Deep inference and probabilistic coherence spaces

Richard Blute*
Department of Mathematics and Statistics
University of Ottawa
Ottawa, Ontario, Canada
rblute@uottawa.ca

Prakash Panangaden*
Department of Computer Science
McGill University
Montreal, Quebec, Canada
prakash@cs.mcgill.ca

Sergey Slavnov*
Moscow State University of Railroad Engineering
Moscow, Russia
sslavnov@yandex.ru

Abstract

This paper proposes a definition of categorical model of the deep inference system BV, defined by Guglielmi. Deep inference introduces the idea of performing a deduction in the interior of a formula, at any depth. Traditional sequent calculus rules only see the roots of formulae. However in these new systems, one can rewrite at any position in the formula tree. Deep inference in particular allows the syntactic description of logics for which there is no sequent calculus. One such system is BV, which extends linear logic to include a noncommutative self-dual connective.

This is the logic our paper proposes to model. Our definition is based on the notion of a *linear functor*, due to Cockett and Seely. A BV-*category* is a linearly distributive category, possibly with negation, with an additional tensor product which, when viewed as a bivariant functor, is linear with a degeneracy condition. We show that this simple definition implies all of the key isomorphisms of the theory.

We consider Girard's category of *probabilistic coherence spaces* and show that it contains a self-dual monoidal structure in addition to the $*$-autonomous structure exhibited by Girard. This structure makes the category a BV-category. We believe this structure is also of independent interest, as well-behaved noncommutative operators generally are.

1 Introduction

This paper is an examination of *deep inference proof theory* [18, 7] from the perspective of categorical logic. In particular, we will propose a general notion of categorical model and give several examples of such structures.

Deep inference is an important new approach to the syntax of logical systems introduced by Guglielmi, and subsequently studied by a number of researchers. We mention the website http://alessio.guglielmi.name as an excellent source of information. Deep inference is not a single logical system, but rather a new approach to considering logic, in which sequent calculus is replaced by a new form of syntax with several attractive features.

*Research supported in part by National Science and Engineering Research Council, CANADA.

There are a number of deep inference logical systems, some corresponding to classical-style logics, and some to linear logics. The one that we consider is BV. This is an extension of multiplicative linear logic, designed to incorporate Retoré's noncommutative, self-dual connective *seq* [24]. We refer the reader to [18] for the inference rules and syntax of BV. As this is a paper on semantics, they will not play a role here.

Proofs in deep inference systems are reversible in the sense that one may invert them and dualize the connectives and still obtain a valid proof. This sort of duality is of course in sharp contrast to the sequent calculus. (This property is shared for example with the *two-sided proof nets* of [4].)

Also important is that deep inference systems allow for a very satisfactory treatment of the notion of context. Within deep inference systems, one can make substitutions within a proof at arbitrary depth. Formalizing a notion of covariant context that allows for such substitutions in the sequent calculus is notoriously difficult.

Perhaps even more importantly, deep inference allows for the consideration of systems which seemingly cannot be considered with sequent calculus at all. Here we are thinking particularly of Retoré's *pomset logic*, which is an extension of Girard's multiplicative linear logic MLL [16] to include a noncommutative self-dual connective. Retoré's work was inspired by semantic considerations. Such a connective, called *seq*, exists on the category of *coherence spaces* [16], and pomset logic was an attempt to capture this structure syntactically. However, pomset logic is not a sequent calculus in any standard sense. Subsequent work of Tiu [27] shows that no sequent calculus could capture this logical connective. However, the deep inference system BV handles the structure quite easily.

One of our goals is to develop a categorical semantics for the various deep inference systems. Here, we only consider the specific system BV. As far as we know, the first person to discuss the categorical structure of the linear logic deep inference systems was Hughes in [19]. Hughes, in considering the deep inference system corresponding to MLL, argues that many of the best features of deep inference are just as true of categorical proof theory. In particular, the ability to make substitutions at arbitrary depth within a proof corresponds categorically to the (bi)functoriality of the logical connectives.

But Hughes does not deal with the coherence issues at all. There is a valid reason not to have considered coherence issues for deep inference, as the usual techniques of categorical proof theory do not apply. Typically when trying to develop categorical semantics, one determines coherence conditions by examining the cut-elimination process. Cut-elimination is typically algorithmic, and there is an evident method of turning the steps of the cut-elimination process into coherence equations. Having done this, it is then evident that the denotation of an arbitrary proof in the logic is equal to the denotation of a cut-free proof.

However, deep inference systems do not have cut rules or satisfy cut-elimination in the usual sense. Thus while it is evident what functorial structure we will need to model the connectives of BV, the appropriate coherence conditions are far from clear. In this paper, we choose to model BV by using standard categorical structures from which much of the desired symmetry can then be derived. We believe that the simplicity of our definition together with the observation that the key isomorphisms are consequences imply that we have captured the structure correctly. In particular, the self-duality of the noncommutative connective seq is a consequence of our structure, as are the linear distributions relating tensor and seq, and seq and par.

The typical starting point for modelling the multiplicative fragment of linear logic is the notion of $*$-*autonomous category*, due to Barr [2, 25]. We instead consider the equivalent notion of *linearly distributive category with negation* [8, 4]. While this notion is equivalent, it is a more natural structure to consider as the multiplicative disjunction *par* is taken as a primitive. Further, the way that negation is added to an LDC is much closer to the way negation is introduced in BV. Indeed the paper [4] introduces a two-sided variant of Danos-Regnier proof nets as a way of analyzing the coherence problem for such categories. These nets are quite close to the syntax of BV and satisfy many of the same desirable symmetries.

We also make use of the notion of morphism between LDCs. These are the *linear functors* of Cockett and Seely [10]. This is not simply a functor commuting with the connectives and isomorphisms of an LDC, but rather a pair of functors, one of which is monoidal with respect to tensor and the other comonoidal with respect to par. There are also further natural transformations and coherence conditions required. The authors show that much of the additional structure one adds to multiplicative linear logic can be described as a linear functor. In particular, both the exponential and additive fragments of linear logic can be viewed as linear functors. In the $*$-autonomous case, the notion of linear functor reduces to that of monoidal functor, as expected. Furthermore, in the case of $*$-autonomous categories or equivalently LDCs with negation, the linear functor satisfies strong commutation properties with respect to negation. We make use of this commutation to derive the self-duality of our seq-connective.

We here introduce the notion of a *degenerate linear functor*, which is a linear functor such that the two functor components are equal. There are several consequences for the coherence conditions to this as well. Then we define a BV-category to be an LDC $\mathbf{C}$, together with an additional monoidal structure, with functor part denoted $\oslash$, such that the functor $\oslash\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$, is a degenerate linear functor. (There are some coherence requirements as well.) We then show that all of the desired properties for our models hold as a consequence of this simple definition. We also show that Retoré's original construction in coherence spaces gives an example.

Finally, to illustrate the generality of our definition, we also give a new example of a seq connective, on Girard's category of *probabilistic coherence spaces*. Girard introduces the category of probabilistic coherence spaces with the intent of using semantic and proof-theoretic ideas in the consideration of structures arising in quantum mechanics. Probabilistic coherence spaces were also studied extensively by Danos and Erhard in [11]. They wish to consider this notion as providing semantics of a probabilistic version of **PCF**, as well as a framework for considering probabilistic games in the sense of [12]. In their semantics, there is a natural interpretation of probabilistic **PCF** terms; in particular, a term of type **Int** is modelled as a subprobability distribution, which gives the probability that the term reduces to integer n.

It is ultimately hoped that the ideas of the present paper, especially the semantic model of probabilistic coherence spaces, can be used to analyze the *discrete quantum causal dynamics* of [6].

Acknowledgements-The authors would like to thank Robin Cockett, Alessio Guglielmi, Francois Lamarche, Robert Seely and Lutz Strassburger for numerous helpful conversations. All three authors received the support of NSERC.

2 Linear functors

In [10], Cockett and Seely introduce the notion of a *linearly distributive functor*, hereafter called *linear functor*. This is the proper notion of morphism between linearly distributive categories (LDCs). This will provide the foundation for our definition of model of BV. So we here review the basic idea, and introduce the new notion of *degenerate linear functor*. We assume familiarity with the notion of linearly distributive category. See [8, 4] where they are called weakly distributive. Some information on them is contained in the appendix.

Definition 2.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be monoidal categories. Then a functor $F\colon \mathbf{X}\to\mathbf{Y}$ is *monoidal* if it is equipped with natural transformations (denoting the tensor unit for both $\mathbf{X}$ and $\mathbf{Y}$ by $\top$):

$$m_\otimes : F(A) \otimes F(B) \to F(A \otimes B)$$
$$m_\top : \top \to F(\top)$$

satisfying standard equations. Conversely F is *comonoidal* if equipped with transformations:

$$n_\otimes : F(A \otimes B) \to F(A) \otimes F(B)$$
$$n_\top : F(\top) \to \top$$

satisfying dual equations.

If the categories in question are also symmetric, then one must also assume commutation with the symmetries, e.g.

$$\begin{array}{ccc}
F(A \otimes B) & \xrightarrow{m_\otimes} & F(A) \otimes F(B) \\
\Big\downarrow F(c) & & \Big\downarrow c \\
F(B \otimes A) & \xrightarrow{m_\otimes} & F(B) \otimes F(A)
\end{array}$$

Definition 2.2 Let $\mathbf{X}$ and $\mathbf{Y}$ be symmetric LDCs. A *linear functor* $F : \mathbf{X} \to \mathbf{Y}$ is a pair of functors $F_\otimes : \mathbf{X} \to \mathbf{Y}$ and $F_\parr : \mathbf{X} \to \mathbf{Y}$ such that $F_\otimes$ is symmetric monoidal with respect to tensor and $F_\parr$ is symmetric comonoidal with respect to par. So there must be natural transformations:

$$\nu_\otimes : F_\otimes(A \parr B) \to F_\parr(A) \parr F_\otimes(B)$$
$$\nu_\parr : F_\otimes(A) \otimes F_\parr(B) \to F_\parr(A \otimes B)$$

All of this data must satisfy a number of coherence conditions as specified in [10].

In [10], it is demonstrated that much of the crucial structure of linear logic falls into the framework of linear functor. For example, both the exponentials and the additives form linear functors. But it is also a familiar notion in the following sense:

Theorem 2.3 (Cockett-Seely) *A linear functor between $*$-autonomous categories (viewed as* LDC*s) is the same thing as a monoidal functor.*

We are interested in a special case of this definition, in which the two functors are equal. We call this a *degenerate linear functor*. In this situation, one can take $\nu_\otimes = m_\parr$ and $\nu_\parr = n_\otimes$. These assumptions also greatly simplify the coherence conditions.

We now record the complete definition:

Definition 2.4 Let $\mathbf{X}$ and $\mathbf{Y}$ be symmetric LDCs. Then a *degenerate linear functor* from $\mathbf{X}$ to $\mathbf{Y}$ is a functor $F : \mathbf{X} \to \mathbf{Y}$ such that F is symmetric monoidal with respect to tensor and symmetric comonoidal with respect to par. This means that there are maps $m_\otimes, m_\top, n_\parr$ and $n_\perp$ as in Definition 2.1. (We will generally drop the subscripts when this causes no confusion.) We further require the following diagrams to commute (noting that as usual δ is the linear distribution):

$$
\begin{array}{ccc}
F(A)\otimes F(B\parr C) & \xrightarrow{id\otimes n} & F(A)\otimes(F(B)\parr F(C)) \\
{\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\
F(A\otimes(B\parr C)) & & (F(A)\otimes F(B))\parr F(C) \\
{\scriptstyle F(\delta)}\big\downarrow & & \big\downarrow{\scriptstyle m\parr id} \\
F((A\otimes B)\parr C) & \xrightarrow[n]{} & F(A\otimes B)\parr F(C)
\end{array}
$$

$$
\begin{array}{ccc}
F(A\parr B)\otimes F(C) & \xrightarrow{n\otimes id} & (F(A)\parr F(B))\otimes F(C) \\
{\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\
F((A\parr B)\otimes C) & & F(A)\parr(F(B))\otimes F(C)) \\
{\scriptstyle F(\delta)}\big\downarrow & & \big\downarrow{\scriptstyle id\parr m} \\
F((A\parr(B\otimes C)) & \xrightarrow[n]{} & F(A)\parr F(B\otimes C)
\end{array}
$$

One of the consequences of this definition is that such a functor commutes with any existing negations:

Lemma 2.5 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be symmetric* LDC*s with negation. If* $F\colon \mathbf{X}\to\mathbf{Y}$ *is a degenerate linear functor, then for all objects* A*, one has* $F(A)^\perp \cong F(A^\perp)$*.*

Proof. This can be proved directly, but it also follows from Remark 6 of [10]. The isomorphisms are constructed as follows.

The map $F(A^\perp)\longrightarrow F(A)^\perp$ is the transpose of the map $F(A^\perp)\otimes F(A)\longrightarrow\perp$ obtained by

$$F(A^\perp)\otimes F(A)\to F(A^\perp\otimes A)\to F(\perp)\to\perp$$

where each map is evident.

The map $F(A)^\perp\to F(A^\perp)$ is the transpose of the map $\top\to F(A)\parr F(A^\perp)$ obtained by

$$\top\to F(\top)\to F(A\parr A^\perp)\to F(A)\parr F(A^\perp)$$

□

2.1 Linear natural transformations

We now review the appropriate notions of transformation between linear functors.

Definition 2.6 • Let $\mathbf{X}$ and $\mathbf{Y}$ be monoidal categories. Let $F, G\colon \mathbf{X} \to \mathbf{Y}$ be monoidal functors. A *monoidal transformation* $\theta\colon F \to G$ is a natural transformation such that

$$m_{\otimes};\theta = \theta \otimes \theta; m_{\otimes}\colon FA \otimes FB \to G(A \otimes B) \qquad m_{\top};\theta = m_{\top}\colon \top \to G(\top)$$

The notion of *comonoidal transformation* is defined dually.

- Let $\mathbf{X}$ and $\mathbf{Y}$ be LDCs. Let $F, G\colon \mathbf{X} \to \mathbf{Y}$ be linear functors. A *linear transformation* is a pair of natural transformations

$$\theta_{\otimes}\colon F_{\otimes} \to G_{\otimes} \qquad \theta_{\invamp}\colon G_{\invamp} \to F_{\invamp}$$

such that $\theta_{\otimes}$ is monoidal with respect to tensor, $\theta_{\invamp}$ is comonoidal with respect to par and several coherence conditions are satisfied. See [10].

3 Definition of BV-category

The fundamental structure which our models of BV will carry is what we call a *weak interchange structure* on an LDC. Basically, this will consist of an additional monoidal structure for which the tensor product, viewed as a 2-variable functor, is degenerate linear and the structure maps are all linear natural transformations.

Definition 3.1 A *pre-*BV*-structure* on a symmetric LDC $\mathbf{C}$ is an additional monoidal structure $(\mathbf{C}, \oslash, I)$ such that the functor (called seq):

$$\oslash\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$$

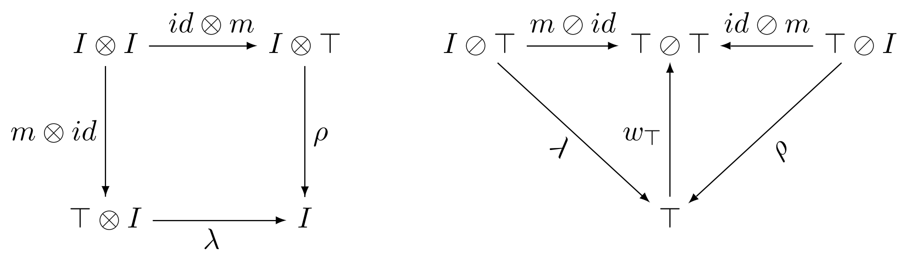
is a symmetric degenerate linear functor, and the structure isomorphism:

$$\alpha\colon A \oslash (B \oslash C) \longrightarrow (A \oslash B) \oslash C$$

is a linear natural transformation, where $A \oslash (B \oslash C)$ is given the evident degenerate linear functor structure.

We call a symmetric LDC with a pre-BV-structure a *pre-*BV*-category.*

A BV-*category* is a pre-BV-category together with an isomorphism $m\colon I \to \top$ such that



The first equation says that m is an isomix map. (For more on this, see the appendix.) The second says that m^{-1} acts as a counit for the comultiplication $w_{\top}$. Together we call these equations the m-equations.

4 Weak interchange structure

While the definitions of pre-BV-category and BV-category are quite concise, they contain a large amount of information, which we now unpack. The key is the notion of a *weak interchange structure.* We begin by focussing on the monoidal case.

The result seems to be a new example of a structure for categories with (multiple) monoidal structures. It obviously takes its name from the interchange rule for double categories, although there we have an equality, and here just a natural transformation. We note that Melliés has also used double categories for the semantics of linear logic, although the setting seems to be entirely different [23].

4.1 Weak interchanges in monoidal categories

Definition 4.1 Suppose that a category $(\mathbf{C}, \otimes, \top)$ is a symmetric monoidal category. Then a *weak interchange* is an additional monoidal structure $(\mathbf{C}, \oslash, I)$ and natural transformations:

$$w = w_{\otimes} \colon (R \oslash U) \otimes (T \oslash V) \to (R \otimes T) \oslash (U \otimes V)$$
$$w_{\top} \colon \top \to \top \oslash \top$$

such that several diagrams commute which we specify now.

- We must first make sure that the weak interchange commutes with the associativities. This amounts to:

$$\begin{array}{ccc}
[(R \oslash U) \otimes (T \oslash V)] \otimes (S \oslash W) & \xrightarrow{\alpha} & (R \oslash U) \otimes [(T \oslash V) \otimes (S \oslash W)] \\
\downarrow{\scriptstyle w \otimes id} & & \downarrow{\scriptstyle id \otimes w} \\
[(R \otimes T) \oslash (U \otimes V)] \otimes (S \oslash W) & & (R \oslash U) \otimes [(T \otimes S) \oslash (V \otimes W)] \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\
[(R \otimes T) \otimes S] \oslash [(U \otimes V) \otimes W] & \xrightarrow{\alpha} & [R \otimes (T \otimes S)] \oslash [U \otimes (V \otimes W)]
\end{array}$$

- The weak interchange must commute with the unit isomorphisms. This amounts to the following equation, and its dual (with the unit on the right):

$$
\begin{CD}
\top \otimes (A \oslash B) @>{\lambda^{-1}}>> A \oslash B \\
@V{w_\top \otimes id}VV @AA{\lambda^{-1} \oslash \lambda^{-1}}A \\
(\top \oslash \top) \otimes (A \oslash B) @>>{w}> (\top \otimes A) \oslash (\top \otimes B)
\end{CD}
$$

- The weak interchange must commute with the commutativity isomorphism of $\otimes$. (We remind the reader that only $\otimes$ is assumed to be commutative, and not $\oslash$.)

$$
\begin{CD}
(R \oslash U) \otimes (T \oslash V) @>{c}>> (T \oslash V) \otimes (R \oslash U) \\
@V{w}VV @VV{w}V \\
(R \otimes T) \oslash (U \otimes V) @>>{c \otimes c}> (T \otimes R) \oslash (V \otimes U)
\end{CD}
$$

- The associativity isomorphism for $\oslash$ must be monoidal with respect to $\otimes$. This leads to the equation:

$$
\begin{CD}
[A \oslash (B \oslash C)] \otimes [A' \oslash (B' \oslash C')] @>{w; id \oslash w}>> (A \otimes A') \oslash [(B \otimes B') \oslash (C \otimes C')] \\
@V{\alpha \otimes \alpha}VV @VV{\alpha}V \\
[(A \oslash B) \oslash C] \otimes [(A' \oslash B') \oslash C'] @>{w; w \oslash id}>> [(A \otimes A') \oslash (B \otimes B')] \oslash (C \otimes C')
\end{CD}
$$

- The map $w_\top$ must be coassociative, i.e.

$$
\begin{CD}
\top @>{w_\top}>> \top \oslash \top \\
@V{w_\top}VV @VV{id \oslash w_\top}V \\
\top \oslash \top @>>{w_\top \oslash id}> \top \oslash \top \oslash \top
\end{CD}
$$

We have the following evident observation:

Theorem 4.2 *Let* $\mathbf{C}$ *be a monoidal category. Weak interchanges on* $\mathbf{C}$ *correspond bijectively to monoidal structures* $(\oslash, I)$ *on* $\mathbf{C}$ *such that the functor* $\oslash\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *is symmetric monoidal, and the associativity isomorphism for* $\oslash$ *is a monoidal natural transformation.*

4.2 Weak interchanges in LDCs

So far, we have only been describing a weak interchange structure on a monoidal category. In the case of a symmetric LDC, we also assume duals of the form

$$
\begin{gathered}
w' = w'_{\invamp} \colon (C \invamp E) \oslash (D \invamp F) \to (C \oslash D) \invamp (E \oslash F) \\
w'_{\perp} \colon \perp \oslash \perp \to \perp
\end{gathered}
$$

satisfying the duals to the above equations. We then also require the following diagram and its symmetric dual:

$$
\begin{array}{ccc}
(A \oslash B) \otimes [(C \invamp E) \oslash (D \invamp F)] & \xrightarrow{\;w\;} & [A \otimes (C \invamp E)] \oslash [B \otimes (D \invamp F)] \\
\Big\downarrow{\scriptstyle id \otimes w'} & & \Big\downarrow{\scriptstyle \delta \oslash \delta} \\
(A \oslash B) \otimes [(C \oslash D) \invamp (E \oslash F)] & & [(A \otimes C) \invamp E] \oslash [(B \otimes D) \invamp F] \\
\Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle w'} \\
[(A \oslash B) \otimes (C \oslash D)] \invamp (E \oslash F) & \xrightarrow{\;id \otimes w\;} & [(A \otimes C) \oslash (B \otimes D)] \invamp (E \oslash F)
\end{array}
$$

Theorem 4.3 *Let* $\mathbf{C}$ *be a symmetric* LDC*. Then weak interchange structures correspond bijectively to pre-*BV*-structures on* $\mathbf{C}$*.*

4.3 Consequences of BV-structure

Theorem 4.4 *Suppose* $(\mathbf{C}, \otimes, \top)$ *and* $(\mathbf{C}, \oslash, I)$ *are monoidal categories connected by a weak interchange, and an isomorphism* $m \colon I \to \top$ *satisfying the* m*-equations of Definition 3.1. Then the natural transformation* δ_L *defined by*

$$
\begin{array}{l}
(R \oslash U) \otimes V \xrightarrow{\;id \otimes \lambda\;} (R \oslash U) \otimes (I \oslash V) \xrightarrow{\;id \otimes (m \oslash id)\;} (R \oslash U) \otimes (\top \oslash V) \\
\xrightarrow{\;w\;} (R \otimes \top) \oslash (U \otimes V) \xrightarrow{\;\rho \oslash id\;} R \oslash (U \otimes V)
\end{array}
$$

together with the symmetric variant δ_R *defined by evident analogy determine a linear distribution.*

Proof. This is a lengthy exercise in diagram chasing. We verify one of the equations. Refering to the numbering of [8], equation 5 of the definition of LDC is:

$$
\begin{array}{ccc}
\top \otimes (A \oslash B) & & \\
\Big\downarrow \delta & \searrow^{\lambda^{-1}} & \\
(\top \otimes A) \oslash B & \xrightarrow[\lambda^{-1} \oslash id]{} & A \oslash B
\end{array}
$$

In our system, this becomes:

$$
\begin{array}{ccc}
\top \otimes (A \oslash B) & \xrightarrow{\lambda^{-1}} & A \oslash B \\
\Big\downarrow \rho \otimes id & & \\
(\top \oslash I) \otimes (A \oslash B) & & \Big\uparrow \lambda^{-1} \oslash \lambda^{-1} \\
\Big\downarrow (id \oslash m) \otimes id & & \\
(\top \oslash \top) \otimes (A \oslash B) & \xrightarrow[w]{} & (\top \otimes A) \oslash (\top \otimes B)
\end{array}
$$

This follows from the observation that $w_\top = (id \oslash m) \circ \rho$, and then one of the basic equations for BV-category. □

Corollary 4.5 *In the case of a* BV*-category, the resulting linearly distributive structure is isomix. (For information on isomix categories, see the Appendix.)*

Proof. This follows immediately from the definition of an *isomix* LDC, as described in the Appendix. □

Also, we have the following fundamental property of Retoré's seq connective.

Theorem 4.6 *Given a pre-*BV*-category* **C** *with negation, the functor* $\oslash$*, hereafter called* seq*, is self-dual. Explicitly, the isomorphism is given by the transpose of the map:*

$$(R^\perp \oslash U^\perp) \otimes (R \oslash U) \xrightarrow{w} (R \otimes R^\perp) \oslash (U^\perp \otimes U) \xrightarrow{\nu \oslash \nu} \perp \oslash \perp \xrightarrow{w'} \perp$$

Exploiting the dualities of a category with negation, we can claim:

Theorem 4.7 *Given a* BV*-category with negation, we have the following:*

- *A linearly distributive structure from* $(\mathbf{C}, \oslash, I)$ *to* $(\mathbf{C}, \invamp, \bot)$*, obtained as the dual of that of Lemma 4.4.*

- $(\mathbf{C}, \oslash, I)$ *and* $(\mathbf{C}, \invamp, \bot)$ *are connected by a weak interchange.*

- *An isomorphism* $m' \colon \bot \to I$ *giving an isomix structure from* $\oslash$ *to* $\invamp$*.*

- *The composite* $m \circ m' \colon \bot \to \top$ *makes the original* LDC *an isomix category.*

5 Retoré's noncommutative operator on coherence spaces

We assume the reader is familiar with Coh, the category of coherence spaces and linear maps, as well as the $*$-autonomous structure of Coh. Christian Retoré in [24] exhibited an additional monoidal structure on Coh.

Definition 5.1 Suppose that $X = (|X|, \frown\!\!\!\smile_X)$ and $Y = (|Y|, \frown\!\!\!\smile_Y)$ are coherence spaces. Define a new coherence space $X \oslash Y$ by defining a symmetric, reflexive relation on $|X| \times |Y|$ by the rule:

$$(x, y) \frown\!\!\!\smile (x', y') \text{ if and only if } (x \frown x' \text{ and } y = y') \text{ or } y \frown y'$$

We call this connective seq.

Theorem 5.2 *(Retoré) The* seq*-connective has the following properties:*

- $\oslash$ *is noncommutative.*

- $\oslash$ *is coherently associative, with unit given by the one-point coherence space. This gives an additional monoidal structure to* Coh*.*

- $\oslash$ *is self-dual, i.e.* $(X \oslash Y)^\bot = X^\bot \oslash Y^\bot$*.*

- *There are canonical linear morphisms:*

$$X \otimes Y \to X \oslash Y \to X \invamp Y$$

In each case, the morphism has the identity as its underlying relation.

We can now more succinctly state:

Theorem 5.3 Coh *is a* BV*-category.*

Proof. All of the necessary structure maps are certainly present. The only issue is the commutativity of the coherence equations. This is straightforward. Note in particular that all of the necessary diagrams commute in the category of relations. $\square$

6 Probabilistic coherence spaces

Jean-Yves Girard, in [17], introduced the notion of a *probabilistic coherence space*, with an eye towards applying ideas from linear logic to the analysis of quantum structure. We begin by reviewing the basic definitions.

Definition 6.1 *(Girard)* Let X be a finite set. In keeping with the language of coherence spaces, we will refer to X as the *carrier*. Let $\mathbf{R}(X)$ denote the set of all functions of the form $f\colon X \to \mathsf{R}^+$ where R^+ denotes the nonnegative reals. These will be referred to as *measures*. Then two elements of $f, g \in \mathbf{R}(X)$ are said to be *polar* (notation: $f \perp g$) if

$$\sum_{x\in X} f(x)g(x) \leq 1$$

We will denote $\sum_{x\in X} f(x)g(x)$ by $\langle f, g\rangle$.
Then, if A is a subset of $\mathbf{R}(X)$, one defines $A^\perp$ in the obvious way, i.e.

$$A^\perp = \{f \in \mathbf{R}(X) | \forall g \in \mathbf{R}(X), f \perp g\}$$

A *probabilistic coherence space* (PCS) is a finite set X with an $A \subseteq \mathbf{R}(X)$ such that $A = A^{\perp\perp}$. The elements of A will be called *allowable measures*. We will frequently denote the PCS (X, A) simply by A.

Theorem 6.2 *(Girard) Let (X, A) be a PCS. Then*

- *A is nonempty.*
- *A is a closed, convex subset of $\mathbf{R}(X)$.*
- *A is downward closed under the pointwise order on R^X.*

Conversely, any subset satisfying these properties is a PCS.

We have the following standard result for constructions of this sort. It can be seen as an instance of the general notion of *abstract orthogonality* due to Hyland and Schalk [20].

Lemma 6.3 *Every subset of $\mathbf{R}(X)$ of the form $A^\perp$ is a PCS.*

In contrast to ordinary coherence spaces, it is easier to define linear implication first, and then use de Morgan duality to define tensor.

Let $\Phi \in \mathbf{R}(X \times Y)$ and $f \in \mathbf{R}(X)$. Define $[\Phi]f \in \mathbf{R}(Y)$ by:

$$[\Phi]f(y) = \sum_{x\in X} \Phi(x, y)f(x)$$

This formula defines a bijection between $\mathbf{R}(X \times Y)$ and the linear maps from $\mathbf{R}(X)$ to $\mathbf{R}(Y)$.

Theorem 6.4 *The set of linear maps from $\mathbf{R}(X)$ to $\mathbf{R}(Y)$ which take A to B is a PCS, when viewed as a subset of $\mathbf{R}(X \times Y)$. This PCS will be denoted $A \multimap B$.*

We fist note that since X and Y are finite, we have the isomorphism of positive cones of vector spaces:

$$\mathbf{R}(X \times Y) \cong \mathbf{R}(X) \otimes \mathbf{R}(Y)$$

So we will now represent elements of $\mathbf{R}(X \times Y)$ by $\Sigma_{i \in I} f_i \otimes g_i$, with $f_i \in \mathbf{R}(X)$ and $g_i \in \mathbf{R}(Y)$.

Now the theorem follows from the observation:

$$A \multimap B = \{f \otimes g | f \in A, g \in B^\perp\}^{\perp\perp}$$

Then one defines

$$A \otimes B = (A \multimap B^\perp)^\perp$$

One can equivalently define $A \otimes B$ as the convex closure of $\{f \otimes g | f \in A, g \in B\}$.

We obtain a category by taking as morphisms from (X, A) to (Y, B) the linear maps in $A \multimap B$. This category will be denoted **PCS**. We have:

Theorem 6.5 *(Girard)* **PCS** *is an isomix ∗-autonomous category.*

Proof. All of the necessary structure is contained in Girard's constructions defined above. We note that the tensor product and closed structure have already been defined. The unit for the tensor is $I = (\{*\}, [0, 1])$, i.e. the carrier is the one-point set and the allowable measures are those mapping $*$ to the closed unit interval.

One can readily verify that this is also the unit for par, as well as a dualizing object. □

We will in fact restrict to a subcategory which is more appropriate for our purpose.

Definition 6.6 Let $A = (X, A)$ be a PCS. Then A is *bounded* if for all $g \in \mathbf{R}(X)$,

$$\sup_{f \in A} \langle f, g \rangle < \infty$$

A is *replete* if for all $g \in \mathbf{R}(X)$, there exists $f \in A$ with $\langle f, g \rangle > 0$.

A PCS which is bounded and replete will be called a brPCS. The full subcategory of bounded replete spaces will be denoted **BRPCS**.

Lemma 6.7 *The properties of being bounded and replete are dual, i.e. if A is bounded, then $A^\perp$ is replete, and vice-versa. In particular, if A is bounded and replete, so is $A^\perp$.*

We also note that the bounded, replete objects are closed under the operations of tensor and negation, and so:

Remark 6.8 **BRPCS** *is a ∗-autonomous subcategory.*

6.1 The seq-connective for bounded, replete PCS

Definition 6.9 We define the seq functor on the category **PCS** by the formula:

$$A \oslash B = \{\sum_{i \in I} f_i \otimes g_i | \forall i\, f_i \in A \text{ and } \sum_{i \in I} g_i \in B\}$$

The intuition here is that this is a linearization of the definition of the operator $\oslash$ in the category of (ordinary) coherence spaces, using $+$ and $\otimes$, instead of union and cartesian product.

We need first to see that we do in fact have an object in **BRPCS**. Hence we give a direct proof.

Theorem 6.10 *For any bounded, replete* PCS*, we have the equation:*

$$(A \oslash B)^{\perp} = A^{\perp} \oslash B^{\perp}$$

Proof. We first show that $A^{\perp} \oslash B^{\perp} \subseteq (A \oslash B)^{\perp}$. So let $\sum f_i \otimes g_i \in A^{\perp} \oslash B^{\perp}$. Thus for all i, $f_i \in A^{\perp}$ and $\sum g_i \in B^{\perp}$. Let $\sum h_j \otimes k_j \in A \oslash B$. So for all j, $h_j \in A$ and $\sum k_j \in B$. We have

$$\langle \sum_i f_i \otimes g_i, \sum_j h_j \otimes k_j \rangle = \sum_{i,j} \langle f_i, h_j \rangle \cdot \langle g_i, k_j \rangle \leq [\max_{i,j} \langle f_i, h_j \rangle] \cdot \sum_{i,j} \langle g_i, k_j \rangle \leq 1$$

So $\sum f_i \otimes g_i \in (A \oslash B)^{\perp}$.

For the converse, let $\sum f_i \otimes g_i \in (A \oslash B)^{\perp}$. We may suppose that the set $\{g_i\}$ is linearly independent, and thus there exists a *dual basis* $\{h_i\}$ such that

$$\langle h_i, g_j \rangle = \delta_{ij}$$

We also suppose that, for all i,

$$S_i = \sup_{f \in A} \langle f_i, f \rangle = 1$$

This is allowable, since we know that $S_i \neq \infty$ because A is bounded. Also $S_i \neq 0$, since A is replete. So if $S_i \neq 1$ for some i, we can replace the original element with

$$\sum \frac{1}{S_i} f_i \otimes S_i g_i$$

We note that given this assumption, we have that $f_i \in A^{\perp}$, as desired. Also since A is a closed set, for every i, there exists an $F_i \in A$ with $\langle f_i, F_i \rangle = 1$. It remains to show that $\sum g_i \in B^{\perp}$.

For contradiction, suppose that this is not the case. Then there exists a $g \in B$ with $\langle \sum g_i, g \rangle > 1$. Let $g = \sum \lambda_j h_j$. We note that this implies $\sum F_i \otimes \lambda_i h_i \in A \oslash B$. We now calculate as follows:

$$\langle \sum F_i \otimes \lambda_i h_i, \sum f_j \otimes g_j \rangle = \sum_i \langle F_i, f_i \rangle \langle \lambda_i h_i, g_i \rangle = \sum_i \langle \lambda_i h_i, g_i \rangle > 1$$

But this contradicts that $\sum f_j \otimes g_j \in (A \oslash B)^{\perp}$. $\square$

Corollary 6.11 *If A and B are bounded, replete* **PCS**, *then so is $A \oslash B$.*

Proof. The bounded and replete properties are straightforward. The fact that the space is a **PCS** follows from the self-duality of the connective. Evidently, $(A \oslash B)^{\perp\perp} = A \oslash B$. □

Note that we have also proven the following, which will be required in the next theorem:

Lemma 6.12 *Each element of $A \oslash B$ can be written in the form $\sum_i f_i \otimes g_i$, where $f_i \in A$, $\sum_i g_i \in B$ and the set $\{g_i\}_{i \in I}$ is linearly independent.*

Theorem 6.13 *The structure $(\mathbf{BRPCS}, \oslash, I)$ is a monoidal category.*

Proof. Let $v = \sum_i (\sum_j f_{ij} \otimes g_{ij}) \otimes h_i \in (A \oslash B) \oslash C$, where

$$\sum_i h_i \in C, \quad \forall i \ \sum_j g_{ij} \in B, \text{ and } \quad \forall ij \ f_{ij} \in A$$

.

Then we have

$$\sum_{ij} g_{ij} \otimes h_i = \sum_i (\sum_j g_{ij}) \otimes h_i \in B \oslash C \text{ and } \quad v = \sum_{ij} f_{ij} \otimes (g_{ij} \otimes h_i)$$

So $v \in A \oslash (B \oslash C)$.

Conversely. let $v = \sum_i f_i \otimes w_i \in A \oslash (B \oslash C)$. So $f_i \in A$ and $w = \sum_i w_i \in B \oslash C$. We note that $w = \sum_j g_j \otimes h_j$ with $g_j \in B$ and $\sum_j h_j \in C$. By the previous lemma, we may assume the set $\{h_j\}_{j \in J}$ is linearly independent, and hence may be completed to a basis. Then for each w_i, we may write:

$$w_i = \sum_j g_{ij} \otimes h_j$$

So:

$$\sum_i w_i = \sum_{ij} g_{ij} \otimes h_j = \sum_j (\sum_i g_{ij}) \otimes h_j \text{ with } \quad \sum_i g_{ij} = g_j \in B$$

Hence:

$$v = \sum_i f_i \otimes (\sum_{ij} g_{ij} \otimes h_j) = \sum_j (\sum_i f_i \otimes g_{ij}) \otimes h_j$$

It is straightforward to verify that this last expression describes an element of $(A \oslash B) \oslash C$.

The unit is a one-point set, with the unit interval as its allowable measures. □

Note that the unit is the same for all three connectives. We next note that the seq-connective is indeed intermediate to tensor and par, i.e.

Theorem 6.14 *In the category* **PCS***, we have inclusions:*

$$A \otimes B \subseteq A \oslash B \subseteq A \parr B$$

Proof. We will show that $A \oslash B \subseteq A^\perp \multimap B$. So let $\sum f_i \otimes g_i \in A \oslash B$. Let $h \in A^\perp$. Then

$$[\sum_i f_i \otimes g_i](h)(y) = \sum_x \sum_i f_i(x) g_i(y) h_i(x) = \sum_i g_i(y) \sum_x f_i(x) h_i(x) \leq \sum_i g_i(y)$$

By downward closure, we conclude that the lefthand side is in B.

For the other inclusion, one can simply apply the dualizing functor to the first inequality to obtain:

$$A^\perp \otimes B^\perp = (A^\perp \multimap B)^\perp \subseteq (A \oslash B)^\perp = A^\perp \oslash B^\perp$$

□

Next we demonstrate the existence of the crucial map that we need:

Lemma 6.15 *The category* **BRPCS** *has a weak interchange, i.e. a map:*

$$w: (R \oslash U) \otimes (T \oslash V) \to (R \otimes T) \oslash (U \otimes V)$$

satisfying the previously stated conditions.

Proof. A typical element of $(R \oslash U) \otimes (T \oslash V)$ is of the form $v = \sum_i s_i \otimes w_i$, with $s_i \in R \oslash U$ and $w_i \in T \oslash V$. Thus $s_i = \sum_j r_j \otimes u_j$ and $r_j \in R, \sum_j u_j \in U$. Also $w_i = \sum_k t_k \otimes v_k$, satisfying similar conditions. Then such an element is mapped by the weak interchange to

$$\sum_{j,k} (r_j \otimes t_k) \otimes (u_j \otimes v_k)$$

It is then straightforward to verify that this vector is in $(R \otimes T) \oslash (U \otimes V)$ □

We now claim the main result for this section.

Theorem 6.16 **BRPCS** *is a* BV*-category.*

Proof. Again, we have established all of the necessary morphisms, and the commutativity of the coherence diagrams is straightforward. □

7 Conclusion

7.1 Related work

There has been other work on categorical versions of various deep inference systems prior to this. In addition to the Hughes work already mentioned [19], there are the works of Führmann-Pym [15], as well as McKinley [22]. (In particular, McKinley also noted that the inference rules for the seq-connective amount to monoidality.) These both consider classical logic deep inference systems rather than BV. The fundamental idea here is to replace the usual notion of equivalence between proofs with inequalities, thereby avoiding the semantic collapse which occurs in naive attempts to model classical logic categorically.

Further work on categorical models of classical logic has focussed on the significance of the *medial rule* as considered in [7]. We mention in particular the works of Lamarche and Strassburger [21, 26]. The medial rule is of a similar type as our *weak interchange*, and occurs in a very different context.

The category of probabilistic coherence spaces is also considered by Danos and Ehrhard in [11]. In that paper, the authors show that this category is not only $*$-autonomous, but supports the structure of a full model of classical linear logic. In particular, they construct a comonad modelling the exponential fragment. They then show that in the associated model of the λ-calculus, one can interpret a probabilistic version of PCF.

7.2 Future work

There are several ideas worth further exploration arising from this paper. First would be the extension of the notion of probabilistic coherence space to general measure spaces, as opposed to finite sets. It is likely that the resulting category will no longer be closed, but rather have a *nuclear ideal* in the sense of [1].

Also we would like to see how general our formula is. In particular, it should be applicable to categories such as Girard's category of *quantum coherence spaces* [17]. Ehrhard's category of *finiteness spaces* [13], or Ehrhard's category of *Köthe spaces* [14].

We are hopeful that we can use the structure of BV to improve upon the *discrete quantum causal dynamics* of [6]. The additional connective of BV should yield a better encoding than MLL, as used in [6]. This is the subject of an ongoing discussion between Blute, Guglielmi, Panangaden and Strassburger.

Finally, it is reasonable to ask whether this definition of model of BV is definitive. This will only be settled when the relationship between proof nets for BV and the free BV-category is established. This issue must be postponed for a later day. We claim here that our equations form a minimal basis for the correct notion of model. Surely any notion of model will satisfy these eqations. We claim here only that they are sufficient to generate the key isomorphisms of the theory, and that the notion of degenerate linear functor provides a succinct and convenient notion for organizing this data.

References

[1] S. Abramsky, R. Blute, P. Panangaden, Nuclear and trace ideals in tensored $*$-categories, Journal of Pure and Applied Algebra 143, pp. 3-47, (2000).

[2] M. Barr. *$*$-autonomous categories*. Springer Lecture Notes in Mathematics 752, (1980).

[3] R. Blute. Linear logic, coherence and dinaturality. Theoretical Computer Science 115 pp. 3–42 (1993).

[4] R. Blute, J. R. B. Cockett, R. A. G. Seely and T. Trimble. Natural deduction and coherence for weakly distributive categories. Journal of Pure and Applied Algebra 13, pp. 229–296, (1996).

[5] R. Blute, J. R. B. Cockett, R. A. G. Seely. The logic of linear functors. Mathematical Structures in Computer Science 12, pp. 513-539, (2002)

[6] R. Blute, I. Ivanov, P. Panangaden. Discrete quantum causal dynamics. International Journal of Theoretical Physics 42, pp. 2025-2041, (2003).

[7] K. Brünner. *Deep inference and symmetry in classical proofs.* Ph. D. Thesis, Dresden Technical University, (2003).

[8] J.R.B. Cockett and R.A.G. Seely. Weakly distributive categories. Journal of Pure and Applied Algebra 114, pp. 133-173, (1997).

[9] J.R.B. Cockett and R.A.G. Seely. Proof Theory for full intuitionistic linear logic, bilinear logic, and MIX categories. Theory and Applications of Categories 5, pp. 85 - 131, (1997).

[10] J.R.B. Cockett and R.A.G. Seely. Linearly Distributive Functors. Journal of Pure and Applied Algebra 143, pp. 155-203, (1999).

[11] V. Danos, T. Ehrhard, On Probabilistic Coherence Spaces. *preprint*, (2008).

[12] V. Danos, R. Harmer, Probabilistic Games Semantics. Logic in Computer Science, 2000. Proceedings.

[13] T. Ehrhard. Finiteness spaces. Mathematical Structures in Computer Science 15, pp. 615-646 (2005).

[14] T. Ehrhard. On Köthe sequence spaces and linear logic. Mathematical Structures in Computer Science 12, 579-623, (2001).

[15] C. Führmann, D. Pym. On categorical models of classical logic and the geometry of interaction, Mathematical Structures in Computer Science 17, pp. 957-1027, (2007).

[16] J.-Y. Girard. Linear logic. Theoretical Computer Science 50, pp. 1–102, (1987).

[17] J.Y. Girard. Between logic and quantic: a tract. *in Linear Logic in Computer Science*, edited by T. Ehrhard et.al., Cambridge UNiversity Press, (2003).

[18] A. Guglielmi. A System of Interaction and Structure. ACM Transactions on Computational Logic, Vol. 8 (1:1) (2007), pp. 1–64.

[19] D. Hughes. Deep inference proof theory equals categorical proof theory minus coherence, preprint, (2004).

[20] M. Hyland and A. Schalk. Glueing and Orthogonality for Models of Linear Logic. Theoretical Computer Science 294, pp. 183–231, (2003).

[21] F. Lamarche. Exploring the gap between linear and classical logic. Theory and Applications of Categories 18, pp. 473-535, (2007).

[22] Richard McKinley. Classical categories and deep inference, preprint, (2005).

[23] P.-A. Melliés. Double categories: a modular model of multiplicative linear logic. Mathematical Structures in Computer Science 12, pp 449-479, (2002).

[24] C. Retoré. Pomset logic: a non-commutative extension of classical linear logic. *Proceedings of* TLCA'97, LNCS vol. 1210, pp. 300–318, (1997).

[25] R.A.G. Seely. Linear logic, $*$-autonomous categories and cofree coalgebras. *Contemporary Mathematics*, Volume 92. American Mathematical Society, (1989).

[26] L. Strassburger. On the axiomatisation of boolean categories with and without medial. Theory and Applications of Categories 18, pp. 536-601, (2007).

[27] A. Tiu. A system of interaction and structure II: The need for deep inference, Logical Methods in Computer Science 2, pp. 1-24, (2006).

8 Appendix: LDCs and the Mix rule

We assume the reader is familiar with the notion of *linearly distributive category with negation* [8, 4]. This will be our starting point. In this first definition, we will use neutral symbols for the connectives, as it will apply to several combinations of connectives in BV. This first definition is due to Cockett and Seely [9].

Definition 8.1 Let $\mathbf{C}$ be a category with monoidal structures $(\mathbf{C}, \bigcirc, \bot)$ and $(\mathbf{C}, \Diamond, \top)$, forming a linearly distributive category. So we have natural transformations of the form

$$\delta_R \colon A \bigcirc (B \Diamond C) \to (A \bigcirc B) \Diamond C$$

$$\delta_L \colon (A \Diamond B) \bigcirc C \to A \Diamond (B \bigcirc C)$$

making the diagrams of [8] commute. We will hereafter refer to both maps as δ, since the type will always be clear from the context.

Then an *isomix structure* for this category consists of an isomorphism $m \colon \bot \to \top$ such that the following diagram commutes:

$$\begin{array}{ccccc}
A \bigcirc B & \xrightarrow{id \bigcirc u} & A \bigcirc (\bot \Diamond B) & \xrightarrow{id \bigcirc (m \Diamond B)} & A \bigcirc (\top \Diamond B) \\
\downarrow{\scriptstyle u \bigcirc id} & & & & \downarrow{\scriptstyle \delta_R} \\
(A \Diamond \bot) \bigcirc B & & & & (A \bigcirc \top) \Diamond B \\
\downarrow{\scriptstyle (id \Diamond m) \bigcirc id} & & & & \downarrow{\scriptstyle u^{-1} \Diamond id} \\
(A \Diamond \top) \bigcirc B & \xrightarrow{\delta_L} & A \Diamond (\top \bigcirc B) & \xrightarrow{id \Diamond u^{-1}} & A \Diamond B
\end{array}$$

In the above, all of the isomorphisms are the coherent isos specified by the monoidal structure. The map m is called the *isomix map*

We mention the following result, which is stated in [9]. It deals with the case in which the two units are equal, and not merely isomorphic.

Lemma 8.2 *If* $\mathbf{C}$*is a* LDC *such that* $\top = \bot$*, then* $\mathbf{C}$ *is an isomix* LDC*.*

We also mention an additional result not explicitly stated in [9] which has proven to be useful. The importance of isomix categories has also been noted by Strassburger and Lamarche [26, 21], which is where the following result can be found. For them, isomix plays a large role in modelling classical logic via the use of the *medial rule* [7].

Theorem 8.3 *An isomophism* $m\colon \bot \to \top$ *is an isomix map if and only if the following diagram commutes:*