Isotropy torsors of an inverse semigroup

Joint work with

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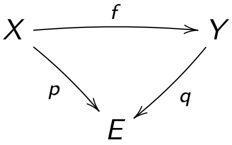
Inverse semigroups

Étale (right) S-sets

S denotes an inverse semigroup; s^* denotes the inverse of s
$E = E(S)$ denotes the subset of idempotents of S
Let $\mathscr{B}(S)$ denote the category whose objects and morphisms are:



X is an S-set: associative right action by S.
$xp(x) = x$
$p(xs) = s^* p(x) s$
f is S-equivariant map: $f(xs) = f(x)s$.

Prehomomorphisms to geometric morphisms

A prehomomorphism $S \xrightarrow{\rho} T$ passes to a geometric morphism (essential)

$$\rho_! \dashv \rho^* \dashv \rho_* : \mathscr{B}(S) \longrightarrow \mathscr{B}(T) \,.$$

The inverse image functor ρ^* acts by pullback along $E(S) \xrightarrow{\rho} E(T)$.

Some étale S-sets $X \xrightarrow{p} E$

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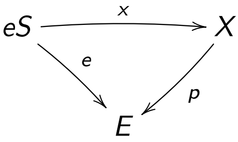
Elements $p(x) = e$ correspond to morphisms



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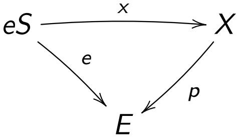

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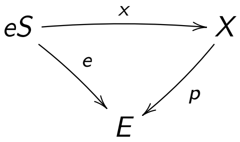

The domain object $S \xrightarrow{\partial} E$: $\partial(s) = s^*s$

The terminal object $E \to E$: $e \mapsto e$

The isotropy group $Z(E) \xrightarrow{\zeta} E\colon \zeta(s) = s^*s = ss^*$

ζ acts in every $X \xrightarrow{p} E\colon \theta_p(x, s) = xs$.
Every morphism of $\mathscr{B}(S)$ is ζ-equivariant.

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The max idem-sep quotient $S/\mu \xrightarrow{\upsilon} E\colon\ \upsilon(\bar{s}) = s^*s$

υ is isotropically trivial in the sense that its action by ζ is trivial.

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$\mathscr{B}(S)/\zeta =$ topos of (unital) crossed sheaves on S
$X \stackrel{p}{\to} Z(E)$ satisfying $p(xs) = s^* p(x) s$ and $xp(x)^* p(x) = x$.
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$$\mathscr{B}(\mathsf{S})/v \simeq$$

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PROPOSITION

The geometric morphism $\mathscr{B}(Z(E)) \longrightarrow \mathscr{B}(S)$ associated with $Z(E) \hookrightarrow S$ is equivalent to $\mathscr{B}(S)/\upsilon \longrightarrow \mathscr{B}(S)$.
Thus, any inverse semigroup has an isotropically trivial étale cover by a Clifford inverse semigroup.

Isotropy theory of toposes

The isotropy group Z of a topos $\mathscr{E}$

classifies the internal symmetries of $\mathscr{E}$ in the sense that morphisms $X \rightarrow Z$ of $\mathscr{E}$ correspond naturally to automorphisms of $\mathscr{E}/X \longrightarrow \mathscr{E}$.

An automorphism t of the left adjoint $\Sigma_X(Y \xrightarrow{\lambda} X) = Y$ is a compatible family $Y \xrightarrow{t_\lambda} Y$ of automorphisms of $\mathscr{E}$.

Z has a universal action θ_X in the every object X of $\mathscr{E}$:

$$\begin{array}{ccc} Y & \xrightarrow{t_Y} & Y \\ {\scriptstyle (\lambda,t)}\downarrow & & \downarrow{\scriptstyle \lambda} \\ X \times Z & \xrightarrow{\theta_X} & X \end{array}$$

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On the other hand, for any $Y \xrightarrow{\lambda} X$ and $X \xrightarrow{t} Z$, we have

$$\begin{array}{ccc} Y & & \\ {\scriptstyle(1_Y,t\lambda)}\big\downarrow & \searrow^{t_\lambda} & \\ Y \times Z & \xrightarrow{\theta_Y} & Y \end{array}$$

All maps of $\mathscr{E}$ are Z-equivariant.

Anisotropic topos, anisotropic object

A topos is anisotropic if Z is trivial.
An object X of a topos $\mathscr{E}$ is anisotropic if $\mathscr{E}/X$ is anisotropic.
X is anisotropic iff

$$X \times Z \rightarrow X \times X \,;\; (x, z) \mapsto (x, xz)\ ,$$

is a monomorphism. $\theta_X(x, z) = xz$.

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Connected object

An object X with global support such that the above map is an epimorphism. Equivalently, $O_X \cong 1$.

Isotropy torsor

A globally supported X such that

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Any two torsors are locally isomorphic.

Slice topos associated with an isotropy torsor

An object X of a topos $\mathscr{E}$ is an isotropy torsor iff the adjoint functors

$$\mathscr{E}/X \underset{X^*\psi^*}{\overset{\psi_!\Sigma_X}{\rightleftarrows}} \mathscr{E}_\theta$$

form an equivalence:

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The isotropy quotient $\mathscr{E}_\theta$ (see below) is anisotropic in this case (higher isotropy is ruled out).
In particular, for any map $Y \to X$

$$\begin{array}{ccc} Y & \longrightarrow & O_Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & 1 \end{array}$$

is a pullback in $\mathscr{E}$.

...back to inverse semigroups

Anisotropic étale S-set $X \xrightarrow{p} E$:

for all $x \in X$ and $s \in Z(E)$ such that $p(x) = ss^*$, we have

$$xs = x \;\Rightarrow\; s = p(x) \; .$$

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Eg.: $S \overset{\partial}{\to} E$ is anisotropic, but it satisfies a stronger property called torsion-free (replace "$s \in Z(E)$" with "$s \in S$" in above definition).

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Connected étale S-set:

(i) p is a surjective map, and
(ii) for every idempotent e and every $x, y \in p^{-1}(e)$, there is $s \in Z(E)$ ($s^*s = e$) such that $xs = y$.

p is connected iff $O_p \cong E$.

Isotropy torsor of an inverse semigroup S :

Anisotropic, connected étale S-set.

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Equivalence of isotropy torsors

An étale S-set $X \xrightarrow{p} E$ is an isotropy torsor of S iff the adjoint functors

$$\mathscr{B}(S)/p \underset{p^*\psi^*}{\overset{\psi_! p_!}{\rightleftarrows}} \mathscr{B}(S/\mu)$$

form an equivalence.

PROPOSITION

$S \xrightarrow{\partial} E$ is an isotropy torsor iff $Z(E) = S$ iff $S/\mu = E$ (Clifford).

PROPOSITION

$S \overset{\partial}{\twoheadrightarrow} E$ is an isotropy torsor iff $Z(E) = S$ iff $S/\mu = E$ (Clifford).

THEOREM

Isotropy torsors of an inverse semigroup S correspond to homomorphic sections of its maximum idempotent-separating quotient $\psi : S \twoheadrightarrow S/\mu$ (split Billhardt transversal).

PROOF

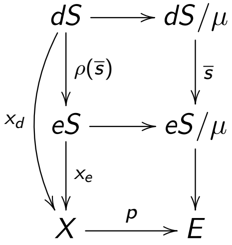
If $S/\mu \stackrel{\rho}{\to} S$ is such a section, then $S \cong T \ltimes G$ where T is a fundamental inverse semigroup and G is a T-group In fact, $T = S/\mu$ and $G = \rho^*\zeta$. Equivalently, G is a group internal to the topos $\mathscr{B}(T)$. We have

$$\mathscr{B}(S) \simeq \mathscr{B}(T \ltimes G) \simeq \mathscr{B}(T)^{G^{\mathrm{op}}} .$$

Because $\mathscr{B}(T)$ is anisotropic, the isotropy group of $\mathscr{B}(T)^{G^{\mathrm{op}}}$ is $\overline{G}$ (conjugation). Moreover, in this case 'Yoneda' G is an isotropy torsor. Ultimately, we obtain an isotropy torsor of S .

On the other hand...

From an isotropy torsor $X \xrightarrow{p} E$ of S we construct a section ρ as follows. Choose a set-theoretic section $\{x_e\}$ of p. If $\bar{s} \in S/\mu$ has domain d and codomain e, then the pullback diagram

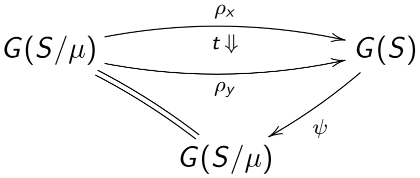


defines $\rho(\bar{s})$, and ultimately a homomorphism ρ that is a section of $\psi : S \to S/\mu$. (ρ preserves the restricted product, and because ψ is an idempotent-separating homomorphism, $\rho(de) = \rho(d)\rho(e)$ holds, so that ρ is a homomorphism.)

Proof continued...

If ρ_x is the homomorphism so obtained from a section $\{x_e\}$, and ρ_y the one obtained from another section $\{y_e\}$, then there is a unique order preserving map $E \xrightarrow{t} Z(E)$ $(t(e)^* t(e) = e)$ such that $xt = y$ and $t_e \rho_y(\bar{s}) = \rho_x(\bar{s}) t_d$, for all $d \xrightarrow{\bar{s}} e$. This defines a central isomorphism $t : \rho_y \Rightarrow \rho_x$.

t is central iff the diagram of ordered functors



commutes over $G(T_\mu)$: $\psi\, t = \mathrm{id}$.

returning to the general theory...

The isotropy quotient $\psi : \mathscr{E} \longrightarrow \mathscr{E}_\theta$

The topos $\mathscr{E}_\theta$ consists of all isotropically trivial objects: those X for which the universal action θ_X is trivial.
The geometric morphism ψ is connected, atomic.
$\psi^*(X) = X$.
$\psi_!(X) =$ orbit space O_X of universal action.

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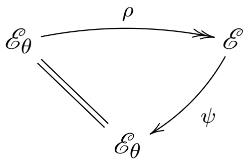
Example

The geometric morphism $\mathscr{B}(S) \longrightarrow \mathscr{B}(S/\mu)$ associated with $S \twoheadrightarrow S/\mu$ is equivalent to the isotropy quotient $\mathscr{B}(S) \longrightarrow \mathscr{B}(S)_\theta$.

Isotropy torsor to étale section of ψ

Let X be an isotropy torsor of $\mathscr{E}$.



For any object Y of $\mathscr{E}$, define

$$\rho^*(Y) = \psi_!(Y \times X) = O_{Y \times X} \ .$$

ρ^* is also given equivalently by

$$\rho^*(Y) = \psi_*(Y^X) \cong \{f \in Y^X \mid f \text{ is } Z\text{-equivariant}\} \ .$$

For any object E of $\mathscr{E}_\theta$ we have

$$\rho_!(E) = \psi^* E \times X \, ; \ \rho_*(E) = (\psi^* E)^X \ .$$

The section ρ is indeed étale

because the equivalence

$$\mathscr{E}/X \simeq (\mathscr{E}/X)_\theta \simeq \mathscr{E}_\theta/O_X \simeq \mathscr{E}_\theta$$

identifies ρ with $\mathscr{E}/X \longrightarrow \mathscr{E}$.

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Étale section to isotropy torsor

If ρ is an étale section of ψ , then $X = \rho_!(1)$ is an isotropy torsor.

Structure of toposes with an isotropy torsor

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THEOREM

If a topos $\mathscr{E}$ has an isotropy torsor with corresponding étale section ρ of $\mathscr{E} \xrightarrow{\psi} \mathscr{E}_\theta$, then $\mathscr{E}_\theta$ is anisotropic, and we have

$$\mathscr{E} \simeq \mathscr{B}(\mathscr{E}_\theta; \rho^* Z) .$$

Conversely, if G is a group internal to an anisotropic topos $\mathscr{F}$, then the isotropy group of $\mathscr{B}(\mathscr{F}; G)$ is $\overline{G}$, 'Yoneda' G is an isotropy torsor of $\mathscr{B}(\mathscr{F}; G)$, and its isotropy quotient is $\mathscr{F}$.

Locally anisotropic topos: a topos with a globally supported anisotropic object

Isotropy torsors play a role in the structure theorem for locally anisotropic toposes.

Structure of locally anisotropic toposes

THEOREM

A topos $\mathscr{E}$ is locally anisotropic iff $\mathscr{E}$ has a globally supported isotropically trivial object O such that

$$\mathscr{E}/O \simeq \mathscr{B}(\mathscr{F}; G) ,$$

where G is a group internal to an anisotropic topos $\mathscr{F}$.

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Afterthought

Thus, O^*Z is isomorphic to $\overline{G}$, modulo an equivalence.