Lecture 15

Review

A complete sup-lattice is a partially ordered set such that all subsets have a least upper bound, or supremum. (We note this implies that it has all infs as well.) A quantale is a complete sup-lattice $Q$ with a function

$$\otimes : Q \times Q \rightarrow Q$$

and an element $e \in Q$ such that the function $\otimes$ preserves all sups, i.e., $(Q, \otimes, e)$ is a monoid.

The first condition means

$$9 \otimes V_{q_1} = \bigvee_{i \in I} V_{q_i, q_1}$$

$$(V_{q_1}) \otimes q = V_{q_i \otimes q}$$

The fact that $\otimes$ preserves sups implies
have right adjoints denoted \( q \cdot (-) \) and \( (-) \cdot q \).

For example

\[
q \cdot q' = \bigvee \{ a \mid q \circ a \leq q' \}
\]

**Important Intuition**

A quantale is like a ring with multiplication \( \circ \) and addition given by \( V \).

2 key differences:

1) We don't have negatives, i.e. there are no additive inverses.
2) "additive is idempotent", i.e.

\[
a \circ a = a
\]

Nonetheless, it is still a good intuition to think of a quantale as a "ring with infinity addition".

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Let \( V \) be a quantale. We'll define the category of \( V \)-relations. An object is a set. An arrow

\[
R : X \to Y
\]

is a function.

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\[
R
\]
Composition is defined by
\[ R : X \to Y \quad S : Y \to Z \]
\[ S \circ R : X \to Z \]
\[ S \circ R(x, z) = \bigvee_{y \in Y} R(x, y) \circ S(y, z) \]

See Ch. 3 of Monoidal Topology.

**Example:** There is a unique 2-element quantale \( D \).
In this case, 2-Rel is equivalent to Rel.

Recall that in an ordered category, \( f : A \to B \)
is a **map** if \( f : B \to A \) s.t.
\[ 1_A \leq g \circ f \quad \text{and} \quad f \circ g \leq 1_B \]

**Theorem:** In Rel = 2-Rel, the maps are precisely the functions viewed as
Relations.

In general, this isn't true. Functions viewed as relations are always maps. The converse is false. The result is much more complicated in general.

See Chapter III, Proposition 1.2.1.

It's still only a partial result.

We talked about enriched categories before. Revisit for a second. If \( C \) is any monoidal category, to construct a category enriched over \( C \), one has a set of objects \( 1_{\mathcal{D}} \), and for all pairs of objects \( A, B \in 1_{\mathcal{D}} \), an object \( \mathcal{D}(A,B) \) and arrows in \( C \)

\[
\mathcal{D}(A,B) \otimes \mathcal{D}(B,C) \to \mathcal{D}(A,C)
\]

tensor in \( C \)

\[
1 \to \mathcal{D}(A,A)
\]

satisfying axioms, which can be found
in Monoidal Topology.

Now, every quandle is a monoidal category, since it is in particular a poset; and every poset is a category.

What is a category enriched over a quandle $\mathbb{V}$?

**Define.** A $\mathbb{V}$-Relation $\alpha : X \to X$ is **transitive** if $\alpha \circ \alpha \leq \alpha$ and **reflexive** if $1 \leq \alpha$.

A $\mathbb{V}$-category is a set $X$ equipped with a transitive, reflexive relation. The two requirements boil down to

\[ \alpha(x,y) \otimes \alpha(y,z) \leq \alpha(x,z) \]
\[ 1 \leq \alpha(x,x). \]

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Make sure you understand how this meets the enriched category definition.

**EX 1.** For the quandle $2$, suppose I have $\alpha : X \times X \to 2$.

$[e, T, F]$
We can write \( x \leq y \) for \( a(xy) = 1 \).

Then transitivity says

\[ x \leq y \text{ and } y \leq z \Rightarrow x \leq z \]

reflexivity says

\[ x \leq x \]

So we get the usual notion of ordered set.
(remember, we don't require antisymmetry.)

**Ex. 2:** Consider \([0, \infty]\) with reverse of the usual order. This is a monoid with \(\otimes\) given by addition.

\[ P_+ = \left( [0, \infty]^{\text{op}}, +, 0 \right) \]

see p. 31.

What is a \( P_+ \) category? \( a : X \times X \to P_+ \)

s.t. 1) \( a(x, y) + a(y, z) \geq a(x, z) \)

This is the triangle inequality, of course.

since we've reversed order.
reflexivity says
\[ q(x,x) \leq 0 \]
which implies \( q(x,x) = 0 \).

This is a metric space, except:
- We allow infinite distance
- We don't require symmetry
- We don't require separation

The book calls these metric spaces anyway.

In addition to \( V \)-categories, we can also talk about \( V \)-functors and \( V \)-natural transformations.

**Defn.** A \( V \)-functor \( f : (X,q) \rightarrow (Y,b) \) is a function \( f : X \rightarrow Y \) such that
\[ q(x,y) \leq b(f(x), f(y)) \]

Remember in ordinary category theory, part of the definition of a functor is a function
\[ \text{Hom}_c(X, Y) \rightarrow \text{Hom}_c(FX, FY) \]
The identity is clearly a V-functor and composite of V-functors, we have an ordinary category \( V\text{-Cat} \).

It is complete and cocomplete.

For details on V-natural transformation see p117.

**Examples**

1) For \( V = \mathcal{O} \), \( v\text{-cat} \) is an order & a V-functor is an order preserving function.

2) For \( V = \mathbb{P}_t \), \( v\text{-cat} \) is a metric space (in generalized sense) and a V-functor is a contraction map

\[ q(x,y) \geq b(f(x),f(y)) \]

**Thm**: For any quantile \( V \), \( V \) is a V-category.

**Proof**: We'll call the V-category \( Q \).

Let \( v, v' \in V \). Define

\[ Q(v,v') = V - v - v' \]
where \( v \otimes w \) is defined by
\[ v \otimes w \text{ iff } v \leq v \otimes w. \]

**Proof:** I need
\[ (v \otimes (v \otimes w)) \otimes (2 \otimes 2) \leq 2 \]

But this is equivalent to
\[ v \otimes (v \otimes w) \otimes (2 \otimes 2) \leq 2 \]

I have
\[ v \otimes (v \otimes w) \otimes (2 \otimes 2) \leq 2 \]

I'm using \( v \otimes (v \otimes w) \leq 2 \)

Similarly \( e \leq 2 \tag*{\Box} \)

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**V-module** Let \( V \) be a quantale.

Let \((X, a)\) and \((Y, b)\) be \( V \)-categories. A **\( V \)-module** (also called a \( V \)-profunctor) from \((X, a)\) to \((Y, b)\) is a \( V \)-relation \( R: X \rightarrow Y \) such that \( R \leq a \) and \( b \leq R \).

We write this as
\[ R: (X, a) \rightarrow (Y, b) \]
Note that since \((X, a)\) is a \(V\)-category, we have \(k \leq r\). So \(r - k \leq r\).

\[ r = r - k \leq r. \]

Thus we in fact have \(r = r \cdot a\) & \(b \cdot r = r\).

Composition of modules is a module, so we get a category \(V\text{-Mod}\) whose objects are \(V\)-category and whose arrows are \(V\)-modules.

Note! The category of profunctors in ordinary category theory is hugely important. See Benabou: Distributors At Work.