Skew Boolean algebras

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Plan of the talk

1. Skew Boolean algebras
   1.1 Skew Boolean algebras and skew Boolean intersection algebras.
   1.2 Non-commutative Stone dualities.
   1.3 Skew Boolean algebras vs Boolean inverse semigroups.
   1.4 Distributive skew lattices vs distributive inverse semigroups.
   1.5 A dualizing object approach to non-commutative Stone duality.
   1.6 Skew Boolean algebras and Boolean inverse semigroups put together.
2. Free skew Boolean algebras.
3. Free skew Boolean intersection algebras.

SKEW BOOLEAN ALGEBRAS

Skew Boolean algebras (SBAs)

A skew Boolean algebra is an algebra $(S; \wedge, \vee, \setminus, 0)$ of type $(2, 2, 2, 0)$ such that for any $a, b, c, d \in S$:

1. (associativity) $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
2. (absorption) $a \vee (a \wedge b) = a$, $(b \wedge a) \vee a = a$, $a \wedge (a \vee b) = a$, $(b \vee a) \wedge a = a$;
3. (distributivity) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $(b \vee c) \wedge a = (b \wedge a) \vee (c \wedge a)$;
4. (properties of 0) $a \vee 0 = 0 \vee a = a$, $a \wedge 0 = 0 \wedge a = 0$;
5. (properties of relative complement) $(a \setminus b) \wedge b = b \wedge (a \setminus b) = 0$, $(a \setminus b) \vee (a \wedge b \wedge a) = a = (a \wedge b \wedge a) \vee (a \setminus b)$;
6. (normality) $a \wedge b \wedge c \wedge d = a \wedge c \wedge b \wedge d$.

Associativity and absorption imply that $(A, \wedge, \vee)$ is a skew lattice and the absorption axiom implies

7. (idempotency) $a \vee a = a$, $a \wedge a = a$.

Skew Boolean intersection algebras (SBIAs)

Let $(S; \wedge, \vee, \setminus, 0)$ be an SBA.

- The natural partial order on S: $a \leq b$ iff $a \wedge b = b \wedge a = a$.
- S has intersections if the meet of a and b with respect to $\leq$ exists for any $a, b \in S$.
- $a \sqcap b$ — the intersection of a and b.
- If $(S; \wedge, \vee, \setminus, 0)$ has intersections, $(S; \wedge, \vee, \sqcap, \setminus, 0)$ is a skew Boolean intersection algebra (SBIA).

Proposition (Bignall and Leech)

Let $(S; \wedge, \vee, \setminus, \sqcap, 0)$ be an algebra of type $(2, 2, 2, 2, 0)$. Then it is an SBIA if and only if $(S; \wedge, \vee, \setminus, 0)$ is an SBA, $(S, \sqcap)$ is a semilattice and the following identities hold:

$$x \sqcap (x \wedge y \wedge x) = x \wedge y \wedge x; \quad x \wedge (x \sqcap y) = x \sqcap y = (x \sqcap y) \wedge x.$$

Historical note

- 1949 - Pascual Jordan: work on non-commutative lattices.
- early 1970's - a series of works by Boris Schein
- late 1970's and early 1980's - Robert Bignall and William Cornish studied non-commutative Boolean algebras
- 1989 - Jonathan Leech initiated the modern study of skew lattices.
- 1995 - Jonathan Leech and Robert Bignall publish the first paper devoted to skew Boolean intersection algebras.
- 1989 - present many aspects of skew lattices and skew Boolean algebras have been studied.
- 2012-2013 - Stone duality was extended to skew Boolean algebras and distributive skew lattices.
- A. Bauer, K. Cvetko-Vah, M. Gehrke, S. van Gool, M. Kinyon, GK, M. V. Lawson, J. Leech, J. Pita-Costa, M.Spinks is an (incomplete) list of researchers who have contributed to the topic.

Left-handed SBAs

An SBA S is left-handed if the normality axiom is replaced by:

(6′) (left normality) $x \wedge y \wedge z = x \wedge z \wedge y$,

and a dual axiom holds for right-handed SBAs.
In this talk, we restrict attention to left-handed SBAs. The general case can be easily obtained applying

$$\mathbf{S} \simeq \mathbf{S}/\mathcal{R} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{L}.$$

for the canonical diagram

$$\begin{array}{ccc} \mathbf{S} & \twoheadrightarrow & \mathbf{S}/\mathcal{R} \\ \downarrow & & \downarrow \\ \mathbf{S}/\mathcal{L} & \twoheadrightarrow & \mathbf{S}/\mathcal{D} \end{array},$$

This is the *Kimura Factorization Theorem*, extended from regular bands to skew lattices.

Structure of finite LSBAs

- Green's relation $\mathcal{D}$: $x \, \mathcal{D} \, y$ if and only if $x \wedge y = x$ and $y \wedge x = y$.
- $\mathcal{D}$ is an SBA congruence and $S/\mathcal{D}$ is the maximum commutative quotient of S. (But, if S is an SBIA, $\mathcal{D}$ is in general not respected by $\sqcap$ operation.)
- Primitive LSBAs: $(\mathbf{k}+\mathbf{1})_L = \{0, \dots, k\}$ ($\wedge$ is left zero multiplication). This generalizes the Boolean algebra $\mathbf{2} = \{0, 1\}$.
- Structure of finite LSBAs. Let S be a finite LSBA. Then

$$S \simeq \mathbf{2}^{k_2} \times \mathbf{3}_L^{k_3} \times \cdots \times (\mathbf{m}+\mathbf{1})_L^{k_{m+1}}$$

  for some m where all $k_i \geq 0$. This generalizes the fact that any finite Boolean algebra is isomorphic to some $\mathbf{2}^k$.

Non-commutative Stone dualities: Boolean case

Theorem 1.
The category of LSBAs is dually equivalent to the category of étale spaces over (locally compact) Boolean spaces where morphisms of étale spaces are cohomomorphisms over continuous proper maps.

Theorem 2.
The category of LSBIAs is dually to the category of Hausdorff étale spaces over Boolean spaces where morphisms are injective étale spaces are cohomomorphisms over continuous proper maps.

Non-commutative Stone dualities: distributive case

A skew lattice S is distributive if it is normal, symmetric and $S/\mathcal{D}$ is a distributive lattice.

Priestley Duality for distributive skew lattices

The category of left-handed distributive[1] skew lattices is dually equivalent to the category of étale spaces over Priestley spaces.

1. If S is distributive, take the dual spectral space $(S/\mathcal{D})^+$ of $S/\mathcal{D}$ with patch topology, and consider sections over increasing compact open sets.
2. The Booleanization of S is the set of all sections of $(S/\mathcal{D})^+$ over all compact open sets.

[1] strongly distributive in the literature

Boolean and distributive left normal bands

- S – a left normal band.
- S is distributive if $S/\mathcal{D}$ is a distributive lattice and joins of compatible families of elements exist in S.
- S is Boolean if $S/\mathcal{D}$ is a Boolean algebra and joins of compatible families of elements exist in S.
- The category of Boolean left normal bands is equivalent to the category of left-handed skew Boolean algebras.
- BUT: The category of distributive left normal bands is not equivalent to the category of distributive left-handed skew Boolean algebras.
- This is because Boolean left normal bands are in a duality with sheaves over spectral spaces, and the set of compact-open sections of such a sheaf $a \vee b$ (non-commutative join!) may not exist for a, b not compatible.

Skew Boolean algebras vs Boolean inverse semigroups

Band setting	Inverse semigroup setting
Skew Boolean algebras	Biases (F. Wehrung)
Boolean left normal bands	Boolean inverse semigroups
Étale spaces over Boolean spaces	Étale groupoids with Boolean spaces of identities

Band setting	Inverse semigroup setting
Skew Boolean intersection algebras	Biases with meets
Boolean left normal bands with meets	Boolean inverse meet semigroups
Hausdorff Étale spaces over Boolean spaces	Hausdorff Étale groupoids with Boolean spaces of identities

Distributive skew Boolean algebras vs distributive Boolean inverse semigroups

Band setting	Inverse semigroup setting
Distributive left normal bands	Distributive inverse semigroups
Étale spaces over spectral spaces	Étale groupoids with spectral spaces of identities

Band setting	Inverse semigroup setting
Distributive skew lattices	?
Étale spaces over Priestley spaces	?

Skew frames?

- A pseudogroup is an inverse semigroup S such that $E(S)$ is a frame and joins of compatible families of elements exist. The category of pseudogroups is dually equivalent to the category of localic étale groupoids.
- An infinitely distributive left normal band is a normal band S such that $S/\mathcal{D}$ is a frame and joins of compatible families of elements exist. These are dually equivalent to localic étale spaces.
- Question: What would be an appropriate notion of a "skew frame"? Perhaps, a normal symmetric skew lattice such that $S/\mathcal{D}$ is a frame. What are the dual objects? Do infinite skew frames exist?

Dualizing object approach to non-commutative Stone duality

The classical Stone duality is induced by a dualizing object, $\{0,1\}$. Does this admit an extension to a non-commutative setting?

- Let S be an LSBA and let S_n^* be the set of all non-zero SBA homomorphisms $S \to \mathbf{n}+\mathbf{1}$ endowed with the subspace topology inherited from the product topology $\{0,1,\dots,n\}^S$. Then S_n^* is a Boolean space. This gives rise to a functor Λ_n from the category of SBAs to the category of Boolean spaces.
- Let X be a Boolean space and consider the set of all proper continuous maps $X \to \{0,1,\dots,n\}$. This inherits an SBA structure from $\mathbf{n}+\mathbf{1} = \{0,1,\dots,n\}$ and gives rise to a functor λ_n from the category of Boolean spaces to the category of SBAs.

Dualizing object approach to non-commutative Stone duality

Theorem
The functor Λ_n is a left adjoint to the functor λ_n, for each $n \geq 0$. The category $\lambda_n(\mathrm{BS})$ is a reflective subcategory of the category of skew Boolean algebras. The reflector (= left adjoint to the inclusion functor) is given by the functor $\lambda_n \Lambda_n$.

GK, A dualizing object approach to non-commutative Stone duality, J. Aust. Math. Soc., 2013.

Skew Boolean algebras and Boolean inverse semigroups put together

- A left generalized inverse semigroup is a regular semigroup whose idempotents form a left normal band.
- Left generalized inverse semigrops are given by the following data: an inverse semigroup S and a presheaf of sets P over $E(S)$. Then

$$(S,P)=\{(s,e)\colon e\in P(s^{-1}s), s\in S\}, \text{ where}$$

$$(s,e)(t,f)=(st, e|_{(st)^{-1}st}),$$

  is a left generalized inverse semigroup and any left generalized inverse semigroup is of this form (essentially due to Yamada).
- One can define a Boolean left generalized inverse semigroup as such an (S,P) for which S is a Boolean inverse semigroup and P is a sheaf. A topological duality follows.

Generalized inverse $*$-semigroups

A semigroup S is called a right $*$-semigroup, if it is equipped with a unary operation $s \mapsto s^*$, satisfying the following axioms:

(S1) $s^{**} = s$;

(S2) $s^* \in V(s)$;

(S3) $(st)^* = t^*(stt^*)^*$;

(S4) If $e^2 = e$ then $e = e^*$.

$*$-semigroups were introduced by J. Funk, in 2007 in order to characterize the classifying topos $\beta(S)$ in semigroup terms.

Theorem (GK and M. V. Lawson, 2012)

Let S be an inverse semigroup. The category of right generalized inverse $*$-semigroups T with $T/\gamma \simeq S$ is equivalent to the classifying topos $\beta(S)$.

This shows that right $*$-semigroups can be in fact chosen to be right generalized inverse $*$-semigroups.

FREE SKEW BOOLEAN ALGEBRAS
(Joint work with Jonathan Leech)

Free generalized Boolean algebras: well known

The free generalized Boolean algebra $\mathbf{GBA}_X$ over the generating set X is the the algebra of all terms over X, where two terms are equal in $\mathbf{GBA}_X$ if one of them can be obtained from another one by a finite number of applications of the identities defining the variety of GBAs.

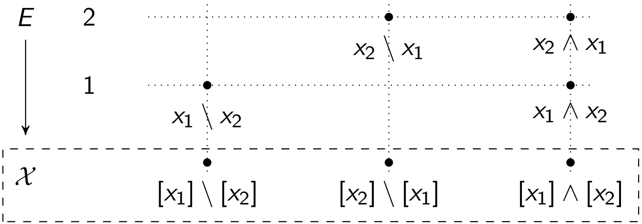
Example. Let $X = \{x_1, x_2\}$. Then $\mathbf{GBA}_X \simeq \mathbf{2}^3$ where $\mathbf{2} = \{0, 1\}$. Indeed, denote $a_{\{1\}} = x_1 \setminus x_2$, $a_{\{2\}} = x_2 \setminus x_1$ and $a_{\{1,2\}} = x_1 \wedge x_2$.

- These terms are pairwise distinct, as they can have distinct evaluations in $\mathbf{2}$.
- $a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3 = 0$.
- $x_1 = (x_1 \setminus x_2) \vee (x_1 \wedge x_2)$ and $x_2 = (x_2 \setminus x_1) \vee (x_1 \wedge x_2)$.
- Thus any element of $\mathbf{GBA}_X$ is a join of a subset of $\{a_1, a_2, a_3\}$.

Similarly, if $|X| = n$, $\mathbf{GBA}_X \simeq \mathbf{2}^{2^n - 1}$, atoms are in a bijections with non-empty subsets of X.

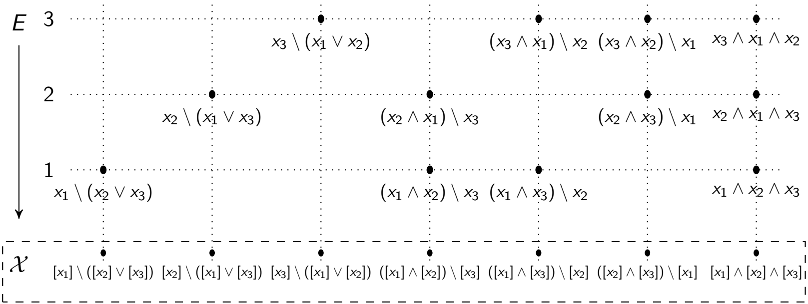
Finite free LSBAs: an example

Atoms of ${}_{\mathcal{L}}\mathbf{SBA}_2$:



Finite free LSBAs: another example

Atoms of ${}_{\mathcal{L}}\mathbf{SBA}_3$:



Finite free LSBAs: finite case

${}_{\mathcal{L}}\mathbf{SBA}_X$ — the free LSBA over the generating set X. Let $X = \{1, 2, \ldots, n\}$.

- Let $Y \subseteq X$ be a non-empty subset and $y \in Y$.
- (X, Y, y) – pointed non-empty subset of X.
- Atoms:

$$e(X, Y, y) = (y \wedge (\wedge\{y \colon y \in Y))) \setminus \vee\{y \colon y \in X \setminus Y\}.$$

  By left normality, this is well defined and $e(X, Y, y) = e(X, Z, z)$ if and only if $Y = Z$ and $y = z$.
- Atoms are in a bijection with pointed non-empty subsets of X.
- Atomic $\mathcal{D}$-classes = atoms of $S/\mathcal{D}$ are in a bijection with non-empty subsets of X.

Free LSBAs: finite case, structure

Theorem (J. Leech and GK, 2015)

Let $n \geq 1$.

1. $$ {}_{\mathcal{L}}\mathbf{SBA}_n \simeq \mathbf{2}^{\binom{n}{1}} \times \mathbf{3}_L^{\binom{n}{2}} \times \mathbf{4}_L^{\binom{n}{3}} \times \cdots \times (\mathbf{n}+\mathbf{1})_L^{\binom{n}{n}}. $$

   Consequently,

   $$ |{}_{\mathcal{L}}\mathbf{SBA}_n| = 2^{\binom{n}{1}} 3^{\binom{n}{2}} 4^{\binom{n}{3}} \ldots (n+1)^{\binom{n}{n}}. $$

2. *The number of atoms of ${}_{\mathcal{L}}\mathbf{SBA}_n$ equals*

   $$ \binom{n}{1}1 + \binom{n}{2}2 + \cdots + \binom{n}{n-1}(n-1) + \binom{n}{n}n = n2^{n-1}. $$

3. *The center of ${}_{\mathcal{L}}\mathbf{SBA}_n$ is isomorphic to $\mathbf{2}^n$.*

Free LSBAs: infinite case

Theorem (J. Leech and GK, 2015)

Let S be a LSBA, let $X \subseteq S$ be a generating set of S and let $\pi\colon S \to S/\mathcal{D}$ be the canonical homomorphism. TFAE:

(i) *S is freely generated by X.*

(ii) *For every subset $\{x_1, \ldots, x_n\}$ of n distinct elements in X, their evaluations in the $n2^{n-1}$ atomic terms on n variables produce $n2^{n-1}$ distinct non-zero outcomes in S.*

(iii) *$S/\mathcal{D}$ is freely generated by $\pi(X)$ and for any $x \neq y \in X$, $x \cap y$ exists and equals 0. Thus ${}_{\mathcal{L}}\mathbf{SBA}_X$ has intersections.*

Proposition

1. *${}_{\mathcal{L}}\mathbf{SBA}_X$ is atomless.*
2. *The center of ${}_{\mathcal{L}}\mathbf{SBA}_X$ equals $\{0\}$.*

Free LSBAs: infinite case: dual étale space

- Let X be infinite and put $\mathcal{X} = \{0,1\}^X \setminus \{f_0\}$ where $f_0 = 0$.
- $\Omega = \{(f,x) \colon f \in \mathcal{X}$ and $x \in X$ is such that $f(x) = 1\}$.
- $p \colon \Omega \to \mathcal{X} \colon p(f,x) = f$, $\mathbf{S}_X = \{A \subseteq \Omega \colon p|_A$ is injective.$\}$
- Define the binary operations $\vee$, $\wedge$ and $\setminus$ on $\mathbf{S}_X$ by:

$$\begin{array}{rcl} A \wedge B & = & \{(f,x) \in A \colon f \in p(A) \cap p(B)\}, \\ A \vee B & = & (A \setminus B) \cup B, \\ A \setminus B & = & \{(f,x) \in A \colon f \in p(A) \setminus p(B)\}. \end{array}$$

- $(\mathbf{S}_X; \wedge, \vee \setminus, \varnothing)$ is a LSBIA.
- $i \colon X \to \mathbf{S}_X$, $i(x) = \{(f,x) \colon f(x) = 1\}$, $\overline{X} = \{i(x) \colon x \in X\}$, $\mathbf{S}_X = \langle \overline{X} \rangle$.
- The evaluation of $e(Z, Y, y)$, where $Z \subseteq X$ is finite, is $\{(f,y) \in \Omega \colon f(x) = 1, x \in Y$ and $f(x) = 0, x \in Z \setminus Y\}$.
- $\mathbf{S}_X$ is freely generated by $\overline{X}$.

FREE SKEW BOOLEAN INTERSECTION ALGEBRAS

Free LSBIAs: finite case, idea

${}_{\mathcal{L}}\mathbf{SBIA}_X$ — the free LSBIA over the generating set X.

- Recall: in free LSBAs atoms are encoded by pointed non-empty subsets of X.
- In free LSBIAs: atoms are encoded by pointed partitions of non-empty subsets of X!

The construction: an example

$X = \{x_1, x_2, x_3\}$. We define:

- For $\alpha = x_1|x_2$ and the marked block $\{x_2\}$:

$$e(X, \alpha, \{x_2\}) = (x_2 \wedge x_1) \setminus (x_3 \vee (x_1 \sqcap x_2)).$$

- For $\alpha = x_1|x_2|x_3$ and the marked block $\{x_2\}$:

$$e(X, \alpha, \{x_2\}) = (x_2 \wedge x_1 \wedge x_3) \setminus ((x_1 \sqcap x_2) \vee (x_1 \sqcap x_3) \vee (x_2 \sqcap x_3)).$$

- For $\alpha = x_1 x_2|x_3$ and the marked block $\{x_1, x_2\}$:

$$e(X, \alpha, \{x_1, x_2\}) = ((x_1 \sqcap x_2) \wedge x_3) \setminus (x_1 \sqcap x_3 \sqcap x_4).$$

- For $\alpha = x_1 x_2 x_3$ (that is, one three-element block) and the marked block $\{x_1, x_2, x_3\}$:

$$e(X, \alpha, \{x_1, x_2, x_3\}) = x_1 \sqcap x_2 \sqcap x_3.$$

The construction (GK, 2016)

Let (X, α, A) be a pointed partition of a non-empty subset of X. Let $\alpha = \{A_1, \ldots, A_k\}$ and $Y = \mathrm{dom}(\alpha)$. Define

$$e(X, \alpha, A) = p \setminus (\vee Q), \quad \text{where}$$

$$p = (\sqcap A) \wedge (\wedge\{\sqcap A_i \colon 1 \leq i \leq k\}) = (\sqcap A) \wedge (\sqcap A_1) \wedge \cdots \wedge (\sqcap A_k) \text{ and}$$

$$Q = (X \setminus Y) \cup \{\sqcap(A_i \cup A_j) \colon 1 \leq i < j \leq k\}.$$

The containment order on partitions.

Let $Z \subseteq Y$. We define: $(Z, \alpha) \preceq (Y, \beta)$ if

$$\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\beta) \subseteq \mathrm{dom}(\alpha) \cup (Y \setminus Z)$$

and for any $x, y \in \mathrm{dom}(\alpha)$: $x \; \alpha \; y$ if and only if $x \; \beta \; y$.

Example

Let $Y = \{x_1, x_2, x_3, x_4\}$, $Z = \{x_1, x_2, x_3\}$. Then

$(Z, x_1|x_2) \preceq (Y, x_1|x_2), (Y, x_1|x_2|x_4), (Y, x_1x_4|x_2), (Y, x_1|x_2x_4)$.

BUT:

$(Z, x_1|x_2) \not\preceq (Y, x_1|x_2x_3|x_4), (Y, x_1x_2), (Y, x_1x_2|x_4)$, etc.

Free LSBIAs: finite case continued

If (X, α, A) is a pointed partition and $(X, \alpha) \preceq (Y, \beta)$ then there is the only block of β which contains A, denoted $A{\uparrow}_{\alpha}^{\beta}$.

Theorem (Decomposition Rule)

Let S be a LSBIA, X, Y be finite non-empty subsets of S and $X \subseteq Y$. Let (X, α, A) be a pointed partition. Then

$$e(X, \alpha, A) = \vee\{e\left(Y, \beta, A{\uparrow}_{\alpha}^{\beta}\right) : (X, \alpha) \preceq (Y, \beta)\}, \qquad (1)$$

the latter join being orthogonal.

Free LSBIAs: finite case, structure

The nth *Bell number*, B_n, equals the number of partitions of an n-element set. The *Stirling number of the second kind*, $\left\{ {n \atop k} \right\}$, equals the number of partitions of an n-element set into k non-empty subsets, $n \geq 1$, $1 \leq k \leq n$.

Theorem (GK, 2016)

1. ${}_{\mathcal{L}}\mathbf{SBIA}_n$ *has precisely* $B_{n+1} - 1$ *atomic* $\mathcal{D}$*-classes. Thus* ${}_{\mathcal{L}}\mathbf{SBIA}_n/\mathcal{D} \simeq \mathbf{2}^{B_{n+1}-1}$.
2. ${}_{\mathcal{L}}\mathbf{SBIA}_n \simeq \mathbf{2}^{\left\{ {n+1 \atop 2} \right\}} \times \mathbf{3}_L^{\left\{ {n+1 \atop 3} \right\}} \times \cdots \times (\mathbf{n+1})_L^{\left\{ {n+1 \atop n+1} \right\}}$, $|{}_{\mathcal{L}}\mathbf{SBIA}_n| = 2^{\left\{ {n+1 \atop 2} \right\}} 3^{\left\{ {n+1 \atop 3} \right\}} \cdots (n+1)^{\left\{ {n+1 \atop n+1} \right\}}$.
3. ${}_{\mathcal{L}}\mathbf{SBIA}_n$ *has precisely* $B_{n+2} - 2B_{n+1}$ *atoms.*
4. *The center of* ${}_{\mathcal{L}}\mathbf{SBIA}_n$ *is isomorphic to* $\mathbf{2}^{2^n-1}$ *which is isomorphic to* $\mathbf{GBA}_n$ *which is isomorphic to the maximum commutative quotient of* ${}_{\mathcal{L}}\mathbf{SBIA}_n$.

Free LSBIAs: infinite case

- $\mathcal{X} = \{(X, \alpha)\colon \alpha \in \mathcal{P}(Y)$ where $Y \subseteq X$ and $Y \neq \varnothing\}$.
- $\Omega = \{(X, \alpha, A)\colon (X, \alpha) \in \mathcal{X}$ and $A \in \alpha\}$.
- $p\colon \Omega \to \mathcal{X}$, $p(X, \alpha, A) = (X, \alpha)$.
- $\mathbf{S}_X = \{U \subseteq \Omega\colon p|_U$ is injective$\}$.
- On $\mathbf{S}_X$ we define the binary operations $\vee$, $\wedge$, $\setminus$ and $\sqcap$ by:

$$
\begin{aligned}
U \wedge V &= \{(X, \alpha, A) \in U\colon (X, \alpha) \in p(U) \cap p(V)\},\\
U \vee V &= (U \setminus V) \cup V,\\
U \setminus V &= \{(X, \alpha, A) \in U\colon (X, \alpha) \in p(U) \setminus p(V)\},\\
U \sqcap V &= U \cap V.
\end{aligned}
$$

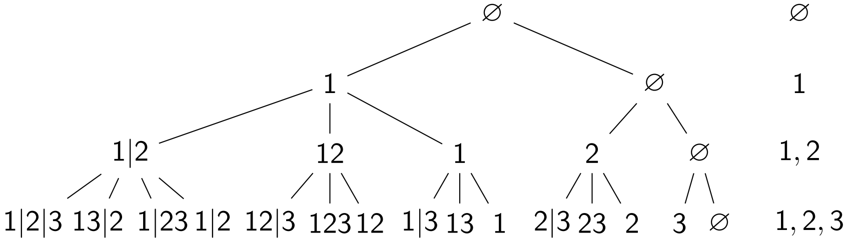
- $(\mathbf{S}_X; \wedge, \vee \setminus, \varnothing, \sqcap)$ is a LSBIA.
- $i\colon X \to \mathbf{S}_X$, $i(x) = \{(X, \alpha, A)\colon x \in \mathrm{dom}(\alpha)$ and $x \in A\}$.
- $\overline{X} = \{i(x)\colon x \in X\}$, $\mathbf{S}_X = \langle \overline{X} \rangle$.
- The evaluation of $e(X_n, \alpha, A)$ on $\{i(x_1), i(x_2), \ldots, i(x_n)\}$:

$$\{(X, \beta, A\!\uparrow_{\alpha}^{\beta}) \in \Omega\colon (X_n, \alpha) \preceq (X, \beta)\}.$$

- $\mathbf{S}_X$ is freely generated by $\overline{X}$.

Free LSBAs: countable generating set

Infinite partition tree: level i: partitions of subsets of $[i] = \{1, 2, \ldots, i\}$. $([i], \alpha)$ is connected with $([i+1], \beta)$ iff $([i], \alpha) \preceq ([i+1], \beta)$.



This tree is Cantorian (G. Michon, Les Cantors réguliers, 1985, J. Pearson, J. Bellissard, 2009), so that its boundary is homeomorphic to the Cantor set. Basis of topology: sets $[v]$ = all paths passing through v, where v ranges over the vertices. Thus ${}_{\mathcal{L}}\mathbf{SBIA}_{\mathbb{N}}/\mathcal{D} \simeq \mathbf{GBA}_{\mathbb{N}}$.

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