Hopf algebras and the Logic of Tensor Categories

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And now for something completely different

- **Categorical proof theory** begins with the idea of forming a category whose objects are formulas in a given logic and whose arrows are proofs.
- Then we study the resulting category to determine its structure. Typically the category will be free in a certain sense.
- As a simple example, in *intuitionistic logic*, conjunction takes on the form of a categorical product and disjunction takes on the form of a coproduct.
- Closed structure (internal homs) provides a model of logical implication.
- In general, logical connectives become functors and inference rules will become natural transformations.
- Categories with the same structure can then be considered as models of that logical system.

Intuitionistic logic

- Began from philosophical concerns (Brouwer).
- Only constructive proofs allowed, so no proof by contradiction or law of excluded middle.
- To prove $\exists x.\varphi(x)$, I have to say what x is and prove it satisfies φ.
- To prove $A \vee B$, I have to specify one of the two and prove it.
- No longer have that $\neg\neg A = A$.
- Philosophical concerns aside, the proof system is much better behaved than classical logic. The villain is the equation $\neg\neg A = A$.

Linear Logic

- *Linear logic* (J.-Y. Girard) provided a great new logic for consideration under this framework.
- Categories of (topological) vector spaces and representations of Hopf algebras can be viewed as models of linear logic.
- In linear logic, conjunction behaves like a tensor product of vector spaces or of representations.
- Linear logic provides a natural framework for studying *noncommutative logic*, i.e. logics where A *and* B does not imply B *and* A.
- Linear logic has had many applications in, for example, computer science and linguistics. In the latter, noncommutative logics are particularly important.

Categorical Proof Theory I

- We use *sequent calculus* as our basic proof system. A sequent is something of the following form with the $\vdash$ representing logical entailment:

  $$\Gamma \vdash A$$

  Here Γ is a finite list of formulas (the premises) in our logic and A is a single formula (the conclusion).

- Sequents are constructed and manipulated using *inference rules*. Here are three examples:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B} \wedge R$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow R$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \wedge L$$

- Every logical connective has a left and right inference rule, explaining how they are introduced to the left or right of the turnstile.
- Proofs are built inductively in the shape of a tree with the *identity sequent* $a \vdash a$ (where a is an atomic formula) as the leaves:
- Proofs are strung together via the *cut rule*:

$$\frac{\Delta \vdash A \qquad \Gamma, A \vdash B}{\Gamma, \Delta \vdash B} \textbf{ CUT}$$

These rules are basically bookkeeping rules and allow us to manage premises:

- *Exchange* says we can rearrange the order of our premises as we like (σ is a permutation):

$$\frac{\Gamma \vdash A}{\sigma(\Gamma) \vdash A}\ \textbf{Ex}$$

- *Contraction* says that it is pointless to have duplicate premises:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \textbf{Con}$$

- *Weakening* says that you can add additional premises.

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \textbf{Weak}$$

- This sequent calculus system for the connectives $\wedge, \Rightarrow$ can be turned into a category whose arrows are formulas and whose objects are equivalence classes of proofs.
- The *Gentzen cut-elimination theorem* says that if a sequent is provable, it is provable without using the cut-rule.
- The categorical reformulation says that every proof is equivalent to a cut-free proof of the same sequent.

Theorem

The category arising as above is the free cartesian closed category generated by the atomic formulas in the logic. Conjunction becomes the categorical product and implication becomes its right adjoint:

$$Hom(A \times B, C) \cong Hom(B, A \Rightarrow C)$$

Categorical Proof Theory IV-Coalgebra Structure

- So a sequent of the form $\Gamma \vdash A$ is interpreted as a map $\times\Gamma \longrightarrow A$.
- To model Contraction

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \ \textbf{Con}$$

  I use the canonical map $\Delta \colon A \to A \times A$

- To model Weakening:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \ \textbf{Weak}$$

  I use the canonical map $A \to \mathbf{1}$ where $\mathbf{1}$ is the terminal object.

- By universality, these maps satisfy the cocommutative coalgebra equations.

Categorical Proof Theory V-Models

- Thus any cartesian closed category can be seen as a model of this logic. In particular, the category of G-sets is cartesian closed.
- Since the category of proofs is free, once I assign a G-set to the atomic formulas, I obtain a unique functor from the category of proofs to G-Sets.
- H. Läuchli developed a notion of *abstract proof theory* where a *proof bundle* was defined to be a G-set, where G was the group of integers. An abstract proof was a fixed point under the G-action. This turns out to be a complete semantics for intuitionistic logic.
- This was updated for linear logic by RB and Phil Scott. In the linear framework, you get a stronger notion called *full completeness*.

- Linear logic begins with a reinterpretation of sequent calculus. We consider the sequent $\Gamma \vdash A$ as expressing a resource requirement. So the sequent is expressing that one needs Γ inputs to produce an output of A. Linear logic is a resource-sensitive logic.
- From this point of view, the rules of contraction and weakening are clearly wrong:

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \textbf{Con}$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\ \textbf{Weak}$$

- When I remove those two rules I get (a fragment of) linear logic, called *multiplicative linear logic*.
- Each object loses its canonical coalgebra structure.
- As a result, conjunction behaves more like a tensor of vector spaces and in fact, we denote the conjunction of linear logic by $\otimes$. The corresponding categories are symmetric monoidal closed categories
- So models of this fragment are categories of vector spaces, Banach spaces, finite-dimensional Hilbert spaces, and various other categories of topological vector spaces.

Linear Logic III-Handling negation

- Linear logic allows one to have classical negation in your categorical models, i.e. $A \cong \neg\neg A$. In any symmetric monoidal closed category, I have a canonical map (k being the unit for the tensor, i.e. the base field).

$$\rho \colon A \longrightarrow (A \Rightarrow k) \Rightarrow k = \neg\neg A$$

- An object for which ρ is an iso is called *reflexive*. A category for which all objects are reflexive is called $*$-*autonomous*.
- These correspond to models where we have a classical-style negation. In these models, we can write all formulas on the right of the turnstile: $\vdash \Gamma$. Such models are harder to come by.

Linear Logic IV-Stereotype spaces [S.S. Akbarov]

Definition

A *stereotype space* is a topological vector space over the complex numbers such that the above map into the second dual space is an isomorphism of topological vector spaces. Here the dual space is defined as the space of all linear continuous functionals endowed with the topology of uniform convergence on totally bounded sets.

Theorem

The category of stereotype spaces is $*$*-autonomous.*

Theorem

A topological vector space is a stereotype space if and only if it is locally convex, pseudo-complete, and pseudo-saturated.

So this is an extremely wide class of spaces, including Fréchet spaces.

Linear Logic IV-Reintroducing contraction and weakening

- It's very important that a logic have sufficient "expressive power" to be of use as a logic. You need to be able to encode all of mathematics within the logic.
- The fragment we have discussed thus far is insufficient for this purpose. You have to reintroduce contraction and weakening, but we'll do so in a controlled fashion.
- To each formula A, we will associate a special formula $!A$. This formula should be thought of as a machine for creating as many copies of A as needed. Here are the rules:

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \textbf{Con}$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \textbf{Weak}$$

Additional rules to introduce these formulas:

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ \textbf{Der}$$

$$\frac{!\Gamma, \vdash B}{!\Gamma, \vdash !B}\ \textbf{Sto}$$

These rules make the functor ! a comonad (cotriple) such that each object $!A$ has a coalgebra structure. These are called *Seely categories*. One way to ensure this is to have a category with products and the equation:

$$!(A \times B) \cong !A \otimes !B$$

- When we eliminate the exchange rule:

$$\frac{\Gamma \vdash A}{\sigma(\Gamma) \vdash A}\ \textbf{Ex}$$

we obtain *noncommutative logic*.

- The idea of non commutative logic originated with the work of Jim Lambek on *categorial grammars*, a form of natural language syntax. This is a highly noncommutative logic (the order of words matters!).
- For a non commutative tensor, you need two implications corresponding to adjoints to $A \otimes -$ and $- \otimes A$. We'll denote these by $\Rightarrow, \Leftarrow$

Noncommutative logic II-Categorial Grammar

- We have two inference rules (Using the categorial grammar notation)

$$\frac{A \Leftarrow B, B}{A} \quad \frac{B, B \Rightarrow A}{A}$$

- We have two basic types **N=Noun** and **S=Sentence**
- Words in your grammar are assigned types:

  Patriots:**N**

  the Super Bowl:**N** (Pretend this is one word!)

  won:($\mathbf{N} \Rightarrow \mathbf{S}$) $\Leftarrow$ **N** (type of a transitive verb)

More Categorial Grammar

To determine whether the string of words "Patriots won the Super Bowl" is a sentence, apply the following deduction:

Patriots won the Super Bowl

$$\dfrac{\mathbf{N} \qquad \dfrac{(\mathbf{N} \Rightarrow \mathbf{S}) \Leftarrow \mathbf{N} \quad \mathbf{N}}{\mathbf{N} \Rightarrow \mathbf{S}}}{\mathbf{S}}$$

Since the conclusion is **S**, it is a sentence.

Models of noncommutative logic

The models of noncommutative linear logic (and hence categorial grammar) are monoidal biclosed categories. (biclosed means having both the implications)

Hopf algebras yield very canonical examples:

Theorem

Let H be a noncocommutative Hopf algebra with bijective antipode. Then the category of left H-modules is a monoidal biclosed category. $V \Rightarrow W$ and $W \Leftarrow V$ are both the space of linear maps with actions given by:

$$(hf)(v) = \sum_{(h)} h_1 f(Sh_2 v)$$

$$(hf)(v) = \sum_{(h)} h_2 f(S^{-1} h_1 v)$$

Varieties of noncommutative logic

Realizing that Hopf algebras provide models leads to thinking about variants of noncommutativity.

Theorem

Let H be a Hopf algebra such that $S^2 = id$. Then for any module V, we have $V \Rightarrow k = k \Leftarrow V$. So in terms of logic, we get noncommutative logic with only one negation.

Logically this corresponds to noncommutative linear logic with the following restricted version of the exchange rule:

$$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)} \ \mathbf{Cyc-Ex}$$

where σ is a *cyclic* permutation.

Shuffle Hopf Algebra

Let X be a set and X^* the free monoid on X. Let $H = k[X^*]$. Define a Hopf algebra structure by:

Multiplication: $$w \otimes w' \mapsto \sum_{v \in \mathrm{SHUF}(w,w')} v$$

Comultiplication: $$w \mapsto \sum_{vv'=w} v \otimes v'$$

Antipode: $$w \mapsto (-1)^{|w|}\overline{w}$$

This is a Hopf algebra with involute antipode and hence a model of cyclic noncommutative linear logic.

Full Completeness (RB, Scott)

Using a notion of topological vector space due to Lefschetz which yields a $*$-autonomous category, we looked at natural transformations between formulas in the multiplicative fragment which were invariant under the action of the shuffle Hopf algebra. This was a vector space which we called the *proof space*.

Theorem

Given a sequent of linear logic, its proof space has the representations of proofs as a basis.

This is a strong form of completeness theorem, called *full completeness*. Ordinary completeness characterizes provability, we actually characterize the proofs.

Differential Linear Logic I

- Given any comonad !, one can always form something called the coKleisli category. An arrow from A to B is an arrow from $!A$ to B in the base category. You can use the structure of the comonad to define composition.
- *Differential linear logic* (Ehrhard, Regnier) begins with the idea that there should be a category of some sort of topological vector space where one can define smoothness and suppose there is a comonad such that the base category consists of linear continuous maps and the coKleisli category are the smooth maps.
- This idea arose from semantic considerations. Ehrhard constructed two models of linear logic where there is just such a decomposition. These were the categories of *Köthe spaces* and *finiteness spaces*. Morphisms had a representation as power series, which could be differentiated.

Differential Linear Logic II

- The important point is that differentiation is represented as an inference rule.
- To see what the inference rule would be, consider the following situation. I have two Euclidean spaces, X and Y, and a smooth map between them. In our model, it would be a map $f\colon !X \to Y$.
- At a point of X, its Jacobian matrix would be a linear map from X to Y. So the process of taking the Jacobian is a smooth map from X to linear maps from X to Y. This suggests an inference rule of the following form:

$$\frac{!X \vdash Y}{!X \vdash X \Rightarrow Y}$$

Differential Linear Logic III

- So a *differential category* (RB, Cockett, Seely) is a model of linear logic modelling an inference rule of the above form satisfying basic differential identities, expressed coalgebraically.
- The derivative of a constant is 0.
- The derivative of a linear function is constant.
- Leibniz rule (Product rule).
- Chain rule.

The goal was to find models where the notion of smoothness matched the standard notion as much as possible.

Convenient vector spaces (Frölicher, Kriegl)

Theorem

Let E be a locally convex vector space. The following statements are equivalent:

- *If $c\colon \mathbb{R} \to E$ is a curve such that $\ell \circ c\colon \mathbb{R} \to \mathbb{R}$ is smooth for every linear, continuous $\ell\colon E \to \mathbb{R}$, then c is smooth.*
- *Every Mackey-Cauchy sequence converges.*
- *Any smooth curve $c\colon \mathbb{R} \to E$ has a smooth antiderivative.*

Definition

Such a vector space is called a *convenient* vector space.

Convenient vector spaces II: Key points

- The category of convenient vector spaces and continuous linear maps forms a symmetric monoidal closed category. The tensor is a completion of the algebraic tensor. There is a convenient structure on the space of linear, continuous maps giving the **internal hom**. So we have a symmetric monoidal closed category.
- Then we can define:

Definition

A function $f : E \to F$ with E, F being convenient vector spaces is *smooth* if it takes smooth curves in E to smooth curves in F.

- We will denote the algebra of smooth functions from E to F by $C^\infty(E, F)$ and the real-valued functionals on E by $C^\infty(E)$.

Convenient vector spaces III: More key points

- The category of convenient vector spaces and smooth maps is cartesian closed. This is an enormous advantage over Euclidean space, as it allows us to consider function spaces.
- There is a comonad such that the smooth maps form the coKleisli category:

Define a map δ (Dirac delta function) as follows:

$$\delta\colon E \to \mathrm{Con}(C^\infty(E), \mathbb{R}) \quad \delta(x)(f) = f(x)$$

Then we define $!E$ to be the Mackey closure of the span of the set $\delta(E)$.

Theorem (Frölicher,Kriegl)

- $!$ *is a comonad.*
- *Each object* $!E$ *has canonical coalgebra structure.*

Convenient vector spaces IV: It's a model

Theorem (Frölicher,Kriegl)

The category of convenient vector spaces and smooth maps is the coKleisli category of the comonad $!$.

Furthermore, we have an operator, which is just a directional derivative, which captures the differentiation inference rule.

Theorem (RB, Ehrhard, Tasson)

Con *is a differential category.*