From Smooth Spaces to Smooth Categories

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Logic, Category Theory and Physics

- Smooth manifolds and convenient categories
- Diffeological spaces
- Convenient properties of diffeological spaces
- Smooth categories

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Outline

Smooth manifolds and convenient categories
- Which category?

Diffeological spaces

Convenient properties of diffeological spaces

Smooth Categories
- Internalization
- Example: the path groupoid and 1-forms

In differential geometry, the most popular category, that of finite-dimensional smooth manifolds, fails to be cartesian closed.

If X and Y are finite-dimensional smooth manifolds, the space of smooth maps $\mathcal{C}^\infty(X, Y)$ usually is not.

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subspaces and quotient spaces,

limits and colimits.

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Synthetic

Lawvere, Kock

Maps in

Chen, Souriau

Maps out

Smith, Sikorski

Maps in & out

Frölicher

- See Andrew Stacey's "Comparitive Smootheology"

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Let Diffeological be the category whose objects are open subsets of $\mathbb{R}^n$, $n \geq 0$, and whose morphisms are smooth maps.

We now define the category of diffeological spaces $\mathcal{C}^\infty$.

Definition

A **diffeological space** is a concrete sheaf

$$X\colon \text{Diffeological}^{\text{op}} \to \text{Set}.$$

Definition

A **smooth map** between diffeological spaces X, Y is a natural transformation $F\colon X \Rightarrow Y$.

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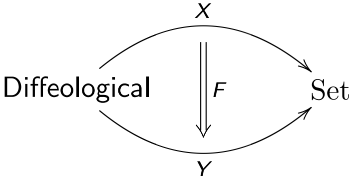

Presheaves

Definition

A **presheaf** on Diffeological is a contravariant functor $X\colon \text{Diffeological}^{op} \to \mathrm{Set}$.

Presheaf condition

Given $\varphi \in X(U)$ and $f\colon U' \to U$ a smooth function between open sets, then $X(f)(\varphi) \in X(U')$.

$$X(f)\colon X(U) \to X(U')$$

$$\varphi \mapsto X(f)(\varphi) := \varphi f$$

We think of $X(1)$ as the underlying set of the diffeological space X.

For any open set U in Diffeological, we call the elements of $X(U)$ **plots in X with domain U**.

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For each open subset U, there is a function

$$\_ \colon X(U) \to Fun(U, X(1))$$

sending plots in $\varphi \in X(U)$ to functions

$$\underline{\varphi} \colon U \to X(1).$$

Definition

We say a presheaf $X \colon \mathrm{Diffeological}^{\mathrm{op}} \to \mathrm{Set}$ is **concrete** if for every object $U \in$ Diffeological, the function $\_ \colon X(U) \to Fun(U, X(1))$ is one-to-one.

The concreteness condition allows us to think of a plot $\varphi \in X(U)$ as a function $\varphi \colon U \to X$.

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Sheaf condition

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Suppose the open sets $U_j \subseteq U$ form an open cover of the open set U, with inclusions $i_j \colon U_j \to U$. If $X(i_j)(\varphi)$ is a plot in X for every j, then φ is a plot in X.

Let X be a diffeological space, $(i_j \colon U_j \to U | j \in J)$ an open cover, and $\{\varphi_j \in X(U_j) | j \in J\}$ a family of plots.

If every two plots agree where both are defined then we can define a unique function $\varphi \colon U \to X$ by gluing together the local plots.

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Recap

- We think of a diffeological space X as a set $X(1)$ with extra structure.
- So, a diffeological space is a set which can be probed by spaces already equipped with smooth structure.
- A map between diffeological spaces $F\colon X \to Y$ is completely determined by the function $F_1\colon X(1) \to Y(1)$.
- A function $f\colon X(1) \to Y(1)$ is smooth if and only if it carries each plot $\varphi\colon U \to X(1)$ to a plot $f\varphi\colon U \to Y(1)$.

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Some examples

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.
- Every diffeological space has a natural topology, and smooth maps between diffeological spaces are automatically continuous.
- Any set X has a **discrete** smooth structure such that the plots $\varphi\colon U \to X$ are just the constant functions.
- Any set X has an **indiscrete** smooth structure where every function $\varphi\colon U \to X$ is a plot.

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Subspaces

Any subset $Y \subseteq X$ of a diffeological space X becomes a diffeological space if we define $\varphi \colon U \to Y$ to be a plot in Y if and only if its composite with the inclusion $i \colon Y \to X$ is a plot in X. We call this the **subspace** smooth structure.

- The inclusion $i \colon Y \to X$ is smooth.
- It is a monomorphism of diffeological spaces.
- Not every monomorphism is of this form.
- **Example** The natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism.

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Quotient spaces

If X is a diffeological space and $\sim$ is any equivalence relation on X, the quotient space $Y = X/\sim$ becomes a diffeological space if we define a plot in Y to be any function of the form

$$U \xrightarrow{\varphi} X \xrightarrow{p} Y$$

where φ is a plot in X.

Products

Given diffeological spaces X and Y, the product $X \times Y$ of their underlying sets becomes a diffeological space where $\varphi\colon U \to X \times Y$ is a plot if and only if its composites with the projections

$$p_X \colon X \times Y \to X, \qquad p_Y \colon X \times Y \to Y$$

are plots in X and Y, respectively.

In fact, diffeological spaces have all limits.

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Coproducts

Given diffeological spaces X and Y, the disjoint union $X + Y$ of their underlying sets becomes a diffeological space where $\varphi\colon U \to X + Y$ is a plot if and only if for each connected component C of U, $\varphi|_C$ is either the composite of a plot in X with the inclusion $i_X\colon X \to X + Y$, or the composite of a plot in Y with the inclusion $i_Y\colon Y \to X + Y$.

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In fact, diffeological spaces have all colimits.

Mapping spaces

Given diffeological spaces X and Y, the set

$$\mathcal{C}^\infty(X, Y) = \{f : X \to Y : f \text{ is smooth}\}$$

becomes a diffeological space where a function $\tilde{\varphi} \colon U \to \mathcal{C}^\infty(X, Y)$ is a plot if and only if the corresponding function $\varphi \colon U \times X \to Y$ given by

$$\varphi(z, x) = \tilde{\varphi}(z)(x)$$

is smooth.

Mapping spaces cont.

With this smooth structure one can show that the natural map

$$\begin{array}{rcl} \mathcal{C}^\infty(X \times Y, Z) & \to & \mathcal{C}^\infty(X, \mathcal{C}^\infty(Y, Z)) \\ f & \mapsto & \tilde{f} \end{array}$$

$$\tilde{f}(x)(y) = f(x, y)$$

is smooth, with a smooth inverse. So, we say the category of diffeological spaces is cartesian closed.

Differential geometry

We can transport many constructions from differential geometry to the setting of diffeological spaces.

Definition

A p-**form** on the diffeological space X is an assignment of a smooth p-form ω_φ on U to each plot $\varphi\colon U \to X$, satisfying this **pullback compatibility condition** for any smooth map $f\colon U' \to U$:

$$(f^*\omega)_\varphi = \omega_{\varphi \circ f}$$

Proposition

Given a smooth map $f\colon X \to Y$ and $\omega \in \Omega^p(Y)$ there is a p-form $f^*\omega \in \Omega^p(X)$ given by

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Diffeological spaces can be used as a tool for defining a notion of 'smooth category'.

A smooth category is a category $\mathcal{C}$ internal to $\mathcal{C}^\infty$.

- $\mathrm{Ob}(\mathcal{C})$ is a diffeological space;
- $\mathrm{Mor}(\mathcal{C})$ is a diffeological space;
- the structure maps are smooth, in particular, the composition map

$$\circ \colon \mathrm{Mor(C)}_t \times_s \mathrm{Mor(C)} \to \mathrm{Mor(C)}$$

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Why pullbacks?

Pullbacks guarantee $\mathrm{Mor(C)}_t \times_s \mathrm{Mor(C)}$ to be a diffeological space.

$$\begin{array}{ccc} \mathrm{Mor(C)}_t \times_s \mathrm{Mor(C)} & \longrightarrow & \mathrm{Mor(C)} \\ \downarrow & & \downarrow s \\ \mathrm{Mor(C)} & \xrightarrow[t]{} & \mathrm{Ob}(\mathcal{C}) \end{array}$$

Smooth functors

While we are at it, we can define smooth functors as functors internal to $\mathcal{C}^\infty$.

Definition

A **functor** $F\colon A \to B$ **internal to** $\mathcal{C}^\infty$ is

- a smooth map of objects $\mathrm{Ob}(F)\colon \mathrm{Ob}(A) \to \mathrm{Ob}(B)$ in $\mathcal{C}^\infty$;
- a smooth map of morphisms $\mathrm{Mor}(F)\colon \mathrm{Mor}(A) \to \mathrm{Mor}(B)$ in $\mathcal{C}^\infty$;

such that diagrams expressing respect for the source, target, identity, and composition maps commute.

Smooth groupoids

Definition

A **smooth groupoid** is a smooth category $\mathcal{C}$ such that every morphism has an inverse, and the map

$$\mathit{inv}\colon \mathrm{Mor(C)} \to \mathrm{Mor(C)}$$

sendng each morphism to its inverse is smooth.

Given a diffeological space X, there is a smooth groupoid $\mathcal{P}_1(X)$, the **path groupoid** of X, such that:

- the objects of $\mathcal{P}_1(X)$ are points of X,
- the morphisms of $\mathcal{P}_1(X)$ are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to X$ such that $\gamma(t)$ is constant near $t = 0$ and $t = 1$.

Thin homotopy can "more or less" be taken to mean equivalence up to reparameterization.

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Thin homotopy can "more or less" be taken to mean equivalence up to reparameterization.

Definition

A **thin homotopy** between smooth paths $\gamma_0, \gamma_1 \colon [0,1] \to X$ is a smooth map $F\colon [0,1]^2 \to X$ such that:

- $F(0,t) = \gamma_0(t)$ and $F(1,t) = \gamma_1(t)$,
- $F(s,t)$ is constant for t near 0 and constant for t near 1,
- $F(s,t)$ is independent of s for s near 0 and for s near 1,
- the rank of the differential $dF(s,t)$ is less than or equal to 1 for all $(s,t) \in [0,1]^2$.

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Smooth groups

Definition

A **smooth group** is a smooth groupoid with one object.

Skipping a bunch of details, we have the following:

- We can define the tangent space at point x in a diffeological space X.
- We can define a concept of smooth vector space: a diffeological space that is also a vector space.
- So we can define a smooth vector space $\mathfrak{g}$ for any smooth group G.
- Exponentiating takes any Lie algebra to a Lie group: a similar concept of **exponentiable group** can be defined in this setting.

Smooth groups

Definition

A **smooth group** is a smooth groupoid with one object.

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Smooth functors and 1-forms

Given an exponentiable smooth group G, a diffeological space X, and a $\mathfrak{g}$-valued 1-form A on X, then for any smooth path $\gamma\colon [0,1] \to X$, we can associate an element

$$g = Pexp \int_\gamma A \in G.$$

The element g describes parallel transport along the path γ using the connection A.

Theorem

(Baez, Schreiber, Waldorf)

Let G be an exponentiable smooth group and X a smooth space. Given any $\mathfrak{g}$-valued 1-form A on X, there is a smooth functor

$$\mathrm{hol}\colon \mathcal{P}_1(X) \to G$$

given by

$$Pexp \int_\gamma A \in G,$$

where γ is any representative of the thin homotopy class $[\gamma]$. This gives a one-to-one correspondence between elements $A \in \Omega^1(X, \mathfrak{g})$ and smooth functors $\mathrm{hol}\colon \mathcal{P}_1(X) \to G$.

Possible directions

- Principal G-bundles with connection and smooth path groupoids
- Higher gerbes
- More differential geometry/topology...