Lecture 3 - Intro Categories

Last time

Let $F, G : C \to D$ be functors. A natural transformation $\alpha : F \to G$ consists of a family of arrows in $D$

$$\alpha_c : Fc \to Gc$$

one for each $c \in I$, such that, given $f : c \to c'$ in $E$,

$$Ff \downarrow \quad \downarrow Gf$$

$$Fc \to Gc$$

$$\downarrow \quad \downarrow$$

$$Fc' \to Gc'$$

Example from last time

Let $C$ be any category. $A \in I$,

$\text{Hom}_C(A, -) : C \to \text{Set}$

is defined by

on objects $\sigma : B \to \text{Hom}_C(A, B)$

on arrows $\sigma$
\[ P : B \to C, \text{ I need} \]
\[ \text{Hom}(A, B) \to \text{Hom}(A, C) \]
\[ (g : A \to B) \mapsto (g ; f : A \to C) \]

Easy to check functor equations.

Now, given \( f : B \to C \), I claim that \( f \) induces a natural transformation

\[ \alpha_f : \text{Hom}(C, -) \to \text{Hom}(B, -) \]

To see this, I need for all objects \( A \),

a function \( \alpha_f (A) : \text{Hom}(C, A) \to \text{Hom}(B, A) \)

\[ g : C \to A \mapsto f ; g : B \to A \]

Let's check naturality, suppose \( h : A \to D \)

\[
\begin{array}{ccc}
\text{Hom}(C, A) & \xrightarrow{\alpha_f (A)} & \text{Hom}(B, A) \\
\downarrow \text{Hom}(C, h) & & \downarrow \text{Hom}(B, h) \\
\text{Hom}(C, D) & \xrightarrow{\alpha_f (D)} & \text{Hom}(B, D)
\end{array}
\]
In both directions, I set
\[ f : g \circ h : B \to D \]
Understand this example!

Note what we've done.

Given \( f : B \to C \), we've constructed a
natural transformation \( \alpha_f : \text{Hom} (C, -) \to \text{Hom} (B, -) \)

Thm: Every natural transformation \( \text{Hom} (C, -) \to \text{Hom} (B, -) \)

is of the form \( \alpha_f \) for a unique \( f : B \to C \)
So \[ \text{Nat}(\text{Hom}(C,-), \text{Hom}(B,-)) \cong \text{Hom}(C,B) \]

This is a consequence of the \textit{Yoneda Lemma}.

Why is naturality important?

Consider

\textbf{Thm}: If \( V \) is a f.d. vector space over \( K \)

\[ V \cong V^* = \text{Lin}(V,K) \]

\textbf{Outline of proof}:

\textbf{Step 1}: Pick a basis for \( V \),

\[ e_1, e_2, \ldots, e_n \]

Define a dual basis for \( V^* \) by

\[ e_1^*, e_2^*, \ldots, e_n^* \]

\[ e_c^*(e_j) = \begin{cases} 1 & \text{if } c=j \\ 0 & \text{if not} \end{cases} \]
You have to check this is a basis, etc.
This determines a linear bijection

\[ V \rightarrow V^* \]

But it required a choice of basis.
Different bases determine different isomorphisms.

So this is not natural.

On the other hand, we constructed a natural transformation

\[ V \rightarrow V^{**} \]

which did not require choosing a basis.

\[ v \mapsto [f \mapsto f(v)] \]

This was natural.
Choosing a basis is problematic from the point of view of constructive mathematics. In fact, the statement:

Every vector space has a basis

is equivalent to

The axiom of choice.

What is constructive mathematics?

Thm: There exist irrational #s $\alpha$ and $\beta$ s.t. $\alpha^\beta$ is rational.

Proof: Consider $\sqrt{2}^\sqrt{2}$. There are 2 possibilities:

1) $\sqrt{2}^\sqrt{2}$ is rational. Then we are done.

   Let $\alpha = \beta = \sqrt{2}$

2) $\sqrt{2}^\sqrt{2}$ is irrational. In this case

   let $\alpha = \sqrt{2}^{\sqrt{2}}$. Let $\beta = \sqrt{2}$. Then

   $$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$$
Since these are the only 2 possibilities, we're done. ☐

Questions:
1) Is this proof correct?
2) What's wrong with it?

Principles of constructive mathematics

1) The only allowable proof of a statement of the form \( \exists x \varphi(x) \), is a statement of what \( x \) is and a proof that it satisfies the property \( \varphi \).

2) The only allowable proof of \( \varnothing \) is a statement of which of the two is true and a proof that it is true.

The villains:
1) The law of excluded middle

Note: Once you eliminate LEM, you lose that $\neg\neg A = A$

2) The axiom of choice.

Proofs that fall apart

1) The usual proof of infinitely many primes

2) Lots of countably vs. uncountable arguments.

See [Bridges & Richman: Varieties Of Constructive Mathematics]

2) A course in constructive algebra by Mines, Richman, Ruitenberg
Next Topic: Adjunctions

Defn: Let \( C \) be any category and \( A, B \) objects in \( C \). Then \( A \) is isomorphic to \( B \) if \( 3 \) morphisms \( f: A \to B, f^{-1}: B \to A \)
\[ f \circ f^{-1} = 1_{A} \text{ and } f^{-1} \circ f = 1_{B}. \]

In particular, we can consider CAT, the 

categories whose objects are (small) categories and whose arrows are functors.

Then \( 2 \) categories are isomorphic if \( 3 \) functors \( F: C \to D \) and \( F^{-1}: D \to C \)
\[ s.t. \quad F \circ F^{-1} = 1_{D} \text{ and } F^{-1} \circ F = 1_{C}. \]

But this idea is too strict to be interesting.

Non-example
$\text{Gr}-\text{Sets}$, $G$ a group

Recall that a $\text{Gr}-\text{Set}$ is a set $X$ and a function

\[ \cdot : G \times X \to X \]

s.t. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

$1 \cdot x = x$

I have a forgetful functor

\[ U : \text{Gr-Set} \to \text{Set} \]

$U(X, \cdot) = X$

There is a functor $F : \text{Set} \to \text{Gr-Set}$

\[ F(X) = (G \times X, \cdot) \]

when

\[ \cdot : G \times G \times X \to G \times X \]

\[ (g_1, g_2, x) \mapsto (g_1 g_2, x) \]
F(X) is called the free Co-Set generated by X.

What is its universal property?

The functions U, F are not inverse to each other. But does have some important property. Approximating leads to the notion of adjunction.