# Semigroups, $P$-graphs, and groupoids

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# Introduction

Many constructions of C*-algebras from algebraic or dynamical systems can be done through a groupoid.

Let us consider the case of a semigroup $P$ (assumed here to be discrete countable and with a unit $e$), is there a semigroup C*-algebra $C^*(P)$? G. Murphy has given a natural answer: the universal C*-algebra for isometric representations. However, this C*-algebra is hard to study.

On the other hand, there exist several C*-algebras associated with $P$ which are much more tractable, for example, the left or right reduced C*-algebras acting on $\ell^2(P)$ or the Wiener-Hopf C*-algebra $\mathcal{W}(P, Q)$ (assuming, as we shall always do, that $P$ is a subsemigroup of a group $Q$). The reason that these C*-algebras are easier to study is that they can be described as groupoid C*-algebras.

# Introduction (cont'd)

I shall concentrate here on the C*-algebra $C^*_{\mathrm{red}}(P)$ of the left regular representation $L$ on $\ell^2(P)$ of a subsemigroup $P$ of a discrete group $Q$ containing the unit element $e$: for $s$, $L(s)$ is the isometry defined by $L(s)\epsilon_x = \epsilon_{sx}$ for all $x \in P$.

We shall also see that the same procedure applies to topological $P$-graphs and gives a groupoid description of the associated Toeplitz and Cuntz algebras.

In each of these situations, we shall first construct an inverse semigroup.

# Inverse semigroups

There is no need here to recall the definition of an inverse semigroup. I do it anyway to set my notation.

## Definition

A semigroup $S$ is called an inverse semigroup if for all $s \in S$, there exist a unique $s^* \in S$ such that

$$ss^*s = s \qquad s^*ss^* = s^*$$

The set of idempotents $E = E(S) = \{e \in S : e^2 = e\}$ is itself a commutative self-adjoint inverse semigroup (equivalently, an inf semi-lattice) and plays an important role.

# Two basic examples

1. Let $X$ be a topological space. Then the set $\mathcal{I}(X)$ of homeomorphisms $\varphi : U \to V$ where $U$ and $V$ are open subsets of $X$ is an inverse semigroup.

2. A family of $S$ of partial isometries closed under involution and composition (then the range and domain projections commute) is an inverse semigroup.

In these examples, the set of idempotents is contained in a Boolean algebra, hence has a natural spectrum.

# Action of an inverse semigroup on a space

**Definition**

An action of an inverse semigroup $S$ on a topological space $X$ is an inverse semigroup morphism $\alpha : S \to \mathcal{I}(X)$.

An abstract inverse semigroup $S$ admits canonical actions, for example, it acts on the spectrum $\hat{E}$ of the commutative C*-algebra $C^*(E)$ and also on the tight spectrum $\mathrm{closure}(\hat{E}_{\max})$ as defined by Exel.

# The groupoid of germs of an inverse semigroup action

Given an action $(X, S, \alpha)$, we define following [Khoshkam-Skandalis 02] and [Exel 08]:

**Definition**

The groupoid of germs $\mathrm{Germ}(X, S, \alpha)$ of the action is the quotient of

$$S * X = \{(s, x) : s \in S, x \in \mathrm{dom}(\alpha_s)\}$$

by the relation $[s, x] = [t, y]$ iff $x = y$ and there exists $e \in E$ such that $x \in \mathrm{dom}(\alpha_e)$ and $se = te$.
It is an étale groupoid.

The choice of $X = \hat{E}$ gives the universal groupoid $G(S)$ of $S$, which gives a groupoid realization of the full and of the reduced C*-algebras of the inverse semigroup $S$.

# Semigroup actions

Let us define now a semigroup action:

## Definition

A left action of a semigroup $P$ on a topological space $X$ is a map

$$T : (n, x) \in P * X \quad \mapsto \quad nx \in X$$

where $P * X$ is an open subset of $P \times X$, such that

1. for all $x \in X$, $(e, x) \in P * X$ and $ex = x$;
2. if $(m, x) \in P * X$, then $(n, mx) \in P * X$ iff $(nm, x) \in P * X$; if this holds, we have $n(mx) = (nm)x$;
3. for all $n \in P$, the map defined by $T_n x = nx$ is a local homeomorphism with domain $U(n) = \{x \in X : (n, x) \in P * X\}$ and range $V(n) = \{nx : (n, x) \in P * X\}$.

# Examples

1. One-sided subshifts of finite type, $z \mapsto z^2$ on the circle: the maps $T_n : U(n) \to V(n)$ are not necessarily injective but they are local homeomorphisms.
2. Rational maps on the sphere: exclude the singular points to get local homeomorphisms.
3. Putnam's algebra $A_U$: start with a self-homeomorphism $T$ of $X$ and consider its restriction $T_{|U}$ where $U$ is an open subset of $X$.
4. infinite graphs (e.g. Exel-Laca's Cuntz-Krieger algebras).
5. We shall see that $P$-graphs lead to such semigroup actions.

# The groupoid of germs of a semigroup action

Assuming that $P \subset Q$, where $Q$ is a group, we now define the groupoid of germs of the action:

We let $\mathcal{S}$ be the sub-inverse semigroup of $Q \times \mathcal{I}(X)$ generated by the elements $(n, T_{n|U})$ where $n \in P$ and $U$ is an open set on which $T_n$ is injective. Then, the restriction $\alpha$ of the second projection of the product $Q \times \mathcal{I}(X)$ is an action of the inverse semigroup $\mathcal{S}$ on $X$. We define the groupoid of germs of $(X, P, T)$ as the groupoid of germs $\mathrm{Germ}(X, \mathcal{S}, \alpha)$ and we denote it by $\mathrm{Germ}(X, P, T)$.

# directed action

Through the groupoid $\mathrm{Germ}(X, P, T)$, we can define the C*-algebra (full or reduced) of the semigroup action $(X, P, T)$. However, very little can be said about it. The following condition will make the groupoid of germs (and its C*-algebra) more tractable.

## Definition

We say that a semigroup action $(X, P, T)$ is directed if for all pairs $(m, n) \in P \times P$ such that $U(m) \cap U(n) \neq \emptyset$, there exist $(a, b) \in P \times P$ such that $am = bn$ and $U(am) \supset U(m) \cap U(n)$.

# The semi-direct product groupoid

Given a semigroup action $(X, P, T)$ where $P \subset Q$, we define $G(X, P, T)$ as the set

$$\{(x, m^{-1}n, y) : m, n \in P, x \in U(m), y \in U(n), T_m x = T_n y\}.$$

When $P = \mathbb{N}$ one retrieves the well-known groupoid of an endomorphism studied by V. Deaconu:

$$G(X, T) = \{(x, m - n, y) : m, n \in \mathbb{N}, x, y \in X, T^m x = T^n y\}.$$

In the general case, $G(X, P, T)$ is not necessarily a groupoid (more precisely a subgroupoid of $X \times Q \times X$).

# Identification of the groupoid of germs

However, when the action is directed, it is a groupoid and it is in fact the groupoid of germs defined earlier

**Theorem (R 2015)**

*Assume that the action $(X, P, T)$ is directed. Then*

1. *$G(X, P, T)$ is a locally compact Hausdorff étale groupoid;*
2. *$G(X, P, T)$ is the groupoid of germs of the action.*

We shall see shortly that this explains a posteriori why both Ore semigroups (e.g. [R-Exel 07] or [R-Sundar 15]) and quasi-lattice ordered semigroups [Nica 92]), rather than more general semigroups have been considered.

# The left inverse hull of a semigroup

Let us consider the left action $L$ of $P$ on itself. If $P$ is left cancellative, the left inverse hull $\mathcal{S}_l(P)$ of $P$ is defined as the inverse semigroup generated by the left translations $L(s) : P \to sP$. Equivalently, it is the inverse semigroup of partial isometries generated by the isometries $L(s) : \ell^2(P) \to \ell^2(sP)$.

The inverse semigroup $\mathcal{S}_l(P)$ acts on $D = C^*_{\mathrm{red}}(P) \cap \ell^\infty(P)$ by $d \mapsto L(s)dL(s)^*$. This induces an action $\alpha$ of $\mathcal{S}_l(P)$ on $X(P) = \hat{D}$.

# The reduced C*-algebra of a semigroup as a groupoid algebra

**Theorem (X. Li 12)**

*Let $P$ be a left cancellative semigroup. Then $C^*_{\mathrm{red}}(P)$ is the reduced C*-algebra of the groupoid of germs $\mathrm{Germ}(X, \mathcal{S}_l(P), \alpha)$ of the action of the left inverse hull on $X(P)$.*

Moreover, if $P$ is a subsemigroup of a group $Q$, there is a cocycle

$$c : \mathrm{Germ}(X, \mathcal{S}_l(P), \alpha) \to Q$$

such that $c([L(s), x]) = s$ for all $s \in P$ and $x \in X(P)$.

# The case of Ore and quasi-lattice ordered semigroups

Assume now that $P$ be a left cancellative semigroup. We have two ways to view the left translation of $P$ on itself .

1. As a left action $L(s)t = st$. This action is directed iff $P$ is right reversible, i.e. for all pairs $(m, n) \in P \times P$, $Pm \cap Pn \neq \emptyset$. This can be written $PP^{-1} \subset P^{-1}P$.
2. As a right action of $P$, by defining $L(s)^{-1}(sx) = x$. This action is directed iff $P$ satisfies the Clifford condition: if $mP \cap nP \neq \emptyset$, it is of the form $rP$ for some $r \in P$.

These conditions define respectively the right Ore and the quasi-lattice ordered semigroups.

# The case of Ore and quasi-lattice ordered semigroups (cont'd)

When we pass to the action of $P$ on the spectrum $X(P)$, the directedness of the action is preserved. Therefore, in these cases, the groupoid of germs can be described as a semi-direct groupoid.

*Examples*

$\mathbb{N}^d \subset \mathbb{Z}^d$ are both Ore and quasi-lattice ordered, which is equivalent to lattice ordered.

If $Q$ is abelian and $P - P = Q$, $P \subset Q$ is Ore.

$S\mathbf{F}_d \subset \mathbf{F}_d$ is quasi-lattice ordered (and not Ore). Graph products of quasi-lattice ordered semigroups are quasi-lattice ordered [Crisp-Laca 02].

# topological category

Let us turn to topological $P$-graphs. We first need the definition of a topological category (or semi-groupoid). It is just like a topological groupoid but without the existence of an inverse.

## Definition

A topological category is a small category $\Lambda$ with set of objects $\Lambda^{(0)}$, range and source maps $r, s : \Lambda \to \Lambda^{(0)}$, composition map

$$m : \Lambda^{(2)} = \Lambda * \Lambda \to \Lambda,$$

endowed with a topology compatible with its structure.

More precisely, we require that all above maps are continuous and that $s$ is a local homeomorphism.

# topological $P$-graphs

## Definition

A topological $P$-graph is a topological category $\Lambda$ endowed with a map, called the degree map, $d : \Lambda \to P$ which satisfies the following properties

1. for all $m \in P$, $\Lambda^m = d^{-1}(m)$ is open;
2. for all $(\mu, \nu) \in \Lambda^{(2)}$, $d(\mu\nu) = d(\mu)d(\nu)$ and for all $v \in \Lambda^{(0)}$, $d(v) = e$;
3. it has the unique factorization property: for all $m, n \in P$, the composition map $\Lambda^m * \Lambda^n \to \Lambda^{mn}$ is a homeomorphism.

# Examples

1. Graphs are $\mathbb{N}$-graphs. More accurately, given a graph $(E, V)$, we let

$$\Lambda = \{x_n \dots x_2 x_1 \quad \text{finite path of the graph}\}$$

Here, $P = \mathbb{N}$ and $d(x_n \dots x_2 x_1) = n$. For example, when $V$ has a single element, $\Lambda$ is the set of all finite words with letters in $E$.

2. Topological graphs are topological $\mathbb{N}$-graphs. Here $E, V$ are topological spaces, $r, s : E \to V$ are continuous and $s$ is a local homeomorphism.

# from $P$-actions to topological $P$-graphs

**Proposition**

*Let $T : X * P \to X$ be a semigroup action as above. Then $\Lambda = X * P$ has a natural structure of a topological P-graph.*

It is given by $\Lambda^{(0)} = X$, the range and source maps $r, s : \Lambda \to X$ are respectively $r(x, n) = x$ and $s(x, n) = xn$. The composition is given by

$$(x, m)(xm, n) = (x, mn)$$

The degree map $d : \Lambda \to P$ is simply $d(x, n) = n$.

# from topological $P$-graphs to $P$-actions

One can view a topological $P$-graph as a generalized $P$-action. We can follow the same pattern as in the construction of the inverse hull of a semigroup: given a $P$-graph $\Lambda$, we let $P$ act on $\Lambda$ by

$$\Lambda * P = \{(\mu\nu, m) \in \Lambda \times P : d(\mu) = m\}$$

and $T : \Lambda * P \to \Lambda$ by $T(\mu\nu, m) = \nu$ if $d(\mu) = m$.

As in the case of the action of $P$ on itself by translation, the action of $P$ can be extended to the "path space" $X$, which is a suitable completion of the "finite path space $\Lambda$". Its groupoid of germs defines the Toeplitz algebra of the $P$-graph. Its reduction to the "boundary path space" $\partial X$ defines the Cuntz algebra of the $P$-graph. Here are the details:

# Assumptions

*This action of $P$ on $\Lambda$ is not so interesting!* it is proper, i.e. the semi-direct product $\Lambda \rtimes P$ is a proper groupoid. However, under additional assumptions, the space $\Lambda$ admits a natural compactification, its order compactification $\overline{\Lambda}$ which we are going to define and things become much more interesting! The ideas goes back to [Nica 92] who considered the case $\Lambda = P$.

Assumptions: We shall assume that

1. $P$ is a quasi-lattice ordered semigroup of a group $Q$.
2. the map $(r, d) : \Lambda \to \Lambda^{(0)} \times P$ sending $\lambda$ to $(r(\lambda), d(\lambda))$ is proper.

# Quasi-lattice ordered semigroups

## Definition

One says that a semigroup $P$ of a group $Q$ is quasi-lattice ordered if $P \cap P^{-1} = \{e\}$ and as soon as $a, b \in P$ have a common upper bound, they have a least common upper bound.

## Proposition

1. *$T$ is an action of $P$ on $\Lambda$ by partial local homeomorphisms.*
2. *If $P$ is quasi-lattice ordered, this action is directed.*

# the path space $\Omega$

We define on $\Lambda$ a pre-order in the same fashion as on $P$:

$$\mu \leq \nu \Leftrightarrow \exists \alpha \in \Lambda : \nu = \mu\alpha.$$

Let $\mathcal{C}(\Lambda)$ be the space of closed subsets of $\Lambda$ endowed with the Fell's topology. Define the map $F : \Lambda \to \mathcal{C}(\Lambda)$ by

$$F(\lambda) = \{\mu \in \Lambda : \mu \leq \lambda\}.$$

Then the closure $\overline{\Lambda}$ of $F(\Lambda)$ is the set of hereditary and directed closed subsets of $\Lambda$.

## Definition

The path space is $\Omega = \overline{\Lambda} \setminus \{\emptyset\}$.

# the action of $P$ on $\Omega$

The action of $P$ on $\Lambda$ extends to $\Omega$: we define

$$\Omega * P = \{(A, n) \in \Omega \times P : \exists \lambda \in A : n \leq d(\lambda)\}$$

$$T : \Omega * P \to \Omega \quad : (A, n) \mapsto A.n$$

## Proposition

*Let $\Lambda$ be a $P$-graph. Assume that $P$ is a quasi-lattice ordered subsemigroup of a group $Q$ and that $\Lambda$ is $(r, d)$-proper. Then,*

1. *$T : \Omega * P \to \Omega$ is an action of $P$ on $\Omega$ by partial local homeomorphisms.*
2. *The action is directed.*

# the Toeplitz algebra of the $P$-graph

Therefore, we can construct the semi-direct product $\Omega \rtimes P$ which is an étale, locally compact and Hausdorff groupoid and we can construct its C*-algebra.

**Proposition**

*The C*-algebra $C^*(\Omega \rtimes P)$ is the Toeplitz C*-algebra $TC^*(\Lambda)$ of the $P$-graph $\Lambda$.*

The groupoid techniques of R-Williams give that if $P$ is a quasi-lattice ordered subsemigroup of an amenable group $Q$ and if $\Lambda$ is $(r, d)$-proper, then the Toeplitz C*-algebra $TC^*(\Lambda)$ is nuclear.

# the boundary action

The definition of the boundary path space $\partial\Omega$, adapted from [Yeend, 2007], is a bit technical: its elements are the subsets $A \in \Omega$ whose elements are *extendable* in $A$, a notion which I do not want to define here. It is very likely that $\partial\Omega$ is the closure of the maximal hereditary directed closed subsets.

**Proposition**

*Under the above assumptions, $\partial\Omega$ is a closed subset of $\Omega$ invariant under the action of $P$ and with respect to the groupoid $G(\Omega, P, T)$.*

The Cuntz $P$-graph C*-algebra is the reduction of $C^*(\Omega \rtimes P)$ to $\partial\Omega$.

# the Cuntz algebra of the $P$-graph

The reduction of the groupoid $G(\Omega, P, T)$ to the boundary path space $\partial\Omega$ is the semidirect product $\partial\Omega \rtimes P$. Again, we can construct its C*-algebra.

**Proposition**

*The C*-algebra $C^*(\partial\Omega \rtimes P)$ is the Cuntz C*-algebra $C^*(\Lambda)$ of the $P$-graph $\Lambda$.*

The same groupoid techniques as above show that if $P$ is a quasi-lattice ordered subsemigroup of an amenable group $Q$ and if $\Lambda$ is $(r, d)$-proper, then the Cuntz C*-algebra $C^*(\Lambda)$ is nuclear.

# application to semigroup actions

We can use this construction to define the C*-algebra of a semigroup action $(X, P, T)$ which is not necessarily directed but where $P$ quasi-lattice ordered.

We introduce the $P$-graph $\Lambda = X * P$ and consider instead the directed action $(\partial\Omega, P, \tilde{T})$. Then we can define $C^*(X, P, T)$ as $C^*(\Lambda)$. When $X$ is reduced to a point, we get Nica's $C^*(P)$.

**Proposition**

*If the action $(X, P, T)$ is already directed, then there exists a $P$-equivariant homeomorphism of $X$ onto $\partial\Omega$ which implements a groupoid isomorphism from $G(X, P, T)$ onto $G(\partial\Omega, P, \tilde{T})$. Therefore $C^*(X, P, T)$ and $C^*(\Lambda)$ are isomorphic.*

# References

R-Williams, arXiv:1501.03027
A. Nica, A. Paterson, R. Exel, X. Li, M. Norling, ...
Khoshkam-Skandalis 02