Groupoids

**Definition (For Sensible People)**

A *groupoid* is a (small) category in which every morphism is invertible.

**Definition (For Functional Analysts)**

A *groupoid* is a pair of sets $G_1$ (arrows) and $G_0$ (objects) with morphisms

- $d, r : G_1 \to G_0$
- $m : G_1 \times G_0 \to G_1 \to G_0$
- $u : G_0 \to G_1$
- $i : G_1 \to G_1$

satisfying evident axioms.

\[
\begin{align*}
G_1 \times G_0 \times G_1 & \xrightarrow{m} G_1 \xleftarrow{r} G_0 \\
\end{align*}
\]
Examples of groupoids

- Any group is a one-object groupoid.
- Any disjoint union of groups. This is called a group bundle.
- The fundamental groupoid of a space.
- An equivalence relation induces a groupoid where there is precisely one arrow between two elements if they are equivalent.
- A group action induces a groupoid:
  Let $G$ act on a set $X$. Let $\mathcal{G}_1 = G \times X$ and $\mathcal{G}_0 = \{e\} \times X$. Then

$$d(g, x) = x, \quad r(g, x) = gx, \quad (g, hx) \cdot (h, x) = (gh, x)$$
More examples of groupoids

These examples illustrate the local nature of groupoids.

- Let $X$ be a set. Objects are subsets of $X$. An arrow is a bijective function (so partial on $X$). This example can be modified to add various sorts of structure on $X$.
- Given a field $K$, define a category whose objects are natural numbers and whose arrows are invertible matrices.
- Let $V$ be a vector space. Define a category whose objects are subspaces of $V$. Morphisms are linear isomorphisms between subspaces.
Deaconu-Renault groupoids

Let $X$ be a set, $\Gamma$ an abelian group and $S \subseteq \Gamma$ a subsemigroup containing $0$. Suppose $S$ acts on $X$. Define a category whose objects are the elements of $X$. An arrow from $x$ to $y$ is of the form $s - t$ with $s, t \in S$ where $s \cdot x = t \cdot y$. Composition uses the group operation of $\Gamma$. Straightforward to verify that this is well-defined.

There are many variations of this, topological and otherwise.
Some category theory

- If $G$ is a groupoid and $x$ is an object of $G$, then $\text{Hom}_G(x, x)$ is a group, called the *isotropy group* of $x$.

- A morphism of groupoids is simply a functor. The category of groupoids is cartesian closed.

- The inclusion of the category of groupoids into the category of categories has both a left and right adjoint. One adjoint is obtained by inverting all maps and the other by taking the wide subcategory of isomorphisms.

- A groupoid is *principal* if the map $G_1 \to G_0 \times G_0$ defined by $f \mapsto (d(f), r(f))$ is injective. A groupoid is principal if and only if it is an equivalence relation.
Internal groupoids

Given a category with finite limits, one can consider groupoids internal to that category, since the definition can be expressed entirely diagrammatically. A localic groupoid is a groupoid internal to the category of locales.

Theorem (Joyal-Tierney)

Every Grothendieck topos is equivalent to a category of sheaves on a localic groupoid.

Theorem (Moerdijk)

The above extends to an equivalence of 2-categories.
Topological groupoids

These can be defined with various levels of generality. We’ll follow A. Sims, *Hausdorff étale groupoids and their C*-algebras*

- A topological groupoid is a groupoid $\mathcal{G}$ in the category of locally compact Hausdorff spaces and continuous maps.

- A topological groupoid is étale if its domain map is a local homeomorphism. (This implies the range map and multiplication are as well.)
Lemma

If \( G \) is a topological groupoid, then \( G_0 \) is closed in \( G \) if and only if \( G \) is Hausdorff.

Lemma

If \( G \) is an étale groupoid, then \( G_0 \) is open in \( G \), and hence clopen.

Lemma

The Deaconu-Renault groupoid is étale if the action of the semigroup is by local homeomorphisms.
Every groupoid is a topological groupoid in the discrete topology.

Every discrete groupoid is étale.

If $X$ is a locally compact Hausdorff space, and $R$ is an equivalence relation on $X$, then $R$ is a topological groupoid in the relative topology inherited from $X \times X$.

The group action groupoid is étale if and only if the acting group is discrete (And $X$ is locally compact Hausdorff.)
Lemma

If $\mathcal{G}$ is an étale groupoid, then for all $x \in \mathcal{G}_0$, the sets $\mathcal{G}_x = \{ \gamma \in \mathcal{G} | d(\gamma) = x \}$ and $\mathcal{G}^x = \{ \gamma \in \mathcal{G} | r(\gamma) = x \}$ are closed and discrete in the subspace topology.

Theorem

Let $\mathcal{C}_c(\mathcal{G}) = \{ f : \mathcal{G} \to \mathbb{C} | \text{supp}(f) \text{ is compact} \}$. Define $f \star g : \mathcal{G} \to \mathbb{C}$ by

$$f \star g(\gamma) = \sum_{\alpha \beta = \gamma} f(\alpha)g(\beta)$$

Then $\mathcal{C}_c(\mathcal{G})$ is a $*$-algebra with above multiplication and $f^*(\gamma) = \overline{f(\gamma^{-1})}$. 
The sum is finite.

The key is showing that the sum is finite. Note that if \( \alpha \beta = \gamma \) then \( \alpha \in \mathcal{G}_r(\gamma) \) and \( \beta \in \mathcal{G}_d(\gamma) \). So

\[
\{(\alpha, \beta) \in \mathcal{G}_1 \times \mathcal{G}_0 \mid \alpha \beta = \gamma \text{ and } f(\alpha)g(\beta) \neq 0\}
\]

is finite, since the intersections of discrete, closed sets and compact sets are finite.

We also note that \( \text{supp}(f \ast g) \subseteq \text{supp}(f)\text{supp}(g) \).
Given the above, it makes sense to ask if there is a class of sufficiently nice topological spaces $X$ such that $(X, U)$ is a finiteness space where $U$ is the set of relatively compact subsets and $U^\perp$ is the set of discrete, closed subspaces. (A subspace is *relatively compact* if its closure is compact in $X$.)

For general topological spaces, this is certainly false. But a reasonable conjecture is the following.

**Conjecture (Conjecture)**

The above determines a finiteness space structure when $X$ is locally compact, Hausdorff.
The conjecture is horribly false. The smallest uncountable ordinal $\omega_1$, with the order topology, is locally compact and Hausdorff but not a finiteness space under the above structure.

But a smaller class of spaces does work.

**Definition**

- $X$ is $\sigma$-compact if it can be covered by a countable family of compact subsets.
- $X$ is $\sigma$-locally compact if it is both $\sigma$-compact and locally compact.
Theorem (B-F,D,D)

- Let $X$ be a $\sigma$-locally compact hausdorff space. Then it is a finiteness space.
- The converse is false. Let $X$ be an uncountable discrete space. Then $X$ is locally compact and hausdorff, but not $\sigma$-compact. But $X$ is a finiteness space.

Nonetheless, the class of $\sigma$-locally compact Hausdorff spaces is quite large. Indeed, it contains every (paracompact second-countable Hausdorff). It also contains every CW-complex with countably many cells, because it is the union of countably many images of disks.

Some of our étale groupoids have underlying spaces which are $\sigma$-locally compact Hausdorff. We will use this fact to give a new approach to constructing algebras for them.