Lecture 4

Let $E$ be a monoidal category. An algebra or a monoid in $E$ is an object $A$ with maps

$$m : A \otimes A \to A$$
$$e : I \to A$$

satisfying associativity and unit equations. Remember that in the category of vector spaces $I = k$, and a map $k \to A$ is the same thing as an element of $A$.

In $V^e$, we get the familiar notion of $k$-algebra.

**Ex.** Let $M$ be a monoid or group. Let $k[M]$ be the vector space with $M$ as basis. Then $k[M] \otimes k[M]$ has as basis elements of the form $m_1 \otimes m_2$ with $m_1, m_2 \in M$. So one can define a multiplication via

$$m_1 \otimes m_2 \mapsto m_1 m_2$$

so this is just the linear extension of the original multiplication of $M$. 
EX2 Let $X$ be a set thought of as an alphabet. Let $X^*$ be the free monoid generated by $X$. So elements of $X^*$ are words generated elements of $X$. Multiplication is just concatenation of words, so it determines an algebraic structure as in example 1.

But there is a second multiplication, called the **shuffle multiplication**.

A shuffle of words $w_1$ and $w_2$ is a permutation of the word $w_1w_2$ such that the internal orders of $w_1$ and $w_2$ are maintained.

For example, let $w_1 = \text{ab}$ and $w_2 = \text{cd}$.

The shuffles are:

- $\text{abc}d$
- $\text{ac}bd$
- $\text{ac}db$
- $\text{cd}ab$
- $\text{cac}d$
- $\text{c}dab$
- $\text{cad}b$
- $\text{c}dab$
- $\text{cd}ab$

So I can define a multiplication on $k[X^*]$ by

$$m(w_1 \otimes w_2) = \sum w$$

where $w$ is a shuffle of $w_1$ and $w_2$. 
so \( m(ab \otimes cd) = abcd + acbd + qcdb + cabd + cadb + cdbd \)

This is an associative multiplication. It's also commutative. The unit is the empty word, the unique word of length 0.

**Ex 3:** Path Algebras

Let \( G \) be a directed graph.

![Diagram of a directed graph with vertices and edges labeled.]

Let \( P \) be the set of all directed paths. Consider \( k[P] \)

Define a multiplication by path concatenation.

So \( m(P_1 \otimes P_2) = \begin{cases} 
0 & \text{if end point of } P_1 \text{ is not starting point of } P_2 \\
\text{path concatenation} & \text{if } \text{starting point of } P_1 \text{ is } \text{ending point of } P_2
\end{cases} \)

Here are 2 paths:

- \( fab \) is a path
- \( ghh \) is a path

The ending point of \( ghh \) is the starting point of \( ghh \).
It's easy to see this is an associative multiplication.
But what about units? Discuss.

**EX 4 :** $M_n(\mathbb{R})$

**EX 5 and 6 :** The complex numbers and the quaternions are algebras over $\mathbb{R}$.

etc. Many books on this subject.

**Representations of algebras**

**First the traditional case.**

Let $k$ be a field and $A$ a $k$-algebra. A representation of $A$ is a $k$-vector space $M$ and a map

$p : A \otimes M \to M$ s.t.

1) $\forall m \in M , 1_A \otimes m = m$
2) $\forall a_1, a_2 \in A, m \in M \Rightarrow a_1 \cdot (a_2 \cdot m) = (a_1 a_2) \cdot m$

We say $A$ acts on $M$ and $M$ is an $A$-module.

**Categorial definition**

Let $\mathbf{C}$ be a monoidal category. Let

$(A, m : A \otimes A \to A, e : I \to A)$ be in $\mathbf{C}$.
This gives us a category $A$-mod. Is it monoidal? Discuss.

A morphism of $A$-modules $f: M \to N$ is a map such that:

$M \otimes_A \text{Hom}(M, N) \to N$.

An $A$-module $p: A \otimes M \to M$ is an object $M$ and a map $p: A \otimes M \to M$ (ignoring associativity).
Categorical Motto: If something is worth doing, it is worth doing backwards.

Coalgebras: We'll focus on the vector space case.

A \( k \)-coalgebra is a vector space \( C \) and maps

\( \Delta : C \rightarrow C \otimes C \), \( \eta : C \rightarrow k \)

s.t.\[
\begin{align*}
\Delta & : C \rightarrow C \otimes C \\
\eta & : C \rightarrow k \\
\Delta \otimes \text{id} & : C \rightarrow C \otimes C \otimes C \\
\end{align*}
\]

**Notes**

1. These are just the equations for \( k \)-algebra reversed.
2. Obviously this can be done in any monoidal category.

But are there examples?

1. Let \( V \) be a vector space with basis \( \{ e_0, e_1 \} \).

Define a coalgebra structure by saying for the basis:

\( \Delta(e_0) = e_0 \otimes e_0 \)

\( \eta(e_0) = 1_k \)

Note this is just for the basis.
One can't define \( \Delta(v) = v \otimes v \) for all \( v \in V \), since \( \mu \) isn't linear.

**Ex 2:** Let \( G \) be a finite group, or monoid. Let \( C = k[G] \). Define
\[
\Delta(g) = \sum g, g_2 \in G, \gamma(g) = \begin{cases} 1_k & \text{if } g = e \\ 0 & \text{if not} \end{cases}
\]

**Ex 3:** Let \( X = \{a, b, c\} \) be a set and \( X^* \) the free monoid on \( X \), as before. Let \( C = k[X^*] \).

Here are two coalgebraic structures on \( C \).

1) As in example 2, we get the **cut coalgebra**. For example, let \( w = abc \)
\[
\Delta(w) = \emptyset \otimes abc + a \otimes b c + b \otimes a c + a b \otimes c + a b c \otimes e
\]
\( \gamma(w) = \begin{cases} 1 & \text{if } w \text{ is empty word} \\ 0 & \text{if not} \end{cases} \)

2) Let \( w_1, w_2 \) be words.
Let \( \text{SHUF}(w_1, w_2) \) be the set of all shuffles of \( w_1 \) and \( w_2 \)
Define
\[ \Delta(w) = \sum_{w \in \text{SHUF}(w_1, w_2)} w_1 \otimes w_2 \]
\[ \eta(w) \text{ as before.} \]
This is called the de킌 coalgebra.

**Ex 4:** Consider \( M_n(k) \) the space of all \( n \times n \) matrices. It has as its basis \[ \{e_{i,j}\}_{1 \leq i,j \leq n} \]
Define a coalgebra structure by
\[ \Delta(e_{i,j}) = \sum_{1 \leq p \leq n} e_{i,p} \otimes e_{p,j} \]
\[ \eta(e_{i,j}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]
Thus, it is the matrix coalgebra.

**Ex 5:** Let \( P \) be a poset. \( P \) is locally finite if \( \forall x, y \in P \), the interval \[ [x, y] = \{ z \in P | x \leq z \leq y \} \]
is finite.
Let $P$ be a locally finite poset. Let $T$ be the set of intervals of $P$

$$T = \left\{ [x, y] \mid x \leq y \right\}$$

Let $C = k[T]$. Define

$$\Delta([x, y]) = \sum_{x \leq z \leq y} [x, z] \otimes [z, y]$$

$$\eta([x, y]) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

**Ex 6**: Let $k<x>$ be the polynomial ring in one variable $x$. So it has as a basis $\{x^n\}_{n \in \mathbb{N}}$. Define a coalgebra structure by

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

$$\eta(x^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

This is the divided power coalgebra.

**Ex 7**: If $V$ is a vector space, let $V^*$ be the dual space,

$$V^*$$
Thm! If $V, W$ are finite-dimensional vector spaces, then

$$(V \otimes W)^* \cong V^* \otimes W^*$$

Why? There is a natural transformation

$$V^* \otimes W^* \rightarrow (V \otimes W)^*$$

$H(f \otimes g)(v \otimes w) = f(v)g(w)$

It is injective and $\dim (V^* \otimes W^*) = \dim ((V \otimes W)^*)$, it is an iso.

So let $A$ be a finite-dimensional algebra. Let $C = A^*$. It's a coalgebra

$$m^* : A^* \rightarrow (A \otimes A)^* \cong A^* \otimes A^*$$

Note contravariance of $(\_)^*$

$$e^* : A^* \rightarrow k^* \cong k$$

Coalgebras are important in combinatorics. See work of Giancarlo Rota
Let \((C, \Delta, \eta)\) be a coalgebra.

A \(C\)-comodule is a vector space \(M\) and a map

\[ \delta : M \to C \otimes M \text{ s.t.} \]

\[ \delta \downarrow \quad \downarrow \text{id} \otimes \delta \]

\[ C \otimes M \to C \otimes C \otimes M \]

\[ \Delta \otimes \text{id} \]

This makes sense in any monoidal category.

A map of \(C\)-comodules is a map

\[ f : M \to N \text{ s.t.} \]

\[ \delta \downarrow \quad \downarrow f \]

\[ C \otimes M \to C \otimes N \]

\[ \text{id} \otimes f \]

So it's a category. But for monoidal structure we need more.