Lecture 17

Marczich's question: For the algebra $k[G]$ with $G$ a group &

\[ \Delta(g) = \sum g_1 \otimes g_2 \quad m(g_1 \otimes g_2) = \begin{cases} g_1 & \text{if } g_1 = g_2 \\ 0 & \text{if not} \end{cases} \]

prove commutativity of

\[ B \otimes B \otimes B \otimes B \quad C_{23} \]

Consider upper legs: $g_1 \otimes g_2$

\[ \sum g_{11} \otimes g_{12} \otimes g_{21} \otimes g_{22} \rightarrow \sum g_{11} \otimes g_{21} \otimes g_{12} \otimes g_{22} \]

\[ \rightarrow \begin{cases} 0 & \text{if } g_{11} \neq g_{21} \text{ or } g_{12} \neq g_{22} \\ \sum g_{11} \otimes g_{21} & \text{if } g_{11} = g_{21} \text{ and } g_{12} = g_{22} \end{cases} \]

So in particular

\[ g_1 \neq g_2 \quad g_{11} = g_{12}, g_{21} = g_{22} \]
Q 2: Functions viewed as relations, when you compose in V-Rel, do you get the right thing? Yes

\[ f : X \to Y, \quad g : Y \to Z \]

functions

\[ f_0 : X \times Y \to V \]
\[ f_0(x, y) = \begin{cases} \top & \text{if } f(x) = y \\ \bot & \text{if } n \neq f(x) \\ \bot \cdot \bot = \bot & \text{for any } n. \end{cases} \]

Remember

So \[ g_0 \circ f_0(x, y) = \begin{cases} \top & \text{unless } g_0(y, z) = \bot \text{ & } f_0(x, y) = \bot \\ \bot & \text{if } f_0(x, y) = \bot \\ \bot \cdot \bot = \bot & \text{for any } n. \end{cases} \]

Last time: All our categories are ordered, so each hom set has an ordering.

Lax functors & Lax natural transformations.
Definition: A lax functor $F : C \to D$

is given by functions

$$F : |C| \to |D|$$

c and

$$\forall A,B \in |C| \quad F : C(A,B) \to D(FA,FB)$$

s.t.

$$\forall f : A \to B$$

1) $f \leq g \Rightarrow F(f) \leq F(g)$

2) $F(f \circ g) \leq F(f) \circ F(g)$

3) $1_{FA} \leq F(1_A)$

An oplax functor has the opposite inequality.

Lax natural transformation

$$\begin{array}{ccc}
FA \xrightarrow{\alpha} & GA \\
Ff & \downarrow & Gf \\
FB & \xrightarrow{\gamma} & GB
\end{array}$$

Oplax NR

$$\begin{array}{ccc}
FA \to GA \\
\downarrow L_1 \downarrow \\
FB \to GB
\end{array}$$
We're interested in lax extensions of

\[ T: \text{Set} \to \text{Set} \to \mathcal{T}: \text{V-Rel} \to \text{V-Rel} \]

See p. 154 for defn.

Then we extended the definition to lax extensions of monads.

**Reminder:** Given a monad \((T, M, \eta)\) on \text{Set},

we can view \(M: T^2 \rightarrow T\) as a relation and \(\eta: X \rightarrow TX\) as \(\text{rel} M\). Then

we require them to be oplax:

\[
\begin{array}{ccc}
\hat{T}^2 & \xrightarrow{M} & \hat{T}^2 \\
\downarrow & & \downarrow \\
\hat{T}^2 & \xrightarrow{\eta} & \hat{T}^2 \\
\end{array}
\]

The most important example is the ultralift monad.

So we'll do a study of ultralifters (no proofs).
Ultrafilters

You've seen convergence of sequences, but one can imagine requiring a notion of convergence for larger sets i.e. uncountable sets. One issue is that once you start dealing with uncountable sets, the axioms of set theory come into play.


See notes by A. Kruckman.

Let $X$ be a set. A filter on $X$ is a set $F$ of subsets of $X$, $F \subseteq \mathcal{P}(X)$ s.t.

1) $X \in F$
2) $\emptyset \notin F$ (not trivial)
3) $A \in F$, $A \subseteq B \Rightarrow B \in F$ (upward closed)
4) $A, B \in F \Rightarrow A \cap B \in F$

Intuition: Elements of $F$ are the "large" subsets.
EX: 1) \( F = \{ X \} \) (trivial filter)
   2) The principal filter generated by \( x \in X \)
   \[ F_x = \{ A \subseteq X | x \in A \} \]
   3) The cofinite filter
   \[ F = \{ A \subseteq X | X \setminus A \text{ is finite} \} \]

Defn: A filter is an ultrafilter if
For all \( A \subseteq X \), either \( A \in F \) or \( X \setminus A \notin F \).
Note: Ultrafilters are the maximal filters.
Adding in one more set causes the result to not be a filter.

EX: The principal filters are ultrafilters.
The trivial and cofinite filters are not.

Thm: There exist nontrivial ultrafilters, but the proof requires the axiom of choice.

Thm: Every filter is contained in an ultrafilter, but the proof requires Zorn's lemma,
which is equivalent to the axiom of choice.
A filter $\mathcal{F}$ on a topological space $Y$ converges to $y \in Y$ if for all open sets $U$ containing $y$, $U \in \mathcal{F}$.

Lemma 2: Let $X$ be a set, $\mathcal{F}$ a filter on $X$, and $f : X \to Y$ be a function. Define

$$f_*(\mathcal{F}) = \{ A \subseteq Y \mid f^{-1}(A) \in \mathcal{F} \}$$

then $f_*(\mathcal{F})$ is a filter. Also, filter can be replaced with ultrafilter in the above.

This lemma allows one to rephrase continuity in terms of filters.

The following is either a definition or a theorem, depending on whose book you read.

1) $Y$ is compact iff every ultrafilter converges to at least one point in $Y$.

2) $Y$ is Hausdorff iff every ultrafilter converges to at most one point in $Y$. 
Idea: So if $X$ is a compact Hausdorff space and $\mathcal{T}X$ is the set of all ultrafilters on $X$, we have a unique function $\alpha: \mathcal{T}X \to X$ sending each ultrafilter to its unique point of convergence.

One can show $\mathcal{T}X$ is monoidal, $\alpha$ is a $T$-closed.
In fact, the category of $T$-closed sets is equivalent to the category of compact Hausdorff spaces. Due to Ernie Manes.

But also recall that every monoid arises from an adjunction.
Let $\mathsf{CH}$ be the category of compact Hausdorff spaces. I have a forgetful functor $U: \mathsf{CH} \to \mathsf{Set}$.

It has a left adjoint, called the Stone–Cech compactification.

$\beta: \mathsf{Set} \to \mathsf{CH}$
\( \beta X \) is the set of ultrafilters on \( X \) with an appropriate topology.

One more nice application of filters:

Thm (Tychonoff)

An arbitrary product of compact spaces is compact.

Compare the filter proof (Krechmer's notes) with the non-filter proof (See Munkres)

So we have a monad on Set, the ultrafilter monad.

Barr noted that there is a lex extension of this monad to \( V\text{-Rel} \), where \( V = \mathcal{C} \).

So \( V\text{-Rel} \) is just \( \text{Rel} \).

Thm (Barr) The resulting category of lex colimits is equivalent to the category of all topological spaces.
I've left out tons of details. All can be found in the book.

What about the other quantities we've worked with: 

\[ P_+ = ([0, \infty])^p_{+0} \] 

Yes, and the resulting category is the category of approach spaces, due to Lowen.

**Definition:** An approach space is a pair \((X, \delta)\) with \(X\) a set and \(\delta: X \times P(X) \to [0, \infty]\) \(\delta\) is called the approach distance. It must satisfy:

1. \(\delta(x, \{x\}) = 0\)
2. \(\delta(x, \emptyset) = \infty\)
3. \(\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}\)
4. \(\delta(x, A) \leq \delta(x, A^{(s)}) + u\)

where \(u \in [0, \infty]\)
and \[ A^u = \{ x \in X \mid \delta(x, A) \leq u \} \]

A map of approach spaces is a function \( f \) s.t. \( \delta'(f(x), f(A)) \leq \delta(x, A) \)

these are the non-expensive maps.

In conclusion:\n
<table>
<thead>
<tr>
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<th>( A )</th>
<th>( P_+ )</th>
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</thead>
<tbody>
<tr>
<td>ID</td>
<td>ordered sets</td>
<td>metric space</td>
</tr>
<tr>
<td>ultra filter</td>
<td>topology</td>
<td>approach space</td>
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See p. 6 the q-entile.