}$ of natural numbers is *omnidivisional* if it satisfies the following properties.

- For all $i$, $n_i \mid n_{i+1}$.
- For all $m \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $m \mid n_i$.

## Example

Let $p_i$ be the $i^{th}$ prime number. Then $\{\prod_{i=1}^{n} p_i^{n-i+1}\}_{n=1}^{\infty}$ is an omnidivisional sequence. The first few members of the sequence are $2, 2^23, 2^33^25, 2^43^35^27, \ldots$.

## Example

The sequence $\{n!\}_{n=1}^{\infty}$.
Theorem (Coordinatization of the rationals)

Let \( D = \{n_i\}_{n=1}^{\infty} \) be an omnidivisional sequence. Then, the directed colimit of the sequence

\[
Q : \mathcal{I}_{n_1} \xrightarrow{\tau_1} \mathcal{I}_{n_2} \xrightarrow{\tau_2} \mathcal{I}_{n_3} \xrightarrow{\tau_3} \mathcal{I}_{n_4} \xrightarrow{\tau_4} \ldots,
\]

coordinatizes \( \mathbb{Q} \cap [0, 1] \).

Proof.

Denote the directed colimit of \( Q \) by \( Q_{\infty} \). Define a map \( w : Q_{\infty}/D \to \mathbb{Q} \cap [0, 1] \) as follows. For \( s \in \mathcal{I}_{n_i} \), define

\[
w([s]/D) = \frac{|\text{dom}(s)|}{n_i}.
\]

Check this is well-defined on several different levels, check it’s an iso, check it’s an MV map.
Coordinatization decomposition theorem

Theorem (Decomposition theorem, part 1)

Let $A$ be an MV algebra. Suppose that $A$ has a chain of subalgebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

such that $A = \bigcup_{i=1}^{\infty} A_i$ and that each $A_i$ is coordinatized by an inverse semigroup $S_i$. Denote inclusions by $\ell_i : A_i \to A_{i+1}$. Choose an explicit isomorphism $f_i : A_i \to S_i/D$ for each $i$.

Suppose there are injective maps $\tau_i : S_i \to S_{i+1}$ such that the maps $t_i : S_i/D \to S_{i+1}/D$, $s/D \to \tau_i(s)/D$ are well defined on $D$-classes, and that the following diagram commutes for all $i$.

\[
\begin{array}{ccc}
A_i & \xrightarrow{\ell_i} & A_{i+1} \\
\downarrow f_i^{-1} & & \downarrow f_{i+1} \\
S_i/D & \xrightarrow{t_i} & S_{i+1}/D
\end{array}
\]

Then, $A$ is coordinatized by $\lim_{\to} (S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots)$. 

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Theorem (Decomposition theorem, part 2)

Suppose \( A \) is an MV algebra coordinatized by the directed colimit of

\[
S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots
\]

Then, \( A \) has a sequence of subalgebras forming a chain of inclusions

\[
A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots
\]

such that \( A = \bigcup_{i=1}^{\infty} A_i \), and each \( A_i \) is coordinatized by \( S_i \).

Some MV algebras yet to be coordinatized, which the decomposition theorems might be useful for:

- The full interval \([0, 1]\).
- The free MV algebra on one (or more) generators.
- Many more challenges/examples given by Mundici.
Thank you for listening!
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