Type monoids

The variety of BISs
ISs from partial functions
BISs and tight maps
Biases

The type monoid
From \( D \) to \( \text{Typ} S \)
Typ \( S \) and equidecomposability types
Dobbertin's Theorem
Abelian \( \ell \)-groups

Type monoids and nonstable K-theory
\( \kappa(s) \)
Typ \( S \) \( \rightarrow \) \( V(\kappa(s)) \)

Type monoids of Boolean inverse semigroups

Friedrich Wehrung

LMNO, CNRS UMR 6139 (Caen)
E-mail: friedrich.wehrung01@unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung

June 2016
Basic definitions

Inverse semigroup

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- The variety of BISs
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Basic definitions

### Inverse semigroup

Semigroup \((S, \cdot)\), where \(\forall x \exists \text{ unique } x^{-1} \text{ (the inverse of } x) \text{ such that } xx^{-1}x = x \text{ and } x^{-1}xx^{-1} = x^{-1}\).
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We set \(d(x) = x^{-1}x\) (the domain of \(x\)), \(r(x) = xx^{-1}\) (the range of \(x\)), \(\text{Idp} S = \{x \in S \mid x^2 = x\}\).
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Fundamental example (symmetric inverse semigroup)
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Inverse semigroups of partial bijections

Vagner-Preston Theorem

Every inverse semigroup embeds into some $\mathcal{I}_\Omega$. Means that every inverse semigroup can be represented as a semigroup of partial bijections on a set. Example constructed from a group action:

If a group $G$ acts on a set $\Omega$, consider all partial bijections $f: X \to Y$ in $\mathcal{I}_\Omega$ that are piecewise in $G$: that is, $\exists$ decompositions $X = \bigcup_{i=1}^n X_i$, $Y = \bigcup_{i=1}^n Y_i$, each $g_i \in G$ and $g_i X_i = Y_i$, and $f(x) = g_i x$ whenever $x \in X_i$.

$\text{Inv}(\mathcal{I}_\Omega \hookrightarrow G) = \{ f \in \mathcal{I}_\Omega | f \text{ is piecewise in } G \}$ is an inverse semigroup.

Idempotents of $\text{Inv}(\mathcal{I}_\Omega \hookrightarrow G)$: they are the identities on all subsets of $\Omega$. They form a Boolean lattice.
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What kind of inverse semigroup is this?

Zero element: the function $0 \in \text{Inv}(\mathcal{B} \hookrightarrow G)$ with empty domain.

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Canonical ordering on an inverse semigroup:

- \( x \leq y \) iff (\( \exists \) idempotent \( e \)) \( x = ye \) (resp., \( x = ey \)),
- \( x = d(x) \),
- \( x = r(x)y \).

For \( S = \text{Inv}(B \hookrightarrow G) \), \( f \leq g \) iff \( g \) extends \( f \).

The latter condition, on \( \exists x \oplus y \), is not redundant (example with \( \text{Idp} S \) the 2-atom Boolean algebra).

Large class of Boolean inverse semigroups: all \( \text{Inv}(B \hookrightarrow G) \).
Boolean inverse semigroups

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For \( S = \text{Inv}(\mathcal{B}, G) \), \( f \leq g \) iff \( g \) extends \( f \).

**Boolean inverse semigroups**

Inverse semigroup \( S \) with zero \( x0 = 0x = 0 \ \forall x \) such that \( \text{Idp} \ S \) is a generalized Boolean algebra, and \( \forall x, y \) with \( x \perp y \), the supremum \( x \oplus y \) of \( \{x, y\} \), with respect to \( \leq \), exists.

The latter condition, on \( \exists x \oplus y \), is not redundant (example with \( \text{Idp} \ S \) the 2-atom Boolean algebra).

Large class of Boolean inverse semigroups: all \( \text{Inv}(\mathcal{B}, G) \).
Distributivity of multiplication and meet on joins

Proposition (folklore).

Let \( S \) be a Boolean inverse semigroup and let \( a \preceq b_1 \preceq \ldots \preceq b_n \in S \).

1. If \( \bigvee_{i=1}^n b_i \) exists, then \( \bigvee_{i=1}^n (a b_i) \) exists and equals \( \bigvee_{i=1}^n b_i a \).

2. If \( \bigvee_{i=1}^n b_i \) exists, then \( \bigwedge_{i=1}^n (a b_i) \) exists iff each \( a \wedge b_i \) exists, and then
   \[ \bigvee_{i=1}^n (a b_i) = a \bigwedge_{i=1}^n b_i. \]

Note: for a Boolean inverse semigroup \( S \) and \( a \preceq b \in S \), \( a \wedge b \) may not exist. Those \( S \) in which \( a \wedge b \) always exists are called inverse meet-semigroups.
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Let $S$ be a Boolean inverse semigroup and let $a, b_1, \ldots, b_n \in S$.

1. $\bigvee_{i=1}^{n} b_i$ exists iff the $b_i$ are pairwise compatible, that is, each $b_i^{-1}b_j$ and each $b_ib_j^{-1}$ is idempotent.
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2. If $\bigvee_{i=1}^{n} b_i$ exists, then $\bigvee_{i=1}^{n} (a b_i)$ and $\bigvee_{i=1}^{n} (b_i a)$ both exist, $\bigvee_{i=1}^{n} (a b_i) = a \bigvee_{i=1}^{n} b_i$, and $\bigvee_{i=1}^{n} (b_i a) = \left( \bigvee_{i=1}^{n} b_i \right) a$. 

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3. If $\bigvee_{i=1}^n b_i$ exists, then $a \wedge \bigvee_{i=1}^n b_i$ exists iff each $a \wedge b_i$ exists, and then $\bigvee_{i=1}^n (a \wedge b_i) = a \wedge \bigvee_{i=1}^n b_i$.
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3. If $\bigvee_{i=1}^{n} b_i$ exists, then $a \land \bigvee_{i=1}^{n} b_i$ exists iff each $a \land b_i$ exists, and then $\bigvee_{i=1}^{n} (a \land b_i) = a \land \bigvee_{i=1}^{n} b_i$.

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Tight homomorphisms

A relevant concept of morphism, for Boolean inverse semigroups, is the following.
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### Tight maps

A semigroup homomorphism $f: S \to T$, between Boolean inverse semigroups, is tight if $x \perp_S y$ implies that $f(x) \perp_T f(y)$ and $f(x \oplus y) = f(x) \oplus f(y)$. (In particular, $f(0_S) = 0_T$.)

Annoying fact: $\oplus$ is only a partial operation.

Derived (full) operations:

- $x \cdot y = (r(x) \cdot r(y)) x (d(x) \cdot d(y))$ (skew difference);
- $x \cdot y = (x \cdot y) \oplus y$ (skew addition).

Both $x \cdot y$ and $x \cdot y$ are always defined.
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Derived (full) operations:

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\begin{align*}
\times \ominus y &= (r(x) \setminus r(y)) \times (d(x) \setminus d(y)) \quad (\text{skew difference}); \\
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**Derived (full) operations:**

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\begin{align*}
x \ominus y &= (r(x) \setminus r(y)) \cdot (d(x) \setminus d(y)) \quad \text{(skew difference);} \\
x \nabla y &= (x \ominus y) \oplus y \quad \text{(skew addition).}
\end{align*}
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Both \( x \ominus y \) and \( x \nabla y \) are always defined.
The variety of all biases

- The structures \((S, \cdot, 0, \oslash, \sqcap)\) can be axiomatized,
The variety of all biases

- The structures \((S, \cdot, 0, \otimes, \nabla)\) can be axiomatized, by finitely many identities (e.g., \(x \otimes y = (x \nabla y)(x \otimes y)^{-1}(x \otimes y)\)).
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The variety of all biases

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- \(\text{Biases}(\cdot, 0, \ominus, \vee) \Rightarrow \text{Boolean inverse semigroups}(\cdot, 0, \oplus)\).
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- Those identities define the variety of all biases.
- \(\text{Biases}(\cdot, 0, \otimes, \nabla) \equiv \text{Boolean inverse semigroups } (\cdot, 0, \oplus)\).
- For Boolean inverse semigroups \(S\) and \(T\), a map \(f : S \to T\) is a homomorphism of biases iff it is tight.
The variety of all biases

- The structures \((S, \cdot, 0, \odot, \triangledown)\) can be axiomatized, by finitely many identities (e.g., \(x \odot y = (x \triangledown y)(x \odot y)^{-1}(x \odot y)\)).
- Those identities define the variety of all biases.
- Biases\((\cdot, 0, \odot, \triangledown)\) \(\iff\) Boolean inverse semigroups \((\cdot, 0, \oplus)\).
- For Boolean inverse semigroups \(S\) and \(T\), a map \(f: S \to T\) is a homomorphism of biases iff it is tight.
- A subset \(S\) in a BIS \(T\) is a sub-bias iff it is a subsemigroup, closed under finite \(\oplus\), and closed under \((x, y) \mapsto x \setminus y\) on \(\text{Idp } S\).
The variety of all biases

- The structures \((S, \cdot, 0, \otimes, \triangledown)\) can be axiomatized, by finitely many identities (e.g., \(x \otimes y = (x \triangledown y)(x \otimes y)^{-1}(x \otimes y)\)).
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- A subset \(S\) in a BIS \(T\) is a sub-bias iff it is a subsemigroup, closed under finite \(\oplus\), and closed under \((x, y) \mapsto x \triangleleft y\) on \(\text{ldp } S\).
- The following term is a Mal’cev term for the variety of all biases:

\[
m(x, y, z) = \left( x (d(x) \otimes d(y)) \triangledown xy^{-1} z \right) \triangledown (r(z) \otimes r(y)) z.
\]
The variety of all biases

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- Therefore, the variety of all biases is congruence-permutable. (Note: it is not congruence-distributive.)
The variety of all biases

- The structures \((S, \cdot, 0, \odot, \triangledown)\) can be axiomatized, by finitely many identities (e.g., \(x \odot y = (x \triangledown y)(x \odot y)^{-1}(x \odot y)\)).
- Those identities define the variety of all biases.
- Biases\((\cdot, 0, \odot, \triangledown)\) \(\iff\) Boolean inverse semigroups \((\cdot, 0, \oplus)\).
- For Boolean inverse semigroups \(S\) and \(T\), a map \(f : S \to T\) is a homomorphism of biases \(\iff\) it is tight.
- A subset \(S\) in a BIS \(T\) is a sub-bias \(\iff\) it is a subsemigroup, closed under finite \(\oplus\), and closed under \((x, y) \mapsto x \triangleleft y\) on Idp \(S\).
- The following term is a Mal’cev term for the variety of all biases:

\[
m(x, y, z) = \left( x(d(x) \odot d(y)) \triangledown xy^{-1}z \right) \triangledown (r(z) \odot r(y))z.
\]

- Therefore, the variety of all biases is congruence-permutable. (Note: it is not congruence-distributive.)
- Hence, Boolean inverse semigroups are much closer to rings than to semigroups.
A Cayley-type theorem for BISs

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Every Boolean inverse semigroup has a tight embedding into some $\mathcal{I}_\Omega$. The embedding preserves all existing finite meets.
A Cayley-type theorem for BISs

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Every Boolean inverse semigroup has a tight embedding into some $\mathcal{J}_\Omega$. The embedding preserves all existing finite meets.

- The $\Omega$ in this representation, denoted by $G_P(S)$ in Lawson and Lenz (2013), is the prime spectrum of $S$. 
A Cayley-type theorem for BISs

### Proposition

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- The $\Omega$ in this representation, denoted by $G_P(S)$ in Lawson and Lenz (2013), is the **prime spectrum** of $S$.
- The result above is contained in a duality theory worked out by Lawson and Lenz (2013).
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- The result above is contained in a duality theory worked out by Lawson and Lenz (2013).
- The set-theoretical content of the result above is the Boolean prime ideal Theorem.
- The representation above is called the regular representation of $S$. 
On any inverse semigroup, we set
Green’s relation $D$

- On any inverse semigroup, we set
  - $x \mathcal{L} y \iff d(x) = d(y)$, $x \mathcal{R} y \iff r(x) = r(y)$, and $D = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.
Green’s relation \( \mathcal{D} \)

- On any inverse semigroup, we set
  
  \[ x \mathcal{L} y \iff d(x) = d(y), \quad x \mathcal{R} y \iff r(x) = r(y), \text{ and} \]
  
  \[ \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}. \]

- For idempotent \( a \) and \( b \), \( a \mathcal{D} b \) iff \( (\exists x) \ (a = d(x) \text{ and } b = r(x)) \).
Green’s relation $\mathscr{D}$

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- For a Boolean inverse semigroup $S$, the quotient $\text{Int } S = S/\mathcal{D}$ (the dimension interval of $S$) can be endowed with a partial addition, given by
Green’s relation $\mathcal{D}$

- On any inverse semigroup, we set
- $x \mathcal{L} y \iff \mathcal{d}(x) = \mathcal{d}(y)$, $x \mathcal{R} y \iff \mathcal{r}(x) = \mathcal{r}(y)$, and
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- For idempotent $a$ and $b$, $a \mathcal{D} b$ iff $(\exists x) \ (a = \mathcal{d}(x) \text{ and } b = \mathcal{r}(x))$.

- For a Boolean inverse semigroup $S$, the quotient $\text{Int } S = S/\mathcal{D}$ (the dimension interval of $S$) can be endowed with a partial addition, given by
  \[(x/\mathcal{D}) + (y/\mathcal{D}) = (x \oplus y)/\mathcal{D}, \text{ whenever } x \oplus y \text{ is defined.}\]
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- On any inverse semigroup, we set
  - $x \mathcal{L} y \iff d(x) = d(y)$, $x \mathcal{R} y \iff r(x) = r(y)$, and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.
- For idempotent $a$ and $b$, $a \mathcal{D} b$ iff $(\exists x) \left( a = d(x) \text{ and } b = r(x) \right)$.
- For a Boolean inverse semigroup $S$, the quotient $\text{Int } S = S/\mathcal{D}$ (the dimension interval of $S$) can be endowed with a partial addition, given by
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- **Important property of** $\text{Int } S$ *(not trivial)*: $x + (y + z)$ is defined iff $(x + y) + z$ is defined, and then both values are the same.
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- On any inverse semigroup, we set

$$x \mathcal{L} y \iff d(x) = d(y), \quad x \mathcal{R} y \iff r(x) = r(y), \quad \text{and} \quad \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}.$$  

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- For a Boolean inverse semigroup $S$, the quotient $\text{Int } S = S/\mathcal{D}$ (the dimension interval of $S$) can be endowed with a partial addition, given by

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- **Important property of** $\text{Int } S$ (not trivial): $x + (y + z)$ is defined iff $(x + y) + z$ is defined, and then both values are the same.

- The type monoid of $S$, denoted by $\text{Typ } S$, is the universal monoid of the partial commutative monoid $\text{Int } S$.  

Let a group $G$ act by automorphisms on a generalized Boolean algebra $\mathcal{B}$. 
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Type monoid of Inv(\(\mathcal{B}, G\))

- Let a group \(G\) act by automorphisms on a generalized Boolean algebra \(\mathcal{B}\).
- \(S = \text{Inv}(\mathcal{B}, G)\) is a Boolean inverse semigroup.
- What is \(\mathcal{D}\) on its idempotents?
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- Let a group $G$ act by automorphisms on a generalized Boolean algebra $\mathcal{B}$.
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- $\text{id}_X \mathcal{D} \text{id}_Y$ iff there is a partial bijection $f$, piecewise in $G$, defined on $X$, such that $f[X] = Y$. 

Dobbertin's Theorem
Abelian $\ell$-groups

Type monoids and nonstable $K$-theory
$K(s)$
Typ $s \rightarrow V(K(s))$
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That is, there are decompositions $X = \bigcup_{i=1}^{n} X_i$, $Y = \bigcup_{i=1}^{n} Y_i$, together with $g_i \in G$, such that each $X_i$, $Y_i \in \mathcal{B}$ and each $Y_i = g_i X_i$. 
Type monoid of \( \text{Inv}(\mathcal{B}, G) \)

- Let a group \( G \) act by automorphisms on a generalized Boolean algebra \( \mathcal{B} \).
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- This means that \( X \) and \( Y \) are \( G \)-equidecomposable, with pieces from \( \mathcal{B} \).
Type monoid of \( \text{Inv}(\mathcal{B}, G) \)

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- This means that \( X \) and \( Y \) are \( G \)-equidecomposable, with pieces from \( \mathcal{B} \).
- Denote by \( \mathbb{Z}^+\langle \mathcal{B} \rangle \| G \) the monoid of [generated by] all equidecomposability types of members of \( \mathcal{B} \) with respect to the action of \( G \).
Let a group $G$ act by automorphisms on a generalized Boolean algebra $\mathcal{B}$.

$S = \text{Inv}(\mathcal{B}, G)$ is a Boolean inverse semigroup.

What is $\mathcal{D}$ on its idempotents?

$id_X \mathcal{D} id_Y$ iff there is a partial bijection $f$, piecewise in $G$, defined on $X$, such that $f[X] = Y$.

That is, there are decompositions $X = \bigcup_{i=1}^{n} X_i$, $Y = \bigcup_{i=1}^{n} Y_i$, together with $g_i \in G$, such that each $X_i, Y_i \in \mathcal{B}$ and each $Y_i = g_i X_i$.

This means that $X$ and $Y$ are $G$-equidecomposable, with pieces from $\mathcal{B}$.

Denote by $\mathbb{Z}^+\langle \mathcal{B} \rangle \! \! / \! \! G$ the monoid of [generated by] all equidecomposability types of members of $\mathcal{B}$ with respect to the action of $G$.

Then the type monoid of $\text{Inv}(\mathcal{B}, G)$ is isomorphic to $\mathbb{Z}^+\langle \mathcal{B} \rangle \! \! / \! \! G$. 
Measurable monoids

- Say that a commutative monoid is **measurable** if it is isomorphic to Typ $S$, for some Boolean inverse semigroup $S$. 

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By the above, every $\mathbb{Z}^+ \rtimes B \rtimes G$ (where a group $G$ acts on a generalized Boolean algebra $B$) is measurable. The converse holds (not so trivial). Starting with a Boolean inverse semigroup $S$, we need to find $G$, $B$ such that Typ $S \cong \mathbb{Z}^+ \rtimes B \rtimes G$. First guess: try $B = \text{Id} p_S$, $G = \text{inner automorphisms}$ (?) of $B$. 

Problem: the map $f : e \mapsto xe - 1$, for $e$ idempotent $\leq d(x)$, may not extend to any automorphism of $B$. Can be solved by representing $B$ as generalized Boolean lattice of subsets of some set $\Omega$, then duplicating $\Omega$. This leaves enough room to extend $f$. 

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**Type monoids**

- The variety of BISs
- ISs from partial functions
- BISs and tight maps
- Biases
- The type monoid
- From $\mathcal{D}$ to Typ $S$
- Typ $S$ and equidecomposability types
- Dobbertin’s Theorem
- Abelian $\ell$-groups

** measurable monoids**

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Measurability versus equidecomposability

Proposition
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A commutative monoid $M$ is measurable (i.e., Typ $S$ for some Boolean inverse semigroup $S$) iff $M \cong \mathbb{Z}^+ \langle B \rangle \rtimes G$ for some action of a group $G$ on a generalized Boolean algebra $B$. 
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- Every measurable monoid is isomorphic to Typ $S$ for a Boolean meet-semigroup (resp., antigroup) $S$. 
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- There is a countable counterexample showing that “meet-semigroup” and “antigroup” cannot be reached simultaneously.
- Every measurable monoid $M$ is conical, that is, has $x + y = 0 \Rightarrow x = y = 0$. 
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- There is a countable counterexample showing that “meet-semigroup” and “antigroup” cannot be reached simultaneously.
- Every measurable monoid $M$ is conical, that is, has $x + y = 0 \Rightarrow x = y = 0$.
- Also, $M$ is a refinement monoid, that is, whenever $a_0 + a_1 = b_0 + b_1$ in $M$, there are $c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1} \in M$ such that each $a_i = c_{i,0} + c_{i,1}$ and each $b_j = c_{0,j} + c_{1,j}$. 
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A commutative monoid $M$ is measurable (i.e., $\text{Typ } S$ for some Boolean inverse semigroup $S$) iff $M \cong \mathbb{Z}^+ \langle \mathcal{B} \rangle /\!// G$ for some action of a group $G$ on a generalized Boolean algebra $\mathcal{B}$.

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- How about the converse?
Dobbertin’s V-measures

**Theorem (Dobbertin, 1983)**

Let $M$ be a countable, conical refinement monoid and let $e \in M$.

Then there are a countable Boolean algebra $B$ and a finitely additive measure $\mu : B \to M$ such that $\mu(1) = e$, $\mu^{-1}\{0\} = \{0\}$, and whenever $\mu(c) = a + b$, there exists a decomposition $c = a \oplus b$ in $B$ such that $\mu(a) = a$ and $\mu(b) = b$.

(We say that $\mu$ is a V-measure.) Moreover, the pair $(B \hookrightarrow \mu)$ is unique up to isomorphism.

Example: $M = (\mathbb{Z} \hookrightarrow + \hookrightarrow 0)$, $e = 1$. Then $B = \{0 \hookrightarrow 1\}$, $\mu(1) = 1$.

Example: $M = (\{0 \hookrightarrow 1\} \hookrightarrow \lor \hookrightarrow 0)$, the two-element semilattice, and $e = 1$. Then $B$ is the unique countable atomless Boolean algebra, $\mu(x) = 1$ iff $x \neq 0$.

Possibilities of extension of Dobbertin’s Theorem: For card $M = \aleph_1$, uniqueness is lost. If card $M \geq \aleph_2$, then existence is lost (W 1998).
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From Dobbertin’s Theorem to type monoids

Proof of Dobbertin’s Theorem: essentially back-and-forth.
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From Dobbertin’s Theorem to type monoids

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Every countable conical refinement monoid is measurable.
From Dobbertin’s Theorem to type monoids

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- **Idea of proof:**
  - $M$ is an *o-ideal* in $M' = M \sqcup \{\infty\}$. Since the o-ideals of Typ $S$ correspond to the tight ideals of $S$, the problem is reduced to the case where $M$ has an order-unit $e$. 
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  - Let $\mu: (B, 1) \rightarrow (M, e)$ be Dobbertin’s V-measure.
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- Let $\mu : (B, 1) \to (M, e)$ be Dobbertin’s V-measure.
- Set $S = \text{Inv}(B, \mu) = \text{semigroup of all } \mu\text{-preserving partial isomorphisms } f : B \downarrow a \to B \downarrow b$, where $a, b \in B$ with $\mu(a) = \mu(b)$. 
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  - $S$ is a Boolean inverse semigroup, with idempotents $\overline{a} = \text{id}_{B \downarrow a}$ where $a \in B$. 
Because of the uniqueness statement in Dobbertin’s Theorem, for any $a, b \in B$, if $\mu(a) = \mu(b)$, there is $f \in S$ (usually not unique) such that $f(a) = b$. 
Because of the uniqueness statement in Dobbertin’s Theorem, for any \( a, b \in B \), if \( \mu(a) = \mu(b) \), there is \( f \in S \) (usually not unique) such that \( f(a) = b \).

Hence, \( \bar{a} \preceq \bar{b} \) within \( S \) iff \( \mu(a) = \mu(b) \).
Measurability of countable CRM's (cont'd)

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- Hence, \( a \not\sim b \) within \( S \) iff \( \mu(a) = \mu(b) \).

- By the definition of \( \text{Typ} \, S \), there is a unique monoid homomorphism \( \varphi: \text{Typ} \, S \to M \) such that \( \varphi(\bar{a}/\mathcal{D}) = \mu(a) \) \( \forall a \in B \).
Measurability of countable CRMs (cont’d)

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Since \( M \) is a refinement monoid and \( \mu \) is a \( V \)-measure, the range of \( \varphi \) (which is also the range of \( \mu \)) generates \( M \) as a submonoid.
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- Moreover, \( \varphi \) is one-to-one on \( \text{Int} S \) (because \( a \not\asymp b \) within \( S \) iff \( \mu(a) = \mu(b) \)).
Because of the uniqueness statement in Dobbertin’s Theorem, for any \(a, b \in B\), if \(\mu(a) = \mu(b)\), there is \(f \in S\) (usually not unique) such that \(f(a) = b\).

Hence, \(a \not\sim b\) within \(S\) iff \(\mu(a) = \mu(b)\).

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By the general properties of refinement monoids, this implies that \(\varphi\) is an isomorphism. Hence \(M \cong \text{Typ } S\).
Representing abelian $\ell$-groups

Theorem (W 2015)

For every abelian $\ell$-group $G$, there is a Boolean inverse semigroup $S$, explicitly constructed, such that $\text{Typ} S \cong G$.

The poset $D = G \cup \{\bot\}$, for a new bottom element $\bot$, is a distributive lattice with zero. Embed $D$ into its enveloping Boolean ring $B = BR(D)$. The elements of $B$ have the form $\sum 0 \leq i < n (a_{2i+1} + a_{2i})$, where all $a_i \in D$ and $\bot \leq a_0 \leq \cdots \leq a_{2n}$.

Adding the condition $a_0 \neq \bot$ (i.e., each $a_i \in G$) yields a Boolean subring $B$ of $B$.

The dimension monoid $\text{Dim} G$ of the (distributive) lattice $(G \rightarrow \vee \rightarrow \wedge)$ is isomorphic to the monoid $\mathbb{Z}^+ \otimes B$ of all nonnegative linear combinations of members of $B$, with $\oplus$ in $B$ turned to $+$ in $\mathbb{Z}^+ \otimes B$. 
Representing abelian $\ell$-groups

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The poset $D = G \uplus \{\perp\}$, for a new bottom element $\perp$, is a distributive lattice with zero. Embed $D$ into its enveloping Boolean ring $B = B R(D)$. The elements of $B$ have the form $\langle 0 \leq i < n \cdot (a_2 i + 1 \cdot a_2 i) \rangle$, where all $a_i \in D$ and $\perp \leq a_0 \leq \cdots \leq a_{2n}$. Adding the condition $a_0 \neq \perp$ (i.e., each $a_i \in G$) yields a Boolean subring $B$ of $B$. The dimension monoid $\text{Dim } G$ of the (distributive) lattice $(G \uplus \vee \uplus \wedge)$ is isomorphic to the monoid $\mathbb{Z}^+ \cdot B \cdot \mathbb{Z}$ of all nonnegative linear combinations of members of $B$, with $\oplus$ in $B$ turned to $+$ in $\mathbb{Z}^+ \cdot B \cdot \mathbb{Z}$. 

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Representing abelian $\ell$-groups

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- Embed \( D \) into its \textit{enveloping Boolean ring} \( \overline{B} = \text{BR}(D) \).
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Representing abelian \(\ell\)-groups

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- Embed \(D\) into its **enveloping Boolean ring** \(\overline{B} = \text{BR}(D)\).
- The elements of \(\overline{B}\) have the form \(\bigvee_{0 \leq i < n} (a_{2i+1} \setminus a_{2i})\), where all \(a_i \in D\) and \(\bot \leq a_0 \leq \cdots \leq a_{2n}\).
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- The **dimension monoid** \(\text{Dim} \, G\) of the (distributive) lattice \((G, \lor, \land)\) is isomorphic to the monoid \(\mathbb{Z}^+\langle B\rangle\) of all nonnegative linear combinations of members of \(B\), with \(\oplus\) in \(B\) turned to \(+\) in \(\mathbb{Z}^+\langle B\rangle\).
Representing abelian $\ell$-groups (cont’d)

- Enables us to define a **V-measure** (as in Dobbertin’s Theorem)
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Representing abelian \( \ell \)-groups (cont’d)

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\[
\mu \left( \bigvee_{0 \leq i < n} (a_{2i+1} \setminus a_{2i}) \right) = \sum_{i < n} (a_{2i+1} - a_{2i})
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(where \( a_0 \leq a_1 \leq \cdots \leq a_{2n} \) in \( G \)).

- Moreover, \( \forall a \in G \), the translation \( x \mapsto x + a \) “extends” to an automorphism \( \tau_a \) of \( B \). So \( \tau_a(y \setminus x) = (a + y) \setminus (a + x) \), \( \forall x \leq y \) in \( G \).
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\[ \overline{G} = \{ \tau_a \mid a \in G \} \text{ is a subgroup of } \text{Aut } B, \text{ isomorphic to } G. \]
Representing abelian $\ell$-groups (cont’d)

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- The desired BIS is $S = \text{Inv}(B, \overline{G})$. 

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*Type monoids*

- The variety of BISs
- ISs from partial functions
- BISs and tight maps
- Biases
- The type monoid
- From $\mathcal{D}$ to $\text{Typ} S$
- $\text{Typ} S$ and equidecomposability types
- Dobbertin’s Theorem
- Abelian $\ell$-groups

*Type monoids and nonstable K-theory*

- $K(S)$
- $\text{Typ} S \to \mathcal{V}(K(S))$
Representing abelian $\ell$-groups (cont’d)

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- $G = \{\tau_a \mid a \in G\}$ is a subgroup of $\text{Aut} \ B$, isomorphic to $G$.

- The desired BIS is $S = \text{Inv}(B, G)$. One must prove that for $x, y \in B$, $\mu(x) = \mu(y)$ iff $x$ and $y$ are equidecomposable modulo translations from $G$ (think of elements of $B$ as disjoint unions of intervals with endpoints from $G$).
Using Mundici’s 1986 result (MV-algebras \(\iff\) unit intervals of abelian \(\ell\)-groups), it thus follows that every MV-algebra is isomorphic to \(\text{Int } S = S/\mathcal{D}\), for some BIS \(S\).
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Loose ends on ℓ-groups

- Using Mundici’s 1986 result (MV-algebras ⇔ unit intervals of abelian ℓ-groups), it thus follows that every MV-algebra is isomorphic to Int S = S/𝔻, for some BIS S. Every such S is factorizable (i.e., ∀x, ∃ unit g, x ≤ g), and has 𝔻 = 𝒯.

- In the countable case, Lawson and Scott get the additional information that S can be taken AF (i.e., countable direct limit of finite products of symmetric inverse semigroups).
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(Proof: mutatis mutandis extend the usual Elliott, Goodearl + Handelman arguments from locally matricial algebras, or C*-algebras, to BISs.)

Getting “locally matricial” in arbitrary cardinality: hopeless for arbitrary dimension groups (counterexamples of size $\aleph_2$), but still open for abelian $\ell$-groups.
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Tight enveloping $K$-algebra of a BIS

Definition

For a unital ring $K$ and a BIS $S$, $K^S$ is the $K$-algebra defined by generators $S$ and relations

\[ \lambda s = s \lambda, s = s, \]

whenever $z = x \oplus y$ (within $S$).

For $S$ a Boolean inverse meet-semigroup, $K^S$ is isomorphic to Steinberg's $KU_T(S)$ (étale groupoid algebra of $U_T(S)$), where $U_T(S)$ is the universal tight groupoid of $S$.

Steinberg's construction extends to Hausdorff inverse semigroups (not necessarily Boolean).

If $K$ is an involutary ring, then $K^S$ is an involutary $K$-algebra (set $(\lambda s)^* = \lambda^* s^{-1}$).

If $X \subseteq S$ generates $S$ as a bias, then it also generates $K^S$ as an involutary subring.

The construction $K^S$ extends known constructions, such as Leavitt path algebras.
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For a unital ring $K$ and a BIS $S$, $K\langle S \rangle$ is the $K$-algebra defined by generators $S$ and relations $\lambda s = s\lambda$, $1s = s$, $z = x + y$ (within $K\langle S \rangle$) whenever $z = x \oplus y$ (within $S$).
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BISs interact with involutary rings

Proposition (W 2015)

\[ \text{Idp}(S \otimes T) \cong (\text{Idp}(S)) \otimes (\text{Idp}(T)) \quad \text{and} \quad U_{\text{mon}}(S \otimes T) \cong U_{\text{mon}}(S) \otimes U_{\text{mon}}(T). \]
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Proposition (W 2015)

Every inverse semigroup $S$, in an involutary ring $R$, is contained in a BIS $\overline{S} \subseteq R$, in which $\oplus$ specializes orthogonal addition ($x + y$, where $x^* y = xy^* = 0$).
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- Can, in certain conditions, be extended to involutary semirings.
- Yields a workable definition of the tensor product $S \otimes T$ of two BISs $S$ and $T$, which is still a BIS and has
  
  $\text{Idp}(S \otimes T) \cong (\text{Idp } S) \otimes (\text{Idp } T)$ and
  
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Embedding properties of $K\langle S \rangle$

- For $S$ a sub-BIS (⇔ sub-bias) of $T$, the canonical map $K\langle S \rangle \rightarrow K\langle T \rangle$ may not be one-to-one.
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- Has to do with so-called transfer properties in lattice theory (getting from $K \leftrightarrow \text{Id} L$ to $K \leftrightarrow L$).
The canonical map $\text{Typ} S \rightarrow V(K\langle S \rangle)$

- For idempotent matrices $a$ and $b$ from a ring $R$, let $a \sim b$ hold if $\exists x, y$, $a = xy$ and $b = yx$ (*Murray - von Neumann equivalence*).
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- MnV classes can be added, *via* $[x] + [y] = [x + y]$ provided $xy = yx = 0$. 
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- MvN classes can be added, via $[x] + [y] = [x + y]$ provided $xy = yx = 0$.
- $V(R) = \{[x] | x \text{ idempotent matrix from } R\}$, the nonstable $K$-theory of $R$. It is a conical commutative monoid.
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- \( V(R) = \{[x] \mid x \text{ idempotent matrix from } R\} \), the nonstable K-theory of \( R \). It is a conical commutative monoid.

Proposition (W 2015)
The canonical map $\text{Typ } S \to V(K\langle S \rangle)$

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**Proposition (W 2015)**

Let $S$ be a BIS and let $K$ be a unital ring. Then there is a unique monoid homomorphism $f : \text{Typ } S \to V(K\langle S \rangle)$ such that $f(x/\mathcal{D}) = [x]_{K\langle S \rangle}$ for all $x \in S$. 

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- Question: does $\text{Typ } S \cong V(\mathbb{Z}\langle S \rangle)$?