

A convenient differential category

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Linear Logic and its Models I

Linear Logic (J.-Y. Girard) is a modification of traditional intuitionistic logic with the following features:

- The traditional rules of *contraction* and *weakening* are eliminated, causing conjunction to behave like the tensor product of vector spaces, rather than cartesian product. We will denote it by \otimes . It still has a right adjoint, modelling implication.
- These rules are partially reintroduced for special formulas of the form $!X$. These formulas thus have the structure of a *cocommutative comonoid*, i.e. are equipped with maps:

$$!X \longrightarrow I \qquad !X \longrightarrow !X \otimes !X$$

Linear Logic and its Models II

- These maps allow us to model the linear versions of *contraction* and *weakening*:

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \text{Cont}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \text{Weak}$$

- The easiest way to ensure this structure is via the *exponential isomorphisms*:

$$!(X \times Y) \cong !X \otimes !Y \quad !1 \cong I$$

Definition

A symmetric monoidal closed category with a comonad denoted $!$, with the above structure and various coherence conditions, is called a *Seely category*.

Example

A boring example is given by the category \mathcal{REL} , whose objects are sets, and an arrow $R: X \rightarrow Y$ is a subset $R \subseteq X \times Y$. The tensor is modelled by cartesian product of objects. $!X$ is the set of finitary multisets on X .

Differential Linear Logic I

- Historically, linear logic came about with Girard's realization that his category of coherence spaces and stable maps decomposed as a coKleisli category of the category of coherence spaces and linear maps.
- *Differential linear logic* (Ehrhard, Regnier) begins with the idea that there is a similar decomposition of a category of *smooth* maps into the coKleisli category of a category of linear maps.
- Again, this idea arose from semantic considerations. Ehrhard constructed two Seely categories where there is just such a decomposition. These were the categories of *Köthe spaces* and *finiteness spaces*. Morphisms had a representation as power series, which could be differentiated.

Differential Linear Logic II

- The important point is that differentiation is represented as an inference rule.
- To see what the inference rule would be, consider the following situation. I have two Euclidean spaces, X and Y , and a smooth map between them. In our model, it would be a map $f: !X \rightarrow Y$. At a fixed point, its Jacobian matrix would be a linear map from X to Y . So the process of taking the Jacobian is a smooth map from X to linear maps from X to Y . This suggests an inference rule of the following form:

$$\frac{!X \vdash Y}{!X \vdash X \multimap Y}$$

- Or, equivalently:

$$\frac{!X \vdash Y}{X \otimes !X \vdash Y}$$

- By naturality of the above inference rule, it suffices to differentiate the identity map. So we require a map $d: X \otimes !X \rightarrow !X$.
- To state axioms, we must have additive structure on the Hom-sets. So we will assume finite biproducts.
- So a *differential category* (RB, Cockett, Seely) is a Seely category with a map of the above form satisfying basic differential identities, expressed coalgebraically.

Differential Linear Logic IV

- Note that combining biproducts and the exponential isomorphisms, we see that objects of the form $!X$ have the structure of a *bialgebra*, not just a coalgebra.
- In this case, it suffices to assume a map called *codereliction*:

$$\text{coder}: X \rightarrow !X$$

Consider $f: !X \rightarrow Y$ then define $df: X \otimes !X \rightarrow Y$ as the composite:

$$X \otimes !X \xrightarrow{\text{coder} \otimes \text{id}} !X \otimes !X \xrightarrow{\nabla} !X \xrightarrow{f} Y$$

Note that ∇ is the multiplication, existing by the bialgebra structure.

Convenient vector spaces (Frölicher, Kriegel)

- Convenient vector spaces are a special class of locally convex spaces.
- Note that in any topological vector space, one can take limits and hence talk about derivatives of curves. A curve is *smooth* if it has derivatives of all orders.
- The analogue of Cauchy sequences in locally convex spaces are called *Mackey-Cauchy sequences*.
- The convergence of Mackey-Cauchy sequences implies the convergence of all Mackey-Cauchy nets.

The following is taken from a long list of equivalences.

Theorem

Let E be a locally convex vector space. The following statements are equivalent:

- If $c: \mathbb{R} \rightarrow E$ is a curve such that $\ell \circ c: \mathbb{R} \rightarrow \mathbb{R}$ is smooth for every linear, continuous $\ell: E \rightarrow \mathbb{R}$, then c is smooth.
- Every Mackey-Cauchy sequence converges.
- Any smooth curve $c: \mathbb{R} \rightarrow E$ has a smooth antiderivative.

Definition

Such a vector space is called a *convenient* vector space.

Convenient vector spaces III: Bornology

The theory of bornological spaces axiomatizes the notion of bounded sets.

Definition

A *convex bornology* on a vector space V is a set of subsets \mathcal{B} (the bounded sets) such that

- \mathcal{B} is closed under finite unions.
- \mathcal{B} is downward closed with respect to inclusion.
- \mathcal{B} contains all singletons.
- If $B \in \mathcal{B}$, then so are $2B$ and $-B$.
- \mathcal{B} is closed under the convex hull operation.

Definition

A linear map between two such spaces is *bornological* if it takes bounded sets to bounded sets.

Convenient vector spaces IV: More bornology

- To any locally convex vector space V , we associate the *von Neumann bornology*. $B \subseteq V$ is bounded if for every neighborhood U of 0 , there is a real number λ such that $B \subseteq \lambda U$.
- This is part of an adjunction between locally convex topological vector spaces and convex bornological vector spaces. The topology associated to a convex bornology is generated by *bornivorous disks*. See Frölicher and Kriegl.

Theorem

Convenient vector spaces can also be defined as the fixed points of these two operations, which satisfy Mackey-Cauchy completeness and a separation axiom.

Convenient vector spaces \mathcal{V} : Key points

- The category \mathbf{Con} of convenient vector spaces and continuous linear maps forms a symmetric monoidal closed category. The tensor is a completion of the algebraic tensor. There is a convenient structure on the space of linear, continuous maps giving the **internal hom**.
- Then we can define:

Definition

A function $f: E \rightarrow F$ with E, F being convenient vector spaces is *smooth* if it takes smooth curves in E to smooth curves in F .

- We will denote the algebra of smooth functions from E to F by $C^\infty(E, F)$ and the real-valued functionals on E by $C^\infty(E)$.

Convenient vector spaces VI: More key points

- The category of convenient vector spaces and smooth maps is cartesian closed. This is an enormous advantage over Euclidean space, as it allows us to consider function spaces.
- There is a comonad on Con such that the smooth maps form the coKleisli category:

Define a map δ (Dirac delta function) as follows:

$$\delta: E \rightarrow \text{Con}(C^\infty(E), \mathbb{R}) \quad \delta(x)(f) = f(x)$$

Then we define $!E$ to be the Mackey closure of the span of the set $\delta(E)$.

Theorem (Frölicher, Kriegl)

- $!$ is a comonad.
- $!(E \oplus F) \cong !E \otimes !F$.
- Each object $!E$ has canonical bialgebra structure.



Convenient vector spaces VII: It's a model

Theorem (Frölicher, Kriegl)

The category of convenient vector spaces and smooth maps is the coKleisli category of the comonad $!$.

One can then prove:

Theorem (RB, Ehrhard, Tasson)

Con is a model of differential linear logic. In particular, it has a codereliction map given by:

$$\text{coder}(v) = \lim_{t \rightarrow 0} \frac{\delta(tv) - \delta(0)}{t}$$

Using this codereliction map, we can build a more general differentiation operator by precomposition:

Consider $f: !E \rightarrow F$ then define $df: E \otimes !E \rightarrow F$ as the composite:

$$E \otimes !E \xrightarrow{\text{coder} \otimes \text{id}} !E \otimes !E \xrightarrow{\nabla} !E \xrightarrow{f} F$$

Theorem (Frölicher, Kriegl)

Let E and F be convenient vector spaces. The differentiation operator

$$d: \mathcal{C}^\infty(E, F) \rightarrow \mathcal{C}^\infty(E, \text{Con}(E, F))$$

defined as

$$df(x)(v) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

is linear and bounded. In particular, this limit exists and is linear in the variable v .

A convenient differential category

The above results show that Con really is an optimal differential category.

- The differential inference rule is really modelled by a directional derivative.
- The coKleisli category really is a category of smooth maps.
- Both the base category and the coKleisli category are closed, so we can consider function spaces.

This seems to be a great place to consider *manifolds*. There is a well-established theory.

Kriegl, Michor-*The convenient setting for global analysis*

Definition

- A *chart* (U, u) on a set M is a bijection $u: U \rightarrow u(U) \subseteq E$ where E is a fixed convenient vector space, and $u(U)$ is an open subset.
- Given two charts (U_α, u_α) and (U_β, u_β) , the mapping $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}$ is called a *chart-changing*.
- An *atlas* or *smooth atlas* is a family of charts whose union is all of M and all of whose chart-changings are smooth.
- A (*convenient*) *manifold* is a set M with an equivalence class of smooth atlases.
- Smooth maps are defined as usual.

Lemma

A function between convenient manifolds is smooth if and only if it takes smooth curves to smooth curves.

This is a complicated subject.

Definition

A manifold M is *smoothly hausdorff* if smooth real-valued functions separate points.

Note that this implies:

- M is hausdorff in its usual topology, **which implies:**
- The diagonal is closed in the manifold $M \times M$.

These three notions are equivalent in finite-dimensions. In the convenient setting, the reverse implications are open. Note that the product topology on $M \times M$ is different than the manifold topology!

Smooth real-compactness

- There is a canonical map:

$$\delta: E \rightarrow \text{Hom}_{\text{Alg}}(C^\infty(E), \mathbb{R})$$

In finite-dimensions, it is a bijection, and allows one to recover the manifold from its C^∞ -algebra.

- For convenient vector spaces, it may or may not be a bijection. We say a convenient vector space is *smoothly real-compact*, if the above map is a bijection.

Theorem (Arias-de-Reyna, Kriegl, Michor)

The following classes of spaces are smoothly real-compact:

- *Separable Banach spaces.*
- *Arbitrary products of separable Fréchet spaces.*
- *Many more.*

Tangent spaces

The many equivalent notions of tangent in finite-dimensions now become distinct. See Kriegl-Michor.

Definition

Let E be a convenient vector space, and let $a \in E$. A *kinematic tangent vector* at a is a pair (a, X) with $X \in E$. Let $T_a E = E$ be the space of all kinematic tangent vectors at a .

The above should be thought of as the set of all tangent vectors at a of all curves through the point a .

For the second definition, let $C_a^\infty(E)$ be the quotient of $C^\infty(E)$ by the ideal of those smooth functions vanishing on a neighborhood of a . Then:

Definition

An *operational tangent vector* at a is a continuous derivation, i.e. a map

$$\partial: C_a^\infty(E) \rightarrow \mathbb{R}$$

such that

$$\partial(f \circ g) = \partial(f) \circ g(a) + f(a)\partial(g)$$

Note that every kinematic tangent vector induces an operational one via the formula

$$X_a(f) = df(a)(X)$$

where d is the directional derivative operator. Let $D_a E$ be the space of all such derivations.

Tangent spaces III

In finite dimensions, the above definitions are equivalent and the described operation provides the isomorphism. That is no longer the case here.

Let $Y \in E''$, the second dual space. Y canonically induces an element of $D_a E$ by the formula $Y_a(f) = Y(df(a))$. This gives us an injective map $E'' \rightarrow D_a E$. So we have:

$$T_a E \hookrightarrow E'' \hookrightarrow D_a E$$

Definition

E satisfies the *approximation property* if $E' \otimes E$ is dense in $\text{Con}(E, E)$ (This is basically the MIX map.).

Theorem (Kriegel, Michor)

If E satisfies the approximation property, then $E'' \cong D_a E$. If E is

- There is a notion of atlas category, due to Cockett and Crutwell. This seems to capture the notion of kinematic tangent vector well.
- But there doesn't seem to be a corresponding categorization of the operational tangent vector. This will make much more explicit use of the differential linear logic structure.
- Models of linear logic with this differential structure should allow for some form of cohomology.