From Smooth Spaces to Smooth Categories

Alexander E. Hoffnung
Joint work with John Baez

Department of Mathematics, University of California, Riverside

May 2, 2009
University of Ottawa
Workshop on Smooth Structures in Logic, Category Theory and Physics
Smooth manifolds and convenient categories

- Diffeological spaces
- Convenient properties of diffeological spaces
- Smooth categories
Smooth manifolds and convenient categories

Diffeological spaces

Convenient properties of diffeological spaces

Smooth categories
Smooth manifolds and convenient categories

Diffeological spaces

Convenient properties of diffeological spaces

Smooth categories
Smooth manifolds and convenient categories
Diffeological spaces
Convenient properties of diffeological spaces
Smooth categories
Outline

Smooth manifolds and convenient categories
  Which category?

Diffeological spaces

Convenient properties of diffeological spaces

Smooth Categories
  Internalization
  Example: the path groupoid and 1-forms
In differential geometry, the most popular category, that of finite-dimensional smooth manifolds, fails to be cartesian closed.

If $X$ and $Y$ are finite-dimensional smooth manifolds, the space of smooth maps $C^\infty(X, Y)$ usually is not.

It is some sort of infinite-dimensional manifold.
In differential geometry, the most popular category, that of finite-dimensional smooth manifolds, fails to be cartesian closed. If $X$ and $Y$ are finite-dimensional smooth manifolds, the space of smooth maps $\mathcal{C}^\infty(X, Y)$ usually is not.

It is some sort of *infinite-dimensional* manifold.
In differential geometry, the most popular category, that of finite-dimensional smooth manifolds, fails to be cartesian closed.

If $X$ and $Y$ are finite-dimensional smooth manifolds, the space of smooth maps $\mathcal{C}^\infty(X, Y)$ usually is not.

It is some sort of *infinite-dimensional* manifold.
The category of finite-dimensional smooth manifolds lacks other desirable features such as:

- subspaces and quotient spaces,
- limits and colimits.

We look for a ‘convenient category’ of smooth spaces in which to do differential geometry.
The category of finite-dimensional smooth manifolds lacks other desirable features such as:

- subspaces and quotient spaces,
- limits and colimits.

We look for a ‘convenient category’ of smooth spaces in which to do differential geometry.
The category of finite-dimensional smooth manifolds lacks other desirable features such as:

subspaces and quotient spaces,

limits and colimits.

We look for a ‘convenient category’ of smooth spaces in which to do differential geometry.
The category of finite-dimensional smooth manifolds lacks other desirable features such as:

- subspaces and quotient spaces,
- limits and colimits.

We look for a ‘convenient category’ of smooth spaces in which to do differential geometry.
Synthetic Lawvere, Kock

Maps in
Chen, Souriau

Maps out
Smith, Sikorski

Maps in & out
Frölicher

- See Andrew Stacey’s “Comparitive Smootheology”
Synthetic
Lawvere, Kock

Maps in
Chen, Souriau

Maps out
Smith, Sikorski

Maps in & out
Frölicher

- See Andrew Stacey’s “Comparitive Smootheology”
**Synthetic**
Lawvere, Kock

**Maps in**
Chen, Souriau

**Maps out**
Smith, Sikorski

**Maps in & out**
Frölicher

- See Andrew Stacey’s “Comparitive Smootheology”
Synthetic
Lawvere, Kock

Maps in
Chen, Souriau

Maps out
Smith, Sikorski

Maps in & out
Frölicher

● See Andrew Stacey’s “Comparitive Smootheology”
Outline

**Smooth manifolds and convenient categories**
Which category?

**Diffeological spaces**

**Convenient properties of diffeological spaces**

**Smooth Categories**
Internalization
Example: the path groupoid and 1-forms
Diffeological spaces

Let Diffeological be the category whose objects are open subsets of $\mathbb{R}^n$, $n \geq 0$, and whose morphisms are smooth maps.

We now define the category of diffeological spaces $C^\infty$.

Definition

A diffeological space is a concrete sheaf

$$X : \text{Diffeological}^{\text{op}} \to \text{Set}.$$  

Definition

A smooth map between diffeological spaces $X$, $Y$ is a natural transformation $F : X \Rightarrow Y$. 

\[
\begin{array}{ccc}
X & \xrightarrow{F} & Y \\
\downarrow & & \downarrow \\
\text{Diffeological} & \xrightarrow{\text{Set}} & \\
\end{array}
\]
Diffeological spaces

Let Diffeological be the category whose objects are open subsets of \( \mathbb{R}^n, n \geq 0 \), and whose morphisms are smooth maps.

We now define the category of diffeological spaces \( C^\infty \).

**Definition**

A **diffeological space** is a concrete sheaf

\[
X : \text{Diffeological}^{\text{op}} \to \text{Set}.
\]

**Definition**

A **smooth map** between diffeological spaces \( X, Y \) is a natural transformation \( F : X \Rightarrow Y \).
Diffeological spaces

Let Diffeological be the category whose objects are open subsets of $\mathbb{R}^n$, $n \geq 0$, and whose morphisms are smooth maps.

We now define the category of diffeological spaces $\mathcal{C}^\infty$.

Definition

A **diffeological space** is a concrete sheaf

$$X : \text{Diffeological}^{\text{op}} \to \text{Set}.$$ 

Definition

A **smooth map** between diffeological spaces $X, Y$ is a natural transformation $F : X \Rightarrow Y$.

$$\begin{align*}
\text{Diffeological} & \quad \xrightarrow{F} \quad \text{Set} \\
X & \quad \downarrow \downarrow \\
Y & \quad \uparrow \uparrow
\end{align*}$$
Presheaves

Definition
A presheaf on Diffeological is a contravariant functor $X : \text{Diffeological}^{\text{op}} \to \text{Set}$.

Presheaf condition
Given $\varphi \in X(U)$ and $f : U' \to U$ a smooth function between open sets, then $X(f)(\varphi) \in X(U')$.

$$X(f) : X(U) \to X(U')$$

$$\varphi \mapsto X(f)(\varphi) := \varphi f$$

We think of $X(1)$ as the underlying set of the diffeological space $X$. For any open set $U$ in Diffeological, we call the elements of $X(U)$ plots in $X$ with domain $U$. 
Presheaves

Definition
A **presheaf** on Diffeological is a contravariant functor $X : \text{Diffeological}^{\text{op}} \to \text{Set}$.

Presheaf condition
Given $\varphi \in X(U)$ and $f : U' \to U$ a smooth function between open sets, then $X(f)(\varphi) \in X(U')$.

$$X(f) : X(U) \to X(U')$$

$$\varphi \mapsto X(f)(\varphi) := \varphi f$$

We think of $X(1)$ as the underlying set of the diffeological space $X$. For any open set $U$ in Diffeological, we call the elements of $X(U)$ **plots** in $X$ with domain $U$. 
Presheaves

Definition
A **presheaf** on Diffeological is a contravariant functor
\( X : \text{Diffeological}^{\text{op}} \to \text{Set} \).

Presheaf condition
Given \( \varphi \in X(U) \) and \( f : U' \to U \) a smooth function between open sets, then \( X(f)(\varphi) \in X(U') \).

\[
X(f) : X(U) \to X(U')
\]

\[
\varphi \mapsto X(f)(\varphi) := \varphi f
\]

We think of \( X(1) \) as the underlying set of the diffeological space \( X \).
For any open set \( U \) in Diffeological, we call the elements of \( X(U) \) plots in \( X \) with domain \( U \).
Presheaves

Definition
A **presheaf** on Diffeological is a contravariant functor
\[ X : \text{Diffeological}^{op} \to \text{Set}. \]

Presheaf condition
Given \( \varphi \in X(U) \) and \( f : U' \to U \) a smooth function between open sets, then \( X(f)(\varphi) \in X(U') \).

\[
X(f) : X(U) \to X(U')
\]

\[
\varphi \mapsto X(f)(\varphi) := \varphi f
\]

We think of \( X(1) \) as the underlying set of the diffeological space \( X \).
For any open set \( U \) in Diffeological, we call the elements of \( X(U) \) **plots in \( X \) with domain \( U \).**
For each open subset $U$, there is a function

$$ _\varphi : X(U) \to \text{Fun}(U, X(1)) $$

sending plots in $\varphi \in X(U)$ to functions

$$ \varphi : U \to X(1). $$

**Definition**

We say a presheaf $X : \text{Diffeological}^{\text{op}} \to \text{Set}$ is **concrete** if for every object $U \in \text{Diffeological}$, the function $ _\varphi : X(U) \to \text{Fun}(U, X(1))$ is one-to-one.

The concreteness condition allows us to think of a plot $\varphi \in X(U)$ as a function $\varphi : U \to X$.

So we think of the plots as the “smooth” functions into $X$. 
For each open subset $U$, there is a function

$$\_ : X(U) \to Fun(U, X(1))$$

sending plots in $\varphi \in X(U)$ to functions

$$\varphi : U \to X(1).$$

**Definition**

We say a presheaf $X : \text{Diffeological}^{\text{op}} \to \text{Set}$ is **concrete** if for every object $U \in \text{Diffeological}$, the function

$$\_ : X(U) \to Fun(U, X(1))$$

is one-to-one.

The concreteness condition allows us to think of a plot $\varphi \in X(U)$ as a function $\varphi : U \to X$.

So we think of the plots as the “smooth” functions into $X$. 
For each open subset $U$, there is a function

$$\_ : X(U) \to \text{Fun}(U, X(1))$$

sending plots in $\varphi \in X(U)$ to functions

$$\varphi : U \to X(1).$$

**Definition**

We say a presheaf $X : \text{Diffeological}^{\text{op}} \to \text{Set}$ is **concrete** if for every object $U \in \text{Diffeological}$, the function

$$\_ : X(U) \to \text{Fun}(U, X(1))$$

is one-to-one.

The concreteness condition allows us to think of a plot $\varphi \in X(U)$ as a function $\varphi : U \to X$.

So we think of the plots as the “smooth” functions into $X$. 
Sheaf condition

Suppose the open sets $U_j \subseteq U$ form an open cover of the open set $U$, with inclusions $i_j : U_j \to U$. If $X(i_j)(\varphi)$ is a plot in $X$ for every $j$, then $\varphi$ is a plot in $X$.

Let $X$ be a diffeological space, $(i_j : U_j \to U | j \in J)$ an open cover, and \{ $\varphi_j \in X(U_j) | j \in J$ \} a family of plots.

If every two plots agree where both are defined then we can define a unique function $\varphi : U \to X$ by gluing together the local plots.
Sheaf condition

Suppose the open sets $U_j \subseteq U$ form an open cover of the open set $U$, with inclusions $i_j: U_j \to U$. If $X(i_j)(\varphi)$ is a plot in $X$ for every $j$, then $\varphi$ is a plot in $X$.

Let $X$ be a diffeological space, $(i_j: U_j \to U | j \in J)$ an open cover, and $\{\varphi_j \in X(U_j) | j \in J\}$ a family of plots.

If every two plots agree where both are defined then we can define a unique function $\varphi: U \to X$ by gluing together the local plots.
We think of a diffeological space $X$ as a set $X(1)$ with extra structure.

So, a diffeological space is a set which can be probed by spaces already equipped with smooth structure.

A map between diffeological spaces $F: X \to Y$ is completely determined by the function $F_1: X(1) \to Y(1)$.

A function $f: X(1) \to Y(1)$ is smooth if and only if it carries each plot $\varphi: U \to X(1)$ to a plot $f\varphi: U \to Y(1)$. 
Recap

- We think of a diffeological space $X$ as a set $X(1)$ with extra structure.
- So, a diffeological space is a set which can be probed by spaces already equipped with smooth structure.
- A map between diffeological spaces $F: X \to Y$ is completely determined by the function $F_1: X(1) \to Y(1)$.
- A function $f: X(1) \to Y(1)$ is smooth if and only if it carries each plot $\varphi: U \to X(1)$ to a plot $f\varphi: U \to Y(1)$. 
Outline

Smooth manifolds and convenient categories
  Which category?

Diffeological spaces

Convenient properties of diffeological spaces

Smooth Categories
  Internalization
  Example: the path groupoid and 1-forms
Some examples

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.

- Every diffeological space has a natural topology, and smooth maps between diffeological spaces are automatically continuous.

- Any set $X$ has a **discrete** smooth structure such that the plots $\varphi: U \to X$ are just the constant functions.

- Any set $X$ has an **indiscrete** smooth structure where every function $\varphi: U \to X$ is a plot.
Some examples

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.

- Every diffeological space has a natural topology, and smooth maps between diffeological spaces are automatically continuous.

- Any set $X$ has a **discrete** smooth structure such that the plots $\varphi: U \to X$ are just the constant functions.

- Any set $X$ has an **indiscrete** smooth structure where every function $\varphi: U \to X$ is a plot.
Some examples

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.

- Every diffeological space has a natural topology, and smooth maps between diffeological spaces are automatically continuous.

- Any set $X$ has a **discrete** smooth structure such that the plots $\varphi : U \to X$ are just the constant functions.

- Any set $X$ has an **indiscrete** smooth structure where every function $\varphi : U \to X$ is a plot.
Some examples

- Every smooth manifold is a smooth space, and a map between smooth manifolds is smooth in the new sense if and only if it is smooth in the usual sense.

- Every diffeological space has a natural topology, and smooth maps between diffeological spaces are automatically continuous.

- Any set $X$ has a **discrete** smooth structure such that the plots $\varphi : U \to X$ are just the constant functions.

- Any set $X$ has an **indiscrete** smooth structure where every function $\varphi : U \to X$ is a plot.
Subspaces

Any subset \( Y \subseteq X \) of a diffeological space \( X \) becomes a diffeological space if we define \( \varphi : U \to Y \) to be a plot in \( Y \) if and only if its composite with the inclusion \( i : Y \to X \) is a plot in \( X \). We call this the **subspace** smooth structure.

- The inclusion \( i : Y \to X \) is smooth.
- It is a monomorphism of diffeological spaces.
- Not every monomorphism is of this form.
- **Example** The natural map from \( \mathbb{R} \) with its discrete smooth structure to \( \mathbb{R} \) with its standard smooth structure is also a monomorphism.
Any subset $Y \subseteq X$ of a diffeological space $X$ becomes a diffeological space if we define $\varphi: U \to Y$ to be a plot in $Y$ if and only if its composite with the inclusion $i: Y \to X$ is a plot in $X$. We call this the **subspace** smooth structure.

- The inclusion $i: Y \to X$ is smooth.
- It is a monomorphism of diffeological spaces.
- Not every monomorphism is of this form.
- **Example** The natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism.
Subspaces

Any subset $Y \subseteq X$ of a diffeological space $X$ becomes a diffeological space if we define $\varphi: U \to Y$ to be a plot in $Y$ if and only if its composite with the inclusion $i: Y \to X$ is a plot in $X$. We call this the **subspace** smooth structure.

- The inclusion $i: Y \to X$ is smooth.
- It is a monomorphism of diffeological spaces.
- Not every monomorphism is of this form.
- **Example** The natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism.
Subspaces

Any subset $Y \subset X$ of a diffeological space $X$ becomes a diffeological space if we define $\varphi: U \to Y$ to be a plot in $Y$ if and only if its composite with the inclusion $i: Y \to X$ is a plot in $X$. We call this the **subspace** smooth structure.

- The inclusion $i: Y \to X$ is smooth.
- It is a monomorphism of diffeological spaces.
- Not every monomorphism is of this form.

**Example** The natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism.
Subspaces

Any subset $Y \subseteq X$ of a diffeological space $X$ becomes a diffeological space if we define $\varphi: U \to Y$ to be a plot in $Y$ if and only if its composite with the inclusion $i: Y \to X$ is a plot in $X$. We call this the **subspace** smooth structure.

- The inclusion $i: Y \to X$ is smooth.
- It is a monomorphism of diffeological spaces.
- Not every monomorphism is of this form.
- **Example** The natural map from $\mathbb{R}$ with its discrete smooth structure to $\mathbb{R}$ with its standard smooth structure is also a monomorphism.
Quotient spaces

If $X$ is a diffeological space and $\sim$ is any equivalence relation on $X$, the quotient space $Y = X/\sim$ becomes a diffeological space if we define a plot in $Y$ to be any function of the form

$$U \xrightarrow{\varphi} X \xrightarrow{p} Y$$

where $\varphi$ is a plot in $X$. 
Given diffeological spaces $X$ and $Y$, the product $X \times Y$ of their underlying sets becomes a diffeological space where $\varphi: U \to X \times Y$ is a plot if and only if its composites with the projections

$$p_X: X \times Y \to X, \quad p_Y: X \times Y \to Y$$

are plots in $X$ and $Y$, respectively.

In fact, diffeological spaces have all limits.
Given diffeological spaces $X$ and $Y$, the product $X \times Y$ of their underlying sets becomes a diffeological space where $\varphi: U \to X \times Y$ is a plot if and only if its composites with the projections

$$p_X: X \times Y \to X, \quad p_Y: X \times Y \to Y$$

are plots in $X$ and $Y$, respectively.

In fact, diffeological spaces have all limits.
Given diffeological spaces $X$ and $Y$, the disjoint union $X + Y$ of their underlying sets becomes a diffeological space where $\varphi: U \rightarrow X + Y$ is a plot if and only if for each connected component $C$ of $U$, $\varphi|_C$ is either the composite of a plot in $X$ with the inclusion $i_X: X \rightarrow X + Y$, or the composite of a plot in $Y$ with the inclusion $i_Y: Y \rightarrow X + Y$.

In fact, diffeological spaces have all colimits.
Coprodcts

Given diffeological spaces $X$ and $Y$, the disjoint union $X + Y$ of their underlying sets becomes a diffeological space where $\varphi: U \to X + Y$ is a plot if and only if for each connected component $C$ of $U$, $\varphi|_C$ is either the composite of a plot in $X$ with the inclusion $i_X: X \to X + Y$, or the composite of a plot in $Y$ with the inclusion $i_Y: Y \to X + Y$.

In fact, diffeological spaces have all colimits.
Mapping spaces

Given diffeological spaces $X$ and $Y$, the set

$$C^\infty(X, Y) = \{ f : X \to Y : f \text{ is smooth} \}$$

becomes a diffeological space where a function $\tilde{\varphi} : U \to C^\infty(X, Y)$ is a plot if and only if the corresponding function $\varphi : U \times X \to Y$ given by

$$\varphi(z, x) = \tilde{\varphi}(z)(x)$$

is smooth.
With this smooth structure one can show that the natural map

\[ C^\infty(X \times Y, Z) \rightarrow C^\infty(X, C^\infty(Y, Z)) \]
\[ f \mapsto \tilde{f} \]

\[ \tilde{f}(x)(y) = f(x, y) \]

is smooth, with a smooth inverse. So, we say the category of
diffeological spaces is cartesian closed.
Differential geometry

We can transport many constructions from differential geometry to the setting of diffeological spaces.

Definition
A \( p \)-form on the diffeological space \( X \) is an assignment of a smooth \( p \)-form \( \omega_\varphi \) on \( U \) to each plot \( \varphi: U \to X \), satisfying this pullback compatibility condition for any smooth map \( f: U' \to U \):

\[
(f^*\omega)_\varphi = \omega_{\varphi \circ f}
\]

Proposition
Given a smooth map \( f: X \to Y \) and \( \omega \in \Omega^p(Y) \) there is a \( p \)-form \( f^*\omega \in \Omega^p(X) \) given by

\[
(f^*\omega)_\varphi = \omega_{\varphi \circ f}
\]

for every plot \( \varphi: U \to X \).

We call \( f^*\omega \) the pullback of \( \omega \) along \( f \).
Differential geometry

We can transport many constructions from differential geometry to the setting of diffeological spaces.

Definition

A \( p \)-form on the diffeological space \( X \) is an assignment of a smooth \( p \)-form \( \omega_\varphi \) on \( U \) to each plot \( \varphi: U \to X \), satisfying this pullback compatibility condition for any smooth map \( f: U' \to U \):

\[
(f^* \omega)_\varphi = \omega_{\varphi \circ f}
\]

Proposition

Given a smooth map \( f: X \to Y \) and \( \omega \in \Omega^p(Y) \) there is a \( p \)-form \( f^* \omega \in \Omega^p(X) \) given by

\[
(f^* \omega)_\varphi = \omega_{\varphi \circ f}
\]

for every plot \( \varphi: U \to X \).

We call \( f^* \omega \) the pullback of \( \omega \) along \( f \).
Outline

Smooth manifolds and convenient categories
  Which category?

Diffeological spaces

Convenient properties of diffeological spaces

Smooth Categories
  Internalization
  Example: the path groupoid and 1-forms
Diffeological spaces can be used as a tool for defining a notion of ‘smooth category’.

A smooth category is a category $C$ internal to $C^\infty$.

- $\text{Ob}(C)$ is a diffeological space;
- $\text{Mor}(C)$ is a diffeological space;
- the structure maps are smooth, in particular, the composition map
  $$\circ : \text{Mor}(C)_t \times_s \text{Mor}(C) \to \text{Mor}(C)$$
  should be smooth.
Diffeological spaces can be used as a tool for defining a notion of ‘smooth category’.

A smooth category is a category $C$ internal to $C^\infty$.

- $\text{Ob}(C)$ is a diffeological space;
- $\text{Mor}(C)$ is a diffeological space;
- the structure maps are smooth, in particular, the composition map

$$\circ: \text{Mor}(C)_t \times_s \text{Mor}(C) \to \text{Mor}(C)$$

should be smooth.
Diffeological spaces can be used as a tool for defining a notion of ‘smooth category’.

A smooth category is a category $\mathcal{C}$ internal to $\mathcal{C}^\infty$.

- $\text{Ob}(\mathcal{C})$ is a diffeological space;
- $\text{Mor}(\mathcal{C})$ is a diffeological space;
- the structure maps are smooth, in particular, the composition map
  $$\circ : \text{Mor}(\mathcal{C})_t \times_s \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C})$$
should be smooth.
Diffeological spaces can be used as a tool for defining a notion of ‘smooth category’.

A smooth category is a category $\mathcal{C}$ internal to $\mathcal{C}^\infty$.

- $\text{Ob}(\mathcal{C})$ is a diffeological space;
- $\text{Mor}(\mathcal{C})$ is a diffeological space;
- the structure maps are smooth, in particular, the composition map
  \[
  \circ : \text{Mor}(\mathcal{C})_t \times_s \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C})
  \]
  should be smooth.
Diffeological spaces can be used as a tool for defining a notion of ‘smooth category’.

A smooth category is a category $\mathcal{C}$ internal to $\mathcal{C}^\infty$.

- $\text{Ob}(\mathcal{C})$ is a diffeological space;
- $\text{Mor}(\mathcal{C})$ is a diffeological space;
- the structure maps are smooth, in particular, the composition map
  $$\circ : \text{Mor}(\mathcal{C})_t \times_s \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C})$$
  should be smooth.
Why pullbacks?

Pullbacks guarantee $\text{Mor}(C)_t \times_s \text{Mor}(C)$ to be a diffeological space.

\[
\begin{array}{c}
\text{Mor}(C)_t \times_s \text{Mor}(C) \longrightarrow \text{Mor}(C) \\
\downarrow \quad \downarrow \quad \downarrow s \\
\text{Mor}(C) \quad \quad \text{Ob}(C)
\end{array}
\]
Smooth functors

While we are at it, we can define smooth functors as functors internal to $C^\infty$.

**Definition**

A functor $F : A \to B$ **internal to** $C^\infty$ is

- a smooth map of objects $\text{Ob}(F) : \text{Ob}(A) \to \text{Ob}(B)$ in $C^\infty$;
- a smooth map of morphisms $\text{Mor}(F) : \text{Mor}(A) \to \text{Mor}(B)$ in $C^\infty$;

such that diagrams expressing respect for the source, target, identity, and composition maps commute.
Smooth groupoids

Definition
A smooth groupoid is a smooth category $\mathcal{C}$ such that every morphism has an inverse, and the map

$$inv: Mor(\mathcal{C}) \to Mor(\mathcal{C})$$

sending each morphism to its inverse is smooth.

Given a diffeological space $X$, there is a smooth groupoid $\mathcal{P}_1(X)$, the path groupoid of $X$, such that:

- the objects of $\mathcal{P}_1(X)$ are points of $X$,
- the morphisms of $\mathcal{P}_1(X)$ are thin homotopy classes of smooth paths $\gamma: [0, 1] \to X$ such that $\gamma(t)$ is constant near $t = 0$ and $t = 1$.

Thin homotopy can “more or less” be taken to mean equivalence up to reparameterization.
Smooth groupoids

Definition
A smooth groupoid is a smooth category $\mathcal{C}$ such that every morphism has an inverse, and the map

$$inv : \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C})$$

sends each morphism to its inverse is smooth.

Given a diffeological space $X$, there is a smooth groupoid $\mathcal{P}_1(X)$, the path groupoid of $X$, such that:

- the objects of $\mathcal{P}_1(X)$ are points of $X$,
- the morphisms of $\mathcal{P}_1(X)$ are thin homotopy classes of smooth paths $\gamma : [0, 1] \to X$ such that $\gamma(t)$ is constant near $t = 0$ and $t = 1$.

Thin homotopy can “more or less” be taken to mean equivalence up to reparameterization.
Smooth groupoids

Definition
A smooth groupoid is a smooth category $\mathcal{C}$ such that every morphism has an inverse, and the map

$$\text{inv}: \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$$

sending each morphism to its inverse is smooth.

Given a diffeological space $X$, there is a smooth groupoid $\mathcal{P}_1(X)$, the path groupoid of $X$, such that:

- the objects of $\mathcal{P}_1(X)$ are points of $X$,
- the morphisms of $\mathcal{P}_1(X)$ are thin homotopy classes of smooth paths $\gamma: [0, 1] \rightarrow X$ such that $\gamma(t)$ is constant near $t = 0$ and $t = 1$.

Thin homotopy can “more or less” be taken to mean equivalence up to reparameterization.
**Definition**

A **thin homotopy** between smooth paths $\gamma_0, \gamma_1 : [0, 1] \to X$ is a smooth map $F : [0, 1]^2 \to X$ such that:

- $F(0, t) = \gamma_0(t)$ and $F(1, t) = \gamma_1(t)$,
- $F(s, t)$ is constant for $t$ near 0 and constant for $t$ near 1,
- $F(s, t)$ is independent of $s$ for $s$ near 0 and for $s$ near 1,
- the rank of the differential $dF(s, t)$ is less than or equal to 1 for all $(s, t) \in [0, 1]^2$.

The path groupoid is a smooth category because the set of morphisms is a quotient of a subspace of the mapping space $C^\infty([0, 1], X)$. 
**Definition**

A **thin homotopy** between smooth paths \( \gamma_0, \gamma_1 : [0, 1] \to X \) is a smooth map \( F : [0, 1]^2 \to X \) such that:

- \( F(0, t) = \gamma_0(t) \) and \( F(1, t) = \gamma_1(t) \),
- \( F(s, t) \) is constant for \( t \) near 0 and constant for \( t \) near 1,
- \( F(s, t) \) is independent of \( s \) for \( s \) near 0 and for \( s \) near 1,
- the rank of the differential \( dF(s, t) \) is less than or equal to 1 for all \( (s, t) \in [0, 1]^2 \).

The path groupoid is a smooth category because the set of morphisms is a quotient of a subspace of the mapping space \( C^\infty([0, 1], X) \).
Smooth groups

Definition

A smooth group is a smooth groupoid with one object.

Skipping a bunch of details, we have the following:

- We can define the tangent space at point $x$ in a diffeological space $X$.
- We can define a concept of smooth vector space: a diffeological space that is also a vector space.
- So we can define a smooth vector space $g$ for any smooth group $G$.
- Exponentiating takes any Lie algebra to a Lie group: a similar concept of exponentiable group can be defined in this setting.
Smooth groups

Definition
A smooth group is a smooth groupoid with one object.

Skipping a bunch of details, we have the following:

- We can define the tangent space at point $x$ in a diffeological space $X$.
- We can define a concept of smooth vector space: a diffeological space that is also a vector space.
- So we can define a smooth vector space $g$ for any smooth group $G$.
- Exponentiating takes any Lie algebra to a Lie group: a similar concept of exponentiable group can be defined in this setting.
Smooth groups

Definition

A smooth group is a smooth groupoid with one object.

Skipping a bunch of details, we have the following:

- We can define the tangent space at point $x$ in a diffeological space $X$.
- We can define a concept of smooth vector space: a diffeological space that is also a vector space.
- So we can define a smooth vector space $g$ for any smooth group $G$.
- Exponentiating takes any Lie algebra to a Lie group: a similar concept of exponentiable group can be defined in this setting.
Smooth groups

Definition
A **smooth group** is a smooth groupoid with one object.

Skipping a bunch of details, we have the following:

- We can define the tangent space at point $x$ in a diffeological space $X$.
- We can define a concept of smooth vector space: a diffeological space that is also a vector space.
- So we can define a smooth vector space $\mathfrak{g}$ for any smooth group $G$.
- Exponentiating takes any Lie algebra to a Lie group: a similar concept of **exponentiable group** can be defined in this setting.
Smooth groups

Definition

A smooth group is a smooth groupoid with one object.

Skipping a bunch of details, we have the following:

- We can define the tangent space at point \( x \) in a diffeological space \( X \).
- We can define a concept of smooth vector space: a diffeological space that is also a vector space.
- So we can define a smooth vector space \( g \) for any smooth group \( G \).
- Exponentiating takes any Lie algebra to a Lie group: a similar concept of exponentiable group can be defined in this setting.
Smooth functors and 1-forms

Given an exponentiable smooth group $G$, a diffeological space $X$, and a $g$-valued 1-form $A$ on $X$, then for any smooth path $\gamma : [0, 1] \to X$, we can associate an element

$$g = \text{Pexp} \int_\gamma A \in G.$$ 

The element $g$ describes parallel transport along the path $\gamma$ using the connection $A$. 
Theorem
(Baez, Schreiber, Waldorf)

Let $G$ be an exponentiable smooth group and $X$ a smooth space. Given any $g$-valued 1-form $A$ on $X$, there is a smooth functor

$$\text{hol}: \mathcal{P}_1(X) \to G$$

given by

$$P\exp \int_{\gamma} A \in G,$$

where $\gamma$ is any representative of the thin homotopy class $[\gamma]$. This gives a one-to-one correspondence between elements $A \in \Omega^1(X, g)$ and smooth functors $\text{hol}: \mathcal{P}_1(X) \to G$. 
Possible directions

- Principal $G$-bundles with connection and smooth path groupoids
- Higher gerbes
- More differential geometry/topology...