

# Derivations in Codifferential Categories

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- *Differential linear logic* (DLL), due to Ehrhard & Regnier, is an extension of linear logic via the addition of an inference rule modelling differentiation.
- It was inspired by models of linear logic discovered by Ehrhard, where morphisms have a natural smooth structure.
- We have a base category of linear maps with a comonad such that the coKleisli category consists of smooth maps.
- The corresponding categorical structures are *differential categories*, due to RB, Cockett and Seely.
- Given this new syntactic/semantic way of thinking about differentiation, we should find models and apply ideas from (categorical) logic to other areas where differential calculus is used.

# A few Canadian projects

- Cockett & Cruttwell are developing manifolds, tangent bundles to manifolds and connections on bundles within the differential category framework.
- They use *cartesian differential categories*, which capture the coKleisli category directly, and *restriction categories*, an axiomatization of partial map categories due to Cockett and Lack.
- Here we focus on ideas from algebraic geometry and work with the linear categories.

- Algebraic geometers are interested in solutions to systems of polynomial equations.
- Even if the field is  $\mathbb{R}$  or  $\mathbb{C}$ , the solution set may or may not be a manifold, due to the existence of singular points.
- We could also be working over fields of characteristic  $p$ .
- One can define differential forms anyway, via Kähler differentials.
- Instead of considering the solution set directly, it is more useful to examine the *coordinate algebra*, i.e.  
 $A = k[x_1, x_2, \dots, x_n]/I$ , where  $I$  is the ideal generated by the polynomials.

# Kähler Differentials II: Definitions

The traditional notion of Kähler differentials defines the notion of a module of  $A$ -differential forms with respect to  $A$ , where  $A$  is a commutative  $k$ -algebra. Let  $M$  be a (left)  $A$ -module.

## Definition

An  $A$ -derivation from  $A$  to  $M$  is a  $k$ -linear map  $\partial : A \rightarrow M$  such that  $\partial(aa') = a\partial(a') + a'\partial(a)$ .

## Definition

Let  $A$  be a  $k$ -algebra. A *module of  $A$ -differential forms* or *Kähler differential forms* is an  $A$ -module  $\Omega_A$  together with an  $A$ -derivation  $d : A \rightarrow \Omega_A$  which is universal in the following sense: for any  $A$ -module  $M$ , for any  $A$ -derivation  $\partial : A \rightarrow M$ , there exists a unique  $A$ -module homomorphism  $f : \Omega_A \rightarrow M$  such that  $\partial = fd$ .

# Kähler Differentials III: Existence Theorem

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega_A \\ & \searrow \partial & \downarrow f \\ & & M \end{array}$$

## Lemma

*For any commutative  $k$ -algebra  $A$ , a module of Kähler differential forms exists.*

One approach is to construct the free  $A$ -module generated by the symbols  $\{da \mid a \in A\}$  divided out by the evident relations, most significantly  $d(aa') = ad(a') + a'd(a)$ .

# Kähler Differentials IV: Example

For the key example, let  $A = k[x_1, x_2, \dots, x_n]$ , then  $\Omega_A$  is the free  $A$ -module generated by the symbols  $dx_1, dx_2, \dots, dx_n$ , so a typical element of  $\Omega_A$  looks like

$$f_1(x_1, x_2, \dots, x_n)dx_1 + f_2(x_1, x_2, \dots, x_n)dx_2 + \dots + f_n(x_1, x_2, \dots, x_n)dx_n.$$

Then we have

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

If  $V$  is an  $n$ -dimensional space and  $\mathbb{S}(V)$  is the free symmetric algebra construction, then there are canonical isomorphisms:

$$\Omega_A \cong \Omega_{\mathbb{S}(V)} \cong \mathbb{S}(V) \otimes V.$$

# Connection to differential categories?

The map  $d$  then is of the form

$$d: \mathbb{S}(V) \longrightarrow \mathbb{S}(V) \otimes V$$

This should remind one of the crucial map in differential linear logic:

$$d: !X \otimes X \longrightarrow !X$$

except it's *backwards*. Hence the need for *codifferential categories*, i.e. the opposites of differential categories.



## Definition

- A symmetric monoidal category  $\mathcal{C}$  is *additive* if every HomSet is an abelian group, and this is preserved by composition.
- An additive symmetric monoidal category has an *algebra modality* if it is equipped with a monad  $(T, \mu, \eta)$  such that for every object  $C$  in  $\mathcal{C}$ , the object,  $T(C)$ , has a commutative associative algebra structure

$$m : T(C) \otimes T(C) \rightarrow T(C), \quad e : I \rightarrow T(C)$$

and this family of associative algebra structures satisfies evident naturality conditions.

Of course, the  $!$  of linear logic is a coalgebra modality.

# Kähler Categories II: Definition (RB, Cockett, Porter, Seely)

## Definition

A *Kähler category* is an additive symmetric monoidal category with

- a (commutative) algebra modality for  $T$ ,
- for all objects  $C$ , a module of  $T(C)$ -differential forms  $\partial_C : T(C) \rightarrow \Omega_{T(C)}$ , i.e. a  $T(C)$ -module  $\Omega_{T(C)}$ , and a  $T(C)$ -derivation,  $\partial_C : T(C) \rightarrow \Omega_{T(C)}$ , which is universal in the following sense: for every  $T(C)$ -module  $M$ , and for every  $T(C)$ -derivation  $\partial' : T(C) \rightarrow M$ , there exists a unique  $T(C)$ -module map  $h : \Omega_{T(C)} \rightarrow M$  such that  $\partial; h = \partial'$ .

$$\begin{array}{ccc} T(C) & \xrightarrow{\partial} & \Omega_{T(C)} \\ & \searrow \partial' & \downarrow h \\ & & M \end{array}$$

## Theorem

*The category of vector spaces over an arbitrary field is a Kähler category, with structure as described above. The monad is the free symmetric algebra monad, and the map  $d$  is the usual differential as applied to polynomials.*

- Note that  $\text{Vec}^{op}$  is a differential category, and the map  $d$  is the canonical deriving transform in the definition of differential category. (Details in a minute.)
- There may be examples which don't arise from models of DLL, but we don't know any.

# Differential Linear Logic

- The important point of DLL is that differentiation is represented as an inference rule.
- To see what the inference rule would be, consider the following situation. I have two Euclidean spaces,  $X$  and  $Y$ , and a smooth map between them. In our model, it would be a map  $f: !X \rightarrow Y$ .
- At a point of  $X$ , its Jacobian matrix would be a linear map from  $X$  to  $Y$ . So the process of taking the Jacobian is a smooth map from  $X$  to linear maps from  $X$  to  $Y$ . This suggests an inference rule of the following form:

$$\frac{!X \vdash Y}{!X \vdash X \multimap Y}$$

- Or, equivalently:

$$\frac{!X \vdash Y}{X \otimes !X \vdash Y}$$

# Differential Categories I

- Categorically, it suffices to differentiate the identity map on  $!X$ . So we require a map

$$D(id) = d: X \otimes !X \longrightarrow !X$$

- Then an arbitrary smooth map  $f: !X \rightarrow Y$  is differentiated by precomposition with  $d$ . So

$$D(f) = X \otimes !X \xrightarrow{d} !X \xrightarrow{f} Y$$

- To state axioms, we must have additive structure on the Hom-sets.
- So a *differential category* is a model of linear logic with a map of the above form satisfying basic differential identities, expressed coalgebraically.

# Differential Categories II

The necessary rules are:

- The derivative of a constant is 0.
- The derivative of a linear function is constant.
- Leibniz rule (Product rule).
- Chain rule.

Here's an example (product rule). The composite

$$X \otimes !X \xrightarrow{d} !X \xrightarrow{\Delta} !X \otimes !X$$

must equal:

$$X \otimes !X \xrightarrow{id \otimes \Delta} X \otimes !X \otimes !X \xrightarrow{d \otimes id} !X \otimes !X$$

+

$$X \otimes !X \xrightarrow{id \otimes \Delta} X \otimes !X \otimes !X \cong !X \otimes X \otimes !X \xrightarrow{id \otimes d} !X \otimes !X$$

Note that this says that in  $\mathcal{C}^{op}$  we have a derivation.

# Looking for examples of differential categories

- Of course,  $\text{Rel}$ , the category of sets and relations, is a (boring) model.
- $\text{Vec}^{op}$ , with structure already described, is a model.
- Ehrhard's two primary models, Köthe spaces and finiteness spaces, are both differential categories.
- *Convenient vector spaces* (Frölicher & Kriegel) form a differential category. (RB, Ehrhard & Tasson)
- If  $C$  has sufficient colimits and  $\otimes$  preserves them, then one can take as monad the symmetric algebra construction  $\mathbb{S}$  and one has a codifferential category, (which is also a Kähler category).

# Kähler Categories IV: Back to Kähler categories

## Definition

- Let  $F$  denote the free associative algebra monad.
- The monad  $T$  satisfies *Property K* if the natural transformation  $\varphi : F \rightarrow T$  is a componentwise epimorphism.

## Theorem (RB, Cockett, Porter, Seely)

*If  $\mathcal{C}$  is a codifferential category, whose monad satisfies Property K, then  $\mathcal{C}$  is a Kähler category, with  $\Omega_{T(\mathcal{C})} = T(\mathcal{C}) \otimes \mathcal{C}$ , and the differential being the map  $d : T(\mathcal{C}) \rightarrow T(\mathcal{C}) \otimes \mathcal{C}$ , the canonical differential arising from Differential Linear Logic.*

## Lemma

*The symmetric algebra construction satisfies property K and hence we get a Kähler category.*



# Kähler Categories V: The Issue

Note that  $\Omega_{T(C)} = T(C) \otimes C$  is the free  $T(C)$  module generated by  $C$ , and so we know we have the following:

$$\begin{array}{ccccc} C & \xrightarrow{\eta} & T(C) & \xrightarrow{\partial} & \Omega_{T(C)} \\ & & & \searrow \partial' & \downarrow \text{---} h \\ & & & & M \end{array}$$

The entire problem is in cancelling the  $\eta$ . Property  $K$  allows for this. But it is clearly unsatisfactory. We will see momentarily that it can be eliminated.

# Kähler differentials for arbitrary $T$ -algebras (O'Neill)

- In a Kähler category, we only have Kähler differentials for free  $T$ -algebras. This is clearly unsatisfactory.
- Note that in the presence of an algebra modality, one can assign a commutative, associative algebra structure to any  $T$ -algebra  $(A, \nu: TA \rightarrow A)$ :

$$A \otimes A \xrightarrow{\eta \otimes \eta} TA \otimes TA \xrightarrow{m} TA \xrightarrow{\nu} A$$

With respect to this commutative algebra structure,  $\nu$  is an algebra map.

# Kähler differentials for arbitrary $T$ -algebras II

There are two steps:

- If  $C$  and  $D$  are algebras with Kähler modules  $\Omega_C$  and  $\Omega_D$  and  $f: C \rightarrow D$  is an algebra map, then there is a unique map of  $C$ -modules  $\Omega_f: \Omega_C \rightarrow \Omega_D$  such that:

$$\begin{array}{ccc} \Omega_C & \xrightarrow{\Omega_f} & \Omega_D \\ d \uparrow & & \uparrow d \\ C & \xrightarrow{f} & D \end{array}$$

- Then the Kähler module for  $(A, \nu)$  is constructed as a coequalizer:

$$\Omega_{T^2 A} \begin{array}{c} \xrightarrow{\Omega_\mu} \\ \xrightarrow{\Omega_{T\nu}} \end{array} \Omega_{TA} \xrightarrow{d} \Omega_{(A, \nu)}$$

## Theorem

*Every codifferential category is Kähler, i.e. no additional property required.*

Key points of proof:

- 1 If  $T$  is an algebra modality, there is a morphism of monads  $\mathbb{S} \rightarrow T$ , which in turn induces a functor  $F: T\text{-Alg} \rightarrow \mathbb{S}\text{-Alg}$ .
- 2  $(\mathcal{C}, \mathbb{S})$  is a Kähler category.
- 3 Every  $\mathbb{S}$ -algebra has a Kähler module, by O'Neil's construction.
- 4 Using the functor  $F$ , we get Kähler modules for all  $T$ -algebras.

# Moral of the Story

In many cases, the category of  $\mathbb{S}$ -algebras will be equivalent to the category of commutative algebras. This is a consequence of *monadicity theorems*. In these cases, we can use this method to get Kähler modules for all commutative algebras.

# Bourbaki-Beck approach to derivations

Let  $A$  be a commutative algebra and  $M$  an  $A$ -module. Define a commutative algebra structure on  $A \oplus M$  as follows:

$$(a, m)(a', m') = (aa', am' + a'm)$$

Then we have the following bijection:

$$\text{Alg}/A(A, A \oplus M) \cong \text{Der}(A, M)$$

Here  $\text{Alg}/A$  is the usual slice category of objects over  $A$ .

Jon Beck, in his thesis, generalized this to arbitrary categories, and made it a fundamental part of his approach to homology.

# Bourbaki-Beck approach to derivations II

As observed by RLW, this idea lifts nicely to the present setting:

## Theorem (Lucyshyn-Wright)

*In a Kähler category, if  $(A, \nu)$  is a  $T$ -algebra and  $M$  is an  $A$ -module, then  $A \oplus M$  has the structure of a  $T$ -algebra as follows. We need maps into  $A$  and into  $M$ :*

$$T(A \oplus M) \xrightarrow{T(\pi_1)} TA \xrightarrow{\nu} A$$

$$T(A \oplus M) \xrightarrow{d} T(A \oplus M) \otimes (A \oplus M) \xrightarrow{T(\pi_1) \otimes \pi_2}$$

$$T(A) \otimes M \xrightarrow{\nu \otimes id} A \otimes M \xrightarrow{act} M$$

*Under this  $T$ -algebra structure, the associated commutative algebra is the one on the previous slide.*

# Bourbaki-Beck approach to derivations III

## Definition

Let  $\mathcal{C}$  be a Kähler category with algebra modality  $T$ , with  $(A, \nu)$  a  $T$ -algebra and  $M$  an  $A$ -module. We define a *Beck  $T$ -derivation* to be a morphism

$$A \longrightarrow A \oplus M$$

in the category  $\mathcal{C}^T/A$ .

Here,  $\mathcal{C}^T$  is the category of  $T$ -algebras. We are again considering the slice category over  $A$ , and  $A \oplus M$  is given the  $T$ -algebra structure of the previous slide.

## Theorem

*All of the previously discussed structure, in particular the existence of universal derivations, lifts to this more general notion of derivation.*



# Homology in differential categories

The module of Kähler differentials should act as 1-forms in an abstraction of de Rham cohomology. In fact all of the categories we have been considering have a great deal of homological structure available. For example:

Let  $A$  be an associative algebra and  $M$  an  $A$ -bimodule. Define a map  $d: M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$ :

$$\begin{aligned} m \otimes a_1 \otimes a_2 \otimes \dots \otimes a_n &\mapsto \\ ma_1 \otimes a_2 \otimes \dots \otimes a_n &- m \otimes a_1 a_2 \otimes \dots \otimes a_n + \\ \dots + (-1)^i m \otimes a_1 \otimes a_2 \otimes a_i a_{i+1} \otimes \dots \otimes a_n &+ \\ \dots (-1)^n a_n m \otimes a_1 \otimes a_2 \otimes \dots \otimes a_{n-1} & \end{aligned}$$

A tedious calculation shows that  $d^2 = 0$ .

# Homology in differential categories II

So we get a chain complex:

$$\dots M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1} \dots \longrightarrow M \otimes A \longrightarrow M \longrightarrow 0$$

This gives the *Hochschild homology of  $A$  with coefficients in  $M$* . In the commutative case, there is an immediate connection to the previous discussion:

$$H_1(A, A) \cong \Omega_A$$

To make  $n$ -forms out of the 1-forms, we set  $\Omega_A^n = \bigwedge^n \Omega_A$ . There is a map, called *antisymmetrization*:

$$\varepsilon: H_n(A, A) \longrightarrow \Omega_A^n$$

which may or may not be an isomorphism. We'll return to this issue later.

# Homology in differential categories III

But even more importantly, we have a **monad**. Thus ideas from (co)triple (co)homology come into play. This is work of Mike Barr and Jon Beck.

## Definition

Let  $\mathcal{C}$  be a category with monad  $T$ . Let  $\mathcal{A}$  be an abelian category, i.e. we can form kernels and quotients and Hom-sets are additive. Let  $E: \mathcal{C} \rightarrow \mathcal{A}$  be a functor.

Then the triple cohomology of  $\mathcal{C}$  with coefficients in  $E$  is the cohomology of the sequence:

$$0 \longrightarrow E(TA) \xrightarrow{d} E(T^2A) \xrightarrow{d} \dots$$

Here the  $d$ 's are alternating sums of expressions involving  $\eta: id \rightarrow T$ , the unit of the monad.

# Homology in differential categories III (Pause)

If  $E$  is contravariant, we get *triple homology*, and similarly *cotriple homology* and *cotriple cohomology*.

Most theories of homology and cohomology can be defined in the monadic approach, although that may not be the most useful computationally.

Note that the category with the (co)monad does not have to have additive structure. So any model of linear logic is open to this sort of analysis if one can construct an interesting functor to an abelian category.

# Homology in differential categories IV: André-Quillen cohomology

Fix a commutative algebra  $R$  and an  $R$ -module  $M$ . We have the adjunction determined by the free commutative algebra on a set. This induces a comonad  $\perp$  on the category  $\text{CommAlg}/R$ . We then take as the functor  $E$ :

$$\text{Der}(\text{---}, M): \text{CommAlg}/R \longrightarrow \text{Vec}$$

André-Quillen homology defined similarly.

What does this measure in the classical setting?

# Homology in differential categories V: Smoothness

Recall that we were originally considering varieties, i.e. solution sets to sets of polynomial equations. Even if the base field is  $\mathbb{R}$  or  $\mathbb{C}$ , the variety may or may not be a manifold. How do we tell from the associated algebra  $A$  whether the variety was a manifold?

**Definition (Grothendieck, See Loday-Cyclic Homology)**

A commutative algebra  $A$  is *formally smooth* if for any pair  $(C, I)$  with  $C$  a commutative algebra and  $I$  an ideal such that  $I^2 = 0$ , the canonical map

$$\mathrm{Hom}_{\mathrm{Alg}}(A, C) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}}(A, C/I)$$

is surjective.

The intuition is that  $C$  is an infinitesimal extension of  $C/I$  and this says that any map from  $A$  into  $C/I$  factors through its infinitesimal extension.

## Theorem (Hochschild-Kostant-Rosenberg)

*If  $A$  is a smooth algebra, then the antisymmetrization map:*

$$\varepsilon: H_n(A, A) \longrightarrow \Omega_A^n$$

*is an isomorphism.*

## Theorem

*If  $A$  is a smooth algebra, then the André-Quillen homology is 0 for  $n \geq 1$ .*

# The Big Theorems in Differential Categories? (In progress)

There has been recent work extending these ideas to general monoidal categories (Ardizzoni, Menini and Stefan). But it seems that the setting of differential categories and Kähler categories is ideal for such explorations.

## Definition

Let  $\mathcal{C}$  be a Kähler category. A  $T$ -algebra is *smooth* if its corresponding algebra structure is smooth.

Let  $\mathcal{C}$  be a Kähler category. Let  $(A, \nu)$  be a  $T$ -algebra.  $T$  induces a comonad on  $\mathcal{C}^T/A$  and there are several reasonable choices for the analogue of  $Der(\text{---}, M)$  to obtain André-Quillen homology.



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We're working on it!