Lecture 19

Let's do one example of a lax extension in detail. We have the powerset functor \( \mathcal{P} : \text{Set} \to \text{Set} \).

Define \( \hat{\mathcal{P}} : \text{Rel} \to \text{Rel} \) by \( R \mapsto \hat{\mathcal{P}}(R) \) such that \( \forall y \in B \exists x \in A \) s.t. \( x R y \).

Here \( A \subseteq X, B \subseteq Y \).

Let's go through the conditions to make \( \hat{\mathcal{P}} \) a lax extension of the functor \( \mathcal{P} \).

1. Given \( R, R' : X \to Y \), suppose \( A \hat{\mathcal{P}}(R) B \)
   - WMS: If \( R = R' \), \( \mathcal{P}(R) \subseteq \mathcal{P}(R') \)
   - Suppose \( A \hat{\mathcal{P}}(R) B \). So \( \forall y \in B \exists x \in A \) s.t. \( x R y \). But this implies \( \forall y \in B \exists x \in A \) s.t. \( x R' y \) since \( R = R' \). So \( A \hat{\mathcal{P}}(R') B \)

2. WMS: \( \hat{\mathcal{P}}(S) \circ \hat{\mathcal{P}}(R) \subseteq \hat{\mathcal{P}}(S \circ R) \)
   - Suppose \( A \hat{\mathcal{P}}(R) B \hat{\mathcal{P}}(S) C \). So \( A \hat{\mathcal{P}}(R) B \hat{\mathcal{P}}(S) C \). So \( A \hat{\mathcal{P}}(R) B \hat{\mathcal{P}}(S) C \). So \( \forall y \in B \exists x \in A \) s.t. \( x R y \) & \( \forall z \in C \exists y \in B \) s.t. \( y S z \).
   - This implies \( \forall y \in C \exists x \in A \) s.t. \( x S - R y \). So \( A \hat{\mathcal{P}}(S \circ R) C \)
3) \( WMS \) if \( f: X \to Y \) is a function

\[ P(f) \leq \hat{P}(f_0) \]

where \( f_0 \) is \( f \) viewed as a relation. Easy

\( A \hat{P}(f) B \) means \( f(A) = B \).

This implies \( A \hat{P}(f_0) \). Similarly for \( P(f)^{\circ} \leq \hat{P}(f) \)

Now we must show that \( \hat{P} \) is a lex extension of the homot. This gives 2 additional equations.

4) \( M_x \circ \hat{P}(\hat{P}(R)) \leq \hat{P}(R) \circ M_x \)

\[
\begin{array}{ccc}
P(\hat{P}(x)) & \xrightarrow{M} & P(x) \\
\hat{P}(\hat{P}(R)) & \subset & \hat{P}(R) \\
\hat{P}(P(y)) & \xrightarrow{M} & P(y)
\end{array}
\]

Let \( A \in P(\hat{P}(x)) \) so \( A \subseteq P(x) \)

Let \( B \in P(Y) \). Suppose \( A \overset{M_y}{\circ} \hat{P}(\hat{P}(R)) B \)

So \( \exists C \subseteq P(P(Y)) \) or \( C \subseteq P(Y) \) s.t.

\( A \overset{\hat{P}(\hat{P}(R))}{\circ} C \) and \( C \overset{M_y}{\circ} B \)

So \( A \subseteq C \subseteq P(A) \) with \( A \hat{P}(R) C \)

So \( \forall y \in C \exists x \in A \) with \( x \overset{R}{\circ} y \)
and \( \mu : \mathcal{P}^2(Y) \to \mathcal{P}(Y) \) is union

So \( \mu_Y(C) = \bigcup_{C \in \mathcal{C}} B \)

Given this, we must show

\[ A \bigcirc_{P(R)} \mu_B \]

i.e.

\[ \exists D \text{ s.t. } A \cap D \text{ and } D \bigcirc_{P(R)} B \]

D must be \( D = \bigcup_{A \in \mathcal{A}} A \) by definition of \( \mu \).

So we have to show \( \forall y \in B \exists x \in D \text{ s.t. } xRy \)

Since \( D = \bigcup_{A \in \mathcal{A}} A \). We need to find \( A \in \mathcal{A} \)

with \( x \in A \).

Let \( C \) be as above. Since \( U \cap C = B \), \( \exists C \)\( \in \mathcal{C} \)

with \( y \in C \). Consider this \( C \in \mathcal{A} \) with \( A \bigcirc_{P(R)} C \). So \( \exists x \in A \) with \( xRy \). \( A \) \( \bigcirc \) the desired subset of \( X \).

\[ \eta_Y \circ R \subset \bigcirc_{P(R)} \circ \eta_X \]

\[ \eta_X : X \to \mathcal{P}(X) \]

\[ \eta_Y : \mathcal{P}(Y) \to \mathcal{P}(X) \]

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Suppose $x \eta_{y} R B$. So $x R x y \& y \eta_{y} B$. But this implies $B = \{ y \}^3$.
So $x \eta_{y} R \{ y \}^3$
Note $x \eta_{x} \{ x \}^3 \& \{ x \}^3 \Theta(\emptyset) \{ y \}^3$. □

Some Interesting Questions related to monoidal topology

### The setup

1) $T: \text{Set} \to \text{Set}$ any functor
we had the notion of lax extension

to $V\text{-Rel}$ where $V$ is a quantale

If $T$ is a monad, we get for
the notion of a lax extension
of he monad to $V\text{-Rel}$

Given such a structure, we can define
the notion of a $(T, V)$-category.
<table>
<thead>
<tr>
<th>Functor (monad)</th>
<th>Ultrafilter</th>
<th>Quantile</th>
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This chart is excellent justification for celling the subject monoidal topology. What else can we do?

1) Consider other monads,
   - Free Group monad
   - any other algebraic varieties

2) Consider other quantiles
   - If $M$ is a monoid, $P(M)$ is a quantile

Fill in the above charts for these choices.

Q: If $T, T': Set \rightarrow Set$ are monads, is there an evident monad structure on $T \circ T': Set \rightarrow Set$?
Answer: No, in general. One needs a "distributive law," that is a natural transformation 
\[ \lambda : T \circ T' \to T' \circ T \]
Assuming one has lax extensions, what can we say about extending the distributive law?

But what about changing the category of sets?

What about replacing sets with [underline]finite spaces? [underline]

Let \( X \) be a set. \( \mathcal{U} \subseteq \mathcal{P}(X) \)
Define \( \mathcal{U}^\perp = \{ x \in X | \forall U \in \mathcal{U}, |X \cup U| < \infty \} \)

Then e. finite spaces \( \Rightarrow \) set \( X \) and \( \mathcal{U} \subseteq \mathcal{P}(X) \) s.t. \( \mathcal{U}^\perp = \mathcal{U} \).

Think of \( (\cdot)^{\perp} \) as a "closure operator."
If \((X,u)\) and \((Y,v)\) are finiteness spaces, a morphism is a relation \(R:X \to Y\) s.t.:

1) If \(x \in U\), then \(x \in V\)

where \(R_u = \{ y \in Y \mid \exists x \in X \text{ with } (x, y) \in R \} \).

2) If \(w \in V\), then \(R^{-1}(w) \in U\)

\[ R^{-1}(w) = \{ x \in X \mid \exists y \in Y \text{ with } (x, y) \in R \} \]

This is a category. Denote the set of all such relations by \(X \Rightarrow Y\), or \(U \Rightarrow V\).

Define a new category as follows. Pick a ring \(R\). The category will be called \(\text{Fin}(R)\).

Objects are finiteness spaces. An arrow \(f: (X,u) \to (Y,v)\) is a function

\[ f: X \times Y \to R \] such that

the support of \(f\)

\[ \text{supp}(f) = \{ (x, y) \mid f(x, y) \neq 0 \} \]

is in \(U \Rightarrow V\).
Composition is as follows

\[ f : (X, \pi) \rightarrow (Y, \rho) \quad \exists \quad f : X \times Y \rightarrow R \]

\[ g : (Y, \sigma) \rightarrow (Z, \omega) \quad \exists \quad g : Y \times Z \rightarrow R \]

\[ g \circ f : (X, \pi) \rightarrow (Z, \omega) \]

must be a function \( X \times Z \rightarrow R \)

\[(g \circ f)(x, z) = \sum_{y \in Y} f(x, y) g(y, z)\]

The key point: This sum is finite!

This formula looks very much like composition in \( V-Rel \)

\[(g \circ f)(x, z) = \bigvee_{y \in Y} f(x, y) g(y, z)\]

The infinite sup in \( V-Rel \) can be replaced by a finite sum in \( \text{Fin}(R) \)
So quantiles can be replaced by rings.

Q1: Why is this sum finite?

Make sure you understand this before thinking about Q2.

Q2: How much of the program of monoidal topology program can be carried out in Fin(\mathbb{R}) rather than V-Bel?

The first issue is how to add order, or do we want something other than order?