Coordinatization of MV algebras and some fun things about effect algebras

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New Directions in Inverse Semigroups workshop
University of Ottawa

June 4, 2015

MV algebras

Definition

An *MV algebra* is a set M with elements $0, 1$, binary operation $\oplus$, and unary operation $\neg$, satisfying

1. *Associativity*: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
2. *Commutativity*: $a \oplus b = b \oplus a$.
3. *Zero law*: $a \oplus 0 = a$.
4. *Involution*: $\neg\neg a = a$.
5. *Absorption law*: $a \oplus 1 = 1$.
6. *Łukasiewicz axiom*: $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

Examples

1. Every boolean algebra is an MV algebra, with $a \oplus b = a \vee b$. Łukasiewicz axiom becomes $\neg(\neg a \vee b) \vee b = \neg(\neg b \vee a) \vee a$.
2. The unit interval $[0, 1]$ with $x \oplus y = \min(1, x + y)$, and $\neg x = 1 - x$.
3. The *Łukasiewicz chains*, $Ł_n = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\} \subseteq [0, 1]$.

Theorem (Chang's completeness theorem)

An MV equation holds in an arbitrary MV algebra if and only if it holds in $[0, 1]$.

Effect algebras

Definition

An *effect algebra* is a set E with elements $0, 1$, **partial** binary operation $\widetilde{\oplus}$, and unary operation $(-)^{\perp}$, satisfying

1. *Commutativity*: if $a \mathbin{\widetilde{\oplus}} b \downarrow$, then $b \mathbin{\widetilde{\oplus}} a \downarrow$, and $a \mathbin{\widetilde{\oplus}} b = b \mathbin{\widetilde{\oplus}} a$.
2. *Associativity*: if $b \mathbin{\widetilde{\oplus}} c \downarrow$ and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) \downarrow$, then $a \mathbin{\widetilde{\oplus}} b \downarrow$, $(a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c \downarrow$, and $a \mathbin{\widetilde{\oplus}} (b \mathbin{\widetilde{\oplus}} c) = (a \mathbin{\widetilde{\oplus}} b) \mathbin{\widetilde{\oplus}} c$.
3. *Zero law*: $a \mathbin{\widetilde{\oplus}} 0 \downarrow$ and $a \mathbin{\widetilde{\oplus}} 0 = a$.
4. *Orthocomplement law*: $a^{\perp}$ is the **unique** element satisfying $a \mathbin{\widetilde{\oplus}} a^{\perp} = 1$.
5. *Zero-one law*: if $a \mathbin{\widetilde{\oplus}} 1 \downarrow$, then $a = 0$.

Effect algebras

Examples

1. Every boolean algebra is an effect algebra, with $p \mathbin{\widetilde{\oplus}} q \downarrow$ iff $p \wedge q = \bot$, in which case $p \mathbin{\widetilde{\oplus}} q$ is $p \vee q$.
2. The unit interval $[0, 1]$ with $x \mathbin{\widetilde{\oplus}} y \downarrow$ iff $x + y \leq 1$, in which case $\widetilde{\oplus}$ is $+$.
3. Denote SA_n the $n \times n$ self-adjoint complex matrices. Define an order on SA_n

   $$M \leq N \Leftrightarrow \langle x, Mx \rangle \leq \langle x, Nx \rangle \text{ for all } x \in \mathbb{C}^n.$$

   Then, *effects* are the self-adjoint matrices

   $$\mathcal{E}_n = \{M \in \mathrm{SA}_n \mid 0 \leq A \leq I_n\},$$

   with $\widetilde{\oplus}$ given by matrix addition and defined iff the sum is in $\mathcal{E}_n$.

Lattice ordering, RDP

Let A be an MV algebra. Define the *algebraic order* $x \leq y$ iff there exists $z \in A$ s.t. $x \oplus z = y$.
(Similarly for an effect algebra E: $x \leq y$ iff there exists $z \in E$ s.t. $x \mathbin{\widetilde{\oplus}} z = y$.)

Fact: MV algebras (w.r.t. $\leq$) are always a lattice. *Not* true for effect algebras.

Definition (Riesz decomposition property)

An effect algebra E has the *Riesz decomposition property* (RDP) if:
For all $a, b_1, b_2 \in E$, if $a \leq b_1 \mathbin{\widetilde{\oplus}} b_2$, then there exist $a_1, a_2 \in E$ such that $a = a_1 \mathbin{\widetilde{\oplus}} a_2$, $a_1 \leq b_1$, and $a_2 \leq b_2$.

MV-effect algebras

Theorem

There is a natural one-to-one correspondence between lattice-ordered effect algebras with RDP and MV algebras.

Proof.

(Sketch): If $(A, \oplus)$ is an MV algebra, we can get an effect algebra $(A, \widetilde{\oplus})$ by defining

$$a \mathbin{\widetilde{\oplus}} b = \begin{cases} a \oplus b & \text{if } a \leq \neg b, \\ \uparrow & \text{otherwise.} \end{cases}$$

This effect algebra is lattice ordered and has RDP.
If $(E, \widetilde{\oplus})$ is an effect algebra with lattice order and RDP, we can get an MV algebra $(E, \oplus)$ by defining

$$a \oplus b = a \mathbin{\widetilde{\oplus}} (a^{\perp} \wedge b).$$

□

Definition (MV-effect algebra)

An effect algebra that has lattice order has RDP is called an *MV-effect algebra*.

Remark: references to and proofs of the above theorem in the present literature is an absolute mess.

But what about the morphisms?

Definition (Homomorphism (of MV algebras))

A function $f\colon A \to B$ is an *MV algebra homomorphism* if for all $a, b \in A$:

1. $f(0_A) = 0_B$.
2. $f(a \oplus b) = f(a) \oplus f(b)$.
3. $f(\neg a) = \neg f(a)$.

Definition (Homomorphism (of effect algebras))

A function $f\colon E \to F$ an *effect algebra homomorphism* if:

1. $f(1_E) = 1_F$.
2. If $a, b \in E$ and $a \mathbin{\widetilde{\oplus}} b \downarrow$, then $f(a) \mathbin{\widetilde{\oplus}} f(b) \downarrow$ and $f(a \mathbin{\widetilde{\oplus}} b) = f(a) \mathbin{\widetilde{\oplus}} f(b)$.

But what about the morphisms?

Let **EA** and **MV** be categories of effect algs and MV algs, respectively.

Proposition

There are a continuum of effect algebra maps from $[0,1]^2 \to [0,1]$.
There are only two MV algebra maps from $[0,1]^2 \to [0,1]$.

Let **MVEA** be the (nonfull) subcategory of **EA** of MV-effect algebras and effect algebra homomorphisms **which preserve the lattice operations** $\wedge$**,** $\vee$**.**

Theorem

There is an isomorphism of categories $\mathbf{MV} \cong \mathbf{MVEA}$.

Limits and colimits

MV algebras are an equational class. Limits and colimits are easy and obvious.

For effect algebras... products and equalizers exactly as for sets. Coproducts also easy; take coproduct of sets and identify all 0s and 1s.

Coequalizers of effect algebras are kinda hard, though.

Problem: congruences on partial structures aren't nice. Can't simply say "Let R be the effect algebra cong. generated by $\{f(a), g(a) \mid a \in E\}$".

Coequalizers — BCMs (Bart Jacobs)

Definition (BCM, BCM homomorphism)

A *barred commutative monoid* (BCM) is a commutative monoid M with an element u called the *bar* of M, s.t. for all $a, b, c \in M$:

1. (Positivity): If $a + b = 0$, then $a = b = 0$.
2. (Cancellation under the bar): If $a + b = a + c = u$, then $b = c$.

A BCM hom $f\colon M \to N$ is a monoid hom which preserves the bar.

Define $\mathcal{T}o\colon \mathbf{EA} \to \mathbf{BCM}$ as follows. For effect algebra $(E, \widetilde{\oplus})$, let $\mathcal{M}(E)$ be free commutative monoid on E. Then, $\mathcal{T}o(E) = (\mathcal{M}(E)/\sim, 1(1_E))$, where $\sim$ is modding out by obvious stuff.

Define $\mathcal{P}a\colon \mathbf{BCM} \to \mathbf{EA}$ by $\mathcal{P}a(M, u) = [0, u] = \{x \in M \mid x \leq u\}$, with $x \widetilde{\oplus} y \downarrow \Leftrightarrow x + y \leq u$, in which case $x \widetilde{\oplus} y = x + y$.

Theorem (B. Jacobs, 2012)

The functors $\mathcal{T}o\colon \mathbf{EA} \rightleftarrows \mathbf{BCM}\colon \mathcal{P}a$ form an adjunction with $\mathcal{T}o$ full and faithful. So $\mathbf{EA} \hookrightarrow \mathbf{BCM}$ is a coreflection.

Coequalizers

Proposition

Let $f, g\colon M \to N$ be BCM homomorphisms. The coequalizer of f, g is $\pi\colon N \to N/\approx$, where $\approx$ is the smallest BCM congruence on N containing $\{(f(x), g(x)) \mid x \in M\}$ and π is the natural projection onto the quotient.

Corollary

*Let $f, g\colon E \to F$ be effect algebra homomorphisms. Let $\pi\colon \mathcal{T}o(F) \to \mathcal{T}o(F)/\approx$ be the coequalizer in **BCM** of $\mathcal{T}o(f), \mathcal{T}o(g)\colon \mathcal{T}o(E) \to \mathcal{T}o(F)$.*

$$\mathcal{T}o(E) \underset{\mathcal{T}o(g)}{\overset{\mathcal{T}o(f)}{\rightrightarrows}} \mathcal{T}o(F) \xrightarrow{\pi} \mathcal{T}o(F)/\approx$$

*Then, the coequalizer of f, g in **EA** is $\mathcal{P}a(\pi) \circ \eta_F\colon F \to \mathcal{P}a(\mathcal{T}o(F)/\approx)$.*

$$E \underset{g}{\overset{f}{\rightrightarrows}} F \xrightarrow{\eta_F} \mathcal{P}a\mathcal{T}o(F) \xrightarrow{\mathcal{P}a(\pi)} \mathcal{P}a(\mathcal{T}o(F)/\approx)$$

Monomorphisms

Non-definition ("Monomorphism")

An effect algebra homomorphism f is a "monomorphism" (definition given in the present literature) if $f(a) \mathbin{\widetilde{\oplus}} f(b) \downarrow$ implies $a \mathbin{\widetilde{\oplus}} b \downarrow$.

Lemma

The monomorphisms of **EA** *are precisely the injective effect algebra homomorphisms.*

Example

Consider the inclusion $i \colon \{0, \frac{1}{4}, \frac{3}{4}, 1\} \to [0,1]$. This is injective, and so a monomorphism. But it's not a "monomorphism", as $i(\frac{1}{4}) \mathbin{\widetilde{\oplus}} i(\frac{1}{4}) \downarrow$ but $\frac{1}{4} \mathbin{\widetilde{\oplus}} \frac{1}{4} \uparrow$.

Theorem

The "monomorphisms" are actually the **regular** *monomorphisms (equalizers).*

Definition (Inverse semigroup)

An *inverse semigroup* is a pair $(S, *)$ consisting of a set S and a binary operation $*\colon S \times S \to S$ satisfying (writing simply xy for $x * y$):

- S1 *Associativity*: for all $x, y, z \in S$, $(xy)z = x(yz)$.
- S2 *Pseudoinverse*: for all $x \in S$, there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

For an inverse semigroup S, write $E(S) = \{x \in S \mid x \text{ is idempotent}\}$.

For $x, y \in S$, define $x \leq y$ to mean there exists $e \in E(S)$ such that $x = ye$. This is a partial order.

Example: partial bijections

Example

Let X be a set. Then the *partial bijections* on X (that is, partially defined functions $X \to X$ which are injective), $\mathcal{I}(X)$, is an inverse semigroup called the *symmetric inverse monoid* on X.

Given $f, g \in \mathcal{I}(X)$, we have $gf = g \circ f$, where $\text{dom}(g \circ f) = f^{-1}(\text{dom}\, g \cap \text{im}\, f)$, and when $x \in \text{dom}(g \circ f)$, then $(g \circ f)(x) = g(f(x))$.

When X is a finite set of n elements, we write $\mathcal{I}(X)$ as $\mathcal{I}_n$.

Facts: The idempotents of $\mathcal{I}(X)$ are the partial identity maps. For $f, g \in \mathcal{I}(X)$, $f \leq g$ means g is an extension of f.

Theorem (Wagner-Preston theorem)

For every inverse semigroup S, there exists a set X and in injective homomorphism $i\colon S \hookrightarrow \mathcal{I}(X)$ such that, for $a, b \in S$,

$$a \leq b \Leftrightarrow i(a) \leq i(b).$$

Boolean inverse monoids

Definition (Boolean inverse monoid)

An inverse semigroup is a *boolean inverse monoid* if it has an identity 1 (note this does not mean $xx^{-1} = 1$), absorbing element 0, and

1. $E(S)$ is a boolean algebra.
2. For all $a, b \in S$, if $ab^{-1} = 0 = a^{-1}b$, then $a \vee b$ exists.
3. Multiplication distributes over binary joins.

Definition (Factorizable)

An inverse monoid is *factorizable* if every element is beneath an element in the group of units i.e.

$$\forall x \in S, \exists y \in S \text{ s.t. } yy^{-1} = 1 = y^{-1}y \text{ and } x \leq y.$$

A factorizable boolean inverse monoid is called a *Foulis monoid*.
Symmetric inverse monoids $\mathcal{I}(X)$ are Foulis monoids iff X is finite.

Green's $\mathcal{D}$ and $\mathcal{J}$ relations

For an inverse semigroup S, $a, b \in S$, define

$$a\mathcal{D}b \iff \exists c \in S \text{ such that } a^{-1}a = c^{-1}c \text{ and } cc^{-1} = b^{-1}b.$$

For $f, g \in \mathcal{I}_n$, $f\mathcal{D}g$ iff $|\operatorname{dom} f| = |\operatorname{dom} g|$.

Proposition

$\mathcal{D}$ is an equivalence relation. Consider a distributive inverse semigroup S with zero and define

$$a \mathbin{\widetilde{\oplus}} b = \begin{cases} a \vee b, & \text{if } ab^{-1} = 0 = a^{-1}b, \\ \uparrow, & \text{otherwise.} \end{cases}$$

- *S with $\widetilde{\oplus}$ is a partial commutative monoid, and so is $S/\mathcal{D}$.*
- *S a Foulis monoid $\Rightarrow$ $S/\mathcal{D}$ an effect algebra with RDP.*
- *S a Foulis monoid with lattice order $\Rightarrow$ $S/\mathcal{D}$ an MV-effect algebra.*

Define $a\mathcal{J}b \iff SaS = SbS$.
Fact: In a Foulis monoid, $\mathcal{D} = \mathcal{J}$.

Coordinatization Theorem

Definition (AF inverse monoid)

Let

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots$$

be a sequence of finite products of finite symmetric inverse monoids and injective maps. The directed colimit of such a sequence is called an *AF inverse monoid*.

Definition (Coordinatizable)

An MV algebra A is said to be *coordinatizable* if there is a Foulis monoid S such that $S/\mathcal{D} \cong A$.

Theorem (Lawson & Scott)

Every countable MV algebra A can be coordinatized by some Foulis monoid S. Moreover, S can be taken to be an AF inverse monoid.

Theorem (Wehrung)

Every MV algebra can be coordinatized. A direct generalization of AF inverse monoids applies at cardinality $\aleph_1$, but not at $\aleph_2$ and beyond.

Coordinatization examples

The *Łukasiewicz chains*, $Ł_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \subseteq [0,1]$ are coordinatized by $\mathcal{I}_n$. Recall that a $\mathcal{D}$-class of $\mathcal{I}_n$ consists of all the partial bijections with the same size domain. The $\mathcal{D}$-class of partial bijections on m elements corresponds to the element $\frac{m}{n}$.

Theorem (Lawson & Scott)

The dyadic rationals in $[0,1]$ (those with denominator 2^k) are coordinatized by the dyadic inverse monoid, which turns out to be isomorphic to the directed colimit of

$$\mathcal{I}_1 \to \mathcal{I}_2 \to \mathcal{I}_4 \to \mathcal{I}_8 \to \mathcal{I}_{16} \to \ldots.$$

How to generalize to get all the rationals in $[0,1]$?

Coordinatizing $\mathbb{Q} \cap [0,1]$

First idea...

$$\mathcal{I}_1 \xrightarrow{\tau_1} \mathcal{I}_2 \xrightarrow{\tau_2} \mathcal{I}_3 \xrightarrow{\tau_3} \mathcal{I}_4 \xrightarrow{\tau_4} \dots$$

Problem: maps don't make sense. Successive maps are "inclusions". For $\tau\colon \mathcal{I}_n \to \mathcal{I}_m$ to make sense, n must divide m.

Idea: if $nq = m$, then each element of the underlying set of $X_n = \{x_1, \dots, x_n\}$ is identified with a subset of q elements in $X_m = \{x_1, \dots x_{nq}\}$ in an obvious way; e.g. identify $x_1 \in X_n$ with $\{x_1, \dots, x_q\} \subseteq X_m$. Extend this to identification to functions.

Coordinatizing $\mathbb{Q} \cap [0,1]$

Definition (Omnidivisional sequence)

A sequence $D = \{n_i\}_{i=1}^{\infty}$ of natural numbers is *omnidivisional* if it satisfies the following properties.

- For all i, $n_i \mid n_{i+1}$.
- For all $m \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $m \mid n_i$.

Example

Let p_i be the i^{th} prime number. Then $\{\Pi_{i=1}^{n} p_i^{n-i+1}\}_{n=1}^{\infty}$ is an omnidivisional sequence. The first few members of the sequence are $2, 2^2 3, 2^3 3^2 5, 2^4 3^3 5^2 7, \ldots$.

Example

The sequence $\{n!\}_{n=1}^{\infty}$.

Coordinatizing $\mathbb{Q} \cap [0,1]$

Theorem (Coordinatization of the rationals)

Let $D = \{n_i\}_{n=1}^{\infty}$ be an omnidivisional sequence. Then, the directed colimit of the sequence

$$Q\colon \mathcal{I}_{n_1} \xrightarrow{\tau_1} \mathcal{I}_{n_2} \xrightarrow{\tau_2} \mathcal{I}_{n_3} \xrightarrow{\tau_3} \mathcal{I}_{n_4} \xrightarrow{\tau_4} \dots,$$

coordinatizes $\mathbb{Q} \cap [0,1]$.

Proof.

Denote the directed colimit of Q by Q_∞. Define a map $w\colon Q_\infty/\mathcal{D} \to \mathbb{Q} \cap [0,1]$ as follows. For $s \in \mathcal{I}_{n_i}$, define

$$w([s]/\mathcal{D}) = \frac{|\operatorname{dom}(s)|}{n_i}.$$

Check this is well-defined on several different levels, check it's an iso, check it's an MV map. □

Coordinatization decomposition theorem

Theorem (Decomposition theorem, part 1)

Let A be an MV algebra. Suppose that A has a chain of subalgebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

such that $A = \bigcup_{i=1}^{\infty} A_i$ and that each A_i is coordinatized by an inverse semigroup S_i. Denote inclusions by $\ell_i \colon A_i \to A_{i+1}$. Choose an explicit isomorphism $f_i \colon A_i \to S_i/\mathcal{D}$ for each i.

Suppose there are injective maps $\tau_i \colon S_i \to S_{i+1}$ such that the maps $t_i \colon S_i/\mathcal{D} \to S_{i+1}/\mathcal{D}, s/\mathcal{D} \to \tau_i(s)/\mathcal{D}$ are well defined on $\mathcal{D}$-classes, and that the following diagram commutes for all i.

$$\begin{array}{ccc} A_i & \xrightarrow{\ \ell_i\ } & A_{i+1} \\ \uparrow{\scriptstyle f_i^{-1}} & & \downarrow{\scriptstyle f_{i+1}} \\ S_i/\mathcal{D} & \xrightarrow[\ t_i\ ]{} & S_{i+1}/\mathcal{D} \end{array}$$

Then, A is coordinatized by $\lim_{\to}(S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots)$.

Coordinatization decomposition theorem

Theorem (Decomposition theorem, part 2)

Suppose A is an MV algebra coordinatized by the directed colimit of

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots .$$

Then, A has a sequence of subalgebras forming a chain of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

such that $A = \bigcup_{i=1}^{\infty} A_i$, and each A_i is coordinatized by S_i.

Some MV algebras yet to be coordinatized, which the decomposition theorems might be useful for:

- The full interval $[0, 1]$.
- The free MV algebra on one (or more) generators.
- Many more challenges/examples given by Mundici.

Thanks!

Thank you for listening!

Thank you to the University of Ottawa and NSERC for financial support.

Thanks to the Fields Institute for funding this workshop.

Thanks to Phil Scott for “inviting” me to give this talk.