On the C^*-algebra of a topos

Simon Henry

College de france

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Note to the reader: this talk was only partly supported by slides, and in particular the present slides are incomplete.

Intuitively, a sheaf of sets over a topological space X, is a "continuously varying family of sets indexed by X".

Definition

A sheaf of sets $\mathcal{F}$ on a topological space X is the data of:

- $\forall U \in \mathcal{O}(X)$, a set $\mathcal{F}(U)$.
- $\forall V \subset U \subset X$ a "restriction" map $x \mapsto x_{|V} : \mathcal{F}(U) \to \mathcal{F}(V)$.

Such that the restrictions are compatible (functorial) and satisfies the sheaf condition with respect to open cover.

Theorem

The data of a sheaf of set $\mathcal{F}$ over X is equivalent to the data of a topological space $Et\mathcal{F}$ together with a local homomorphism (an étale map) $\pi : Et\mathcal{F} \to X$.
$\mathcal{F}$ can be reconstructed out of $Et\mathcal{F}$ by:

$$\mathcal{F}(U) = \{f : U \to Et\mathcal{F} | f \text{ is continuous and } \pi(f(x)) = x\}$$

Examples: $X \to X$ corresponds to the constant sheaf equal to $\{*\}$ denoted 1_X. The subsheaves of of 1_X corresponds to the open subset $U \to X$.

Any sheaf is obtained by gluing open subsets of X together.

Catch phrase: A sheaf is a generalized open subset.

The idea of topos theory is to do general topology with sheaves instead of open subsets.

General topology or Point-set topology	Topological spaces	A set of points X, a subset $\mathcal{O}(X) \subset \mathcal{P}(X)$
Point-free topology	Locales	A Frame $\mathcal{O}(X)$.
Topos theory	Toposes	A category of sheaves $Sh(\mathcal{T})$

From this perspective, a site for a topos is a “basis” of its “topology” (i.e. of its category of sheaves).

If G is an oriented graph (locally finite) one can attach to it the topos $\mathcal{T}_G$ defined by: A sheaf $\mathcal{F}$ over $\mathcal{T}_G$ is the data of:

- For each vertices v of G, a set $\mathcal{F}_v$.
- For each arrow $s : v' \to v$ of G a map $\mathcal{F}_v \to \mathcal{F}_{v'}$.
- For each vertices v the natural map:

$$\mathcal{F}_v \to \prod_{s:v' \to v} \mathcal{F}_{v'}$$

  is an isomorphism.

We will need at some point to use the “internal logic” of a topos.

Roughly, it is the fact any category of sheaves is in particular the category of sets of some model of intuitionist set theory.

Any construction that can be performed or theorem that can be proved within intuitionist set theory yield a construction or a theorem about the sheaves over a topos.

For example, the definition of real number using (two sided) Dedekind cut or the Cauchy filter completion of $\mathbb{Q}$ defines a sheaf on every topos, called the sheaf of real number. On a topological space X it is the sheaf of continuous functions with values in $\mathbb{R}$. Other definitions of the real numbers (for example sequential completion of $\mathbb{Q}$ or one sided dedekind cut) can be non-equivalent to this one in intuitionist mathematics and give rise to different “sheaf of real numbers”.

Let $\mathbb{R}_{\mathcal{T}}$ be the sheaf of real numbers over $\mathcal{T}$.

Theorem

If $\mathcal{T}$ is a "locally nice" topos, then there is a (non-unital) ring $\mathcal{C}_c(\mathcal{T})$ such that the category of sheaves of $\mathbb{R}_{\mathcal{T}}$-module over $\mathcal{T}$ is equivalent to the category of non-degenerate right $\mathcal{C}_c(\mathcal{T})$-modules.

This theorem characterize $\mathcal{C}_c(\mathcal{T})$ "up to Morita equivalence".

At least three things remains to be explained:

- What is a (locally) nice topos ?
- How $\mathcal{C}_c(\mathcal{T})$ is constructed explicitly ?
- What is the $*$-involution on $\mathcal{C}_c(\mathcal{T})$?

Comparison to the usual groupoid C^* algebra construction:

- At first sight, it does not produces new examples, and it miss some examples (the C^*-algebras of connected topological groups for example).
- It is easier to perform when we don't have a groupoid directly at hand (for example, with graphs and generalizations of graphs).
- It seems more conceptual (at least to me) and a lot simpler (once you are familiarised with topos theory).
- It doesn't loose that many examples: conjecturally, only the connected component of the isotropy groups is lost (Known Lie groupoid).
- The construction of the algebras does not involve any analysis: only finite sums and products.
- There is a fully constructive version of this construction (in the sense that it is valid within intuitionist mathematics).
- In particular, one has a relative version: from a morphism $f : \mathcal{E} \rightarrow \mathcal{T}$ between toposes that satisfies a relative form of the condition, one can construct a sheaf of algebras over $\mathcal{T}$ that satisfies a similar universal property.

A “nice” a topos is a topos $\mathcal{T}$ such that:

- $\mathcal{T}$ is separated, i.e. $\Delta : \mathcal{T} \rightarrow \mathcal{T} \times \mathcal{T}$ is a proper map.
- The localic reflection of $\mathcal{T}$ is locally compact.
- Every sheaf over $\mathcal{T}$ can be covered by decidable sheaves.

This is a very restrictive condition: in terms of groupoids it corresponds to a proper and separated locally compact groupoid.

A topos is locally nice it it admit an étale covering $EtX = \mathcal{T}_{/X} \rightarrow \mathcal{T}$ by a nice topos. It corresponds essentially to the idea of a locally compact groupoid, all the examples we are interested in are locally nice.

The construction of $\mathcal{C}_c(\mathcal{T})$:

$\mathcal{C}_c(\mathcal{T})$ has to be a sub-algebra of endomorphisms of a certain sheaf of $\mathbb{R}_\mathcal{T}$ module: the one corresponding to the $\mathcal{C}_c(\mathcal{T})$-module $\mathcal{C}_c(\mathcal{T})$.

Les X be a sheaf over $\mathcal{T}$ such that $EtX := \mathcal{T}/X$ is a "nice" cover of $\mathcal{T}$. Let R_X be the free $\mathbb{R}_\mathcal{T}$ module generated by X. If R is another sheaf of $\mathbb{R}_\mathcal{T}$-modules a morphism of sheaf of $\mathbb{R}_\mathcal{T}$-modules from R_X to R is the same as a function from X to R, i.e. a section of R over X.

Theorem

If X is a bound of $\mathcal{T}$, then $\mathcal{C}_c(\mathcal{T})$ can be chosen as algebra of endomorphisms of R_X that are compactly supported as functions from X to R_X.

The equivalence from sheaves of $\mathbb{R}_\mathcal{T}$-modules to $\mathcal{C}_c(\mathcal{T})$-module is given by associating to a sheaf R the set of morphisms from X to R with compact support over X.

What is the $*$-operation ?

First answer:In the case where the sheaf X such that $\mathcal{T}_{/X}$ is nice is decidable, one can see the sheaf R_X internally as "the set of finitely supported functions on X with values in $\mathbb{R}_{\mathcal{T}}$ "

In particular, a function from X to R_X can be represented by a function from $X \times X$ to $\mathbb{R}_{\mathcal{T}}$ (internally, it is the function of matrix elements). It appears that the set of compactly supported functions from X to R_X is stable under exchange of the two variables.

This does not works when X is not decidable !

This assumption (X decidable) corresponds to Hausdorff groupoids.

There is a better (and deeper answer).

A right $\mathcal{C}_c(\mathcal{T})$-module is the same as a sheaf of $\mathbb{R}_{\mathcal{T}}$-modules,and a $*$-operation is a construction that turn right $\mathcal{C}_c(\mathcal{T})$-modules into left $\mathcal{C}_c(\mathcal{T})$-modules.

So what are left $\mathcal{C}_c(\mathcal{T})$-modules in terms of $\mathcal{T}$?
What is the geometric meaning of this operation that turns left modules into right modules ?

It follows from our main theorem and general categorical non-sense that left $\mathcal{C}_c(\mathcal{T})$-modules are *co-sheaf* of $\mathbb{R}_{\mathcal{T}}$-modules.

The operation we are looking for is (a form of) *Verdier duality*.