Conformal field theory as a nuclear functor

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Abstract

We consider Segal's categorical approach to conformal field theory (CFT). Segal constructed a category whose objects are finite families of circles, and whose morphisms are Riemann surfaces with boundary compatible with the families of circles in the domain and codomain. A CFT is then defined to be a functor to the category of Hilbert spaces, preserving the appropriate structure. In particular, morphisms in the geometric category must be sent to trace class maps.
However, Segal's approach is not quite categorical, as the geometric structure he considers has an associative composition, but lacks identities. We begin by demonstrating that an appropriate method of dealing with the lack of identities in this situation is the notion of *nuclear ideal*, as defined by Abramsky and the first two authors. More precisely, we show that Segal's structure is contained in a larger category as a nuclear ideal. While it is straightforward to axiomatize categories without identities, the theory of nuclear ideals further captures the idea of the identity as a singular object. An excellent example of a singular identity to keep in mind is the Dirac delta "function." We argue that this sort of singularity is precisely what is occurring in conformal field theory.
We then show that Segal's definition of CFT can be defined as a nuclear functor to the category of Hilbert spaces and bounded linear maps, equipped with its nuclear structure of Hilbert-Schmidt maps.
As a further example, we examine Neretin's notion of *correct linear relation* (CLR), and show that it also contains a nuclear ideal. We then present Neretin's construction of a functor from the geometric category to CLR, the category of Hilbert spaces and correct linear relations, and show that it is a nuclear functor and hence a generalized CFT.
We conclude by noting that composition in Neretin's category can also be seen in Girard's *geometry of interaction* construction.

Keywords: Monoidal categories, nuclear ideals, conformal field theory, geometry of interaction

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1 Dedication

The breadth of Gordon Plotkin's interests can be seen in the variety of articles in this volume. Our contribution is not in one of the areas where he has made a decisive impact but is one in which he has long had an interest. By his numerous questions in private conversations and his style, he has stimulated many contributions where he is not actually an author. We hope that he enjoys this foray into the boundary between mathematical physics and category theory and the connections to linear logic.

2 Introduction

The use of categories as an organizing structure in various branches of physics has been one of the most remarkable and unexpected applications of category theory to date. Nowhere is this application more evident than in *topological quantum field theory* or *TQFT*. TQFT as envisaged by Witten [34,35] was reformulated by Atiyah in [6,7]. Atiyah viewed TQFT as a structure-preserving functor on a geometric category whose objects are families of circles (in the 2-dimensional case we will consider here) and whose morphisms are 2-manifolds with boundary such that the boundaries are compatible with the families of circles in the domain and codomain. The target category in this definition is the category of finite-dimensional Hilbert spaces and linear maps.

One additional structure that the functor must preserve is the monoidal structure. A *monoidal category* is a category with a bifunctor, denoted $(A, B) \mapsto A \otimes B$, which satisfies certain associativity and unit constraints. The key example to keep in mind is the category of vector or Hilbert spaces, with the usual tensor product. Given a monoidal category, one may readily define the notion of a dual object $A \mapsto A^*$, analogous to a dual space. The notion of duality that is relevant to TQFT is the notion of a *compact closed category* [24]. Compact closed categories can be thought of as an axiomatization of the category of finite-dimensional vector spaces, or equivalently of categories whose morphisms are matrices. So in such a category one has an isomorphism:

$$A \multimap B \cong A^* \otimes B$$

Here $A \multimap B$ is the *internal hom functor*, i.e. the internalization of the "set" or "space" of arrows from A to B. So compact closed categories are in particular monoidal closed [25]. Both the cobordism category and the category of finite-dimensional Hilbert spaces are compact closed. We require the TQFT functor to preserve the compact closed structure.

In this paper, we are interested in Segal's definition of *conformal field theory*. See [31] and the longer recently published manuscript [32]. Segal's work actually predates Atiyah's version of TQFT, and has a similar structure. Objects in the geometric category are again families of circles, but now morphisms are Riemann surfaces with compatible boundary. The additional conformal structure on morphisms makes for a substantially different and more complex categorical structure. We similarly see a different structure in the target category as well. In this setting, one allows (separable) Hilbert spaces of arbitrary dimension, and requires that the Riemann surfaces of the domain category are mapped to trace-class maps.

The primary goal of this paper is to introduce a general categorical framework which captures Segal's construction. As it stands, Segal's structure is not quite categorical, as it has the necessary associative composition to be a category, but is lacking identities. The framework we propose is the notion of a *nuclear ideal*, as introduced in [2]. These are ideals that live inside monoidal categories and, were it not for the lack of identities, would be compact closed categories. Thus they are more than just categories without the identity arrows, they are structures that "would have been compact closed if they only had identity arrows."

We further argue that there is a fundamental reason for the lack of identities in structures such as these, which the theory of nuclear ideals is designed to capture. Consider for example the following "category". Its objects are open regions in Euclidean space. A morphism $\varphi\colon X \to Y$ will be a function $\varphi\colon X \times Y \to \mathsf{R}$. Composition is then defined as

$$\varphi;\psi(x,z) = \int\limits_{Y} \varphi(x,y)\psi(y,z)dy$$

Under suitable assumptions on the functions, this gives a well-defined associative composition. But this structure lacks identities. This problem is well-known and led to the definition of distributions as generalized functions. In particular, one has the Dirac delta distribution which would act as the identity for the above composition. From this viewpoint, the identity is seen as a singular object, and the passage from functions to distributions amounts to allowing such singularities.

A second example is seen in the "category" of Hilbert spaces and Hilbert-Schmidt maps. The definition of Hilbert-Schmidt maps is given below, but for now we note that while the composition of two such maps is again Hilbert-Schmidt, the identity on an infinite-dimensional Hilbert space fails to be Hilbert-Schmidt.

It is of course trivial to define the notion of a category without identities and functors between such, but nuclear ideals were defined in order to capture this idea of identity as a singular object. The categories without identities mentioned above live inside actual categories as ideals, and the passage from the ideal to the category is seen as a completion in which the missing singularities are added. Indeed, nuclear ideals originally arose from considering the category of tame distributions on Euclidean space, as introduced in the same reference. But it was quickly realized that this structure occurs in many other places, as in the Hilbert-Schmidt example. Subsequent examples can be found in [9]. Note that in general one needs to add more than just identities in order to obtain the larger category for a nuclear ideal.

Another important feature of nuclear ideals is their interaction with monoidal structure. The category of sets and relations and the category of Hilbert spaces and bounded linear maps share a great deal of common structure. Both categories are monoidal and have a †-functor which is involutive and the identity on objects. This is the notion of a *symmetric monoidal dagger category.*[5] A key further property of the category of relations is that one has a notion of *transfer of variables*, *i.e.*, one can use the closed structure and involution to move variables from "input" to "output" and vice-versa. This is represented by the adjunction:

$$\mathsf{Hom}(A,B) \cong \mathsf{Hom}(I, A^* \otimes B) \cong \mathsf{Hom}(A \otimes B^*, I)$$

(In the category of relations, the functor $A \mapsto A^*$ will be the identity, while for Hilbert spaces, it will be complex conjugation or dual space.) This is precisely the adjunction defining the notion of compact closed category mentioned above. While the category of Hilbert spaces and bounded linear maps is also a tensored $*$-category, one only has an "adjunction" of the above sort for the Hilbert-Schmidt maps. Hilbert-Schmidt maps exist within the category of Hilbert spaces as a two-sided ideal, and a two-sided ideal satisfying such an adjunction is the definition of a nuclear

[5] The theory of nuclear ideals was originally defined using the notion of a tensored $*$-category [10], which is strictly weaker. We here reformulate the notion, in these stronger terms.

ideal. It is precisely this notion of partial adjunction, or adjunction with respect to an ideal, that is appropriate for conformal field theory.

The theory of symmetric monoidal dagger categories was introduced by Abramsky-Coecke [5] and Selinger [33] in formulations of an axiomatic approach to quantum mechanics. Both of those papers were especially interested in the compact closed case (to be discussed below). The paper [5] particularly demonstrates the advantages of this axiomatic approach, as a number of quantum protocols are shown to be encodable in this framework. It is reasonable to think of the Abramsky-Coecke and Selinger work as dealing with the finite-dimensional structure, as their primary example is the category of finite-dimensional Hilbert spaces, while the theory of nuclear ideals deals with infinite-dimensional structure.

We will analyze a construction of Neretin [29] which inspired Segal's definition, and show that it can be extended to a proper category, which contains the Segal structure as a nuclear ideal. We then redefine a conformal field theory to be a nuclear functor from Neretin's geometric category to the category of Hilbert spaces. Of course, a nuclear functor is a functor preserving all of the tensored $*$-structure, and taking nuclear maps to nuclear maps. We show that this is sufficient to ensure the crucial property of Segal's definition that all morphisms are sent to trace class maps.

Given a categorical framework for defining CFT, the next evident step is to define a generalized CFT as a nuclear functor to any symmetric monoidal dagger category with a nuclear ideal. There are several examples of nuclear ideals in the reference [2] which might be of interest for such a study, in particular a category of probabilistic relations. For now, we consider another construction of Neretin, the category of *correct linear relations* (CLR) [29]. We show that CLR is also a tensored $*$-category with a nuclear ideal, and that Neretin's prescription for assigning correct linear relations to geometric structures is indeed a generalized conformal field theory.

Neretin's construction is very much like constructions used by computer scientists to model feedback. The notion of feedback is one instance of the general notion of *trace* [22]. The usual trace of linear algebra is another. This perspective of Neretin's construction links it with linear logic and in particular the geometry of interaction [14,15]. Linear logic is a logical system, which, from the perspective of categorical logic, is ideal for the analysis of monoidal categories. It seems possible then that ideas from linear logic may be useful in studying some aspects of conformal field theory.

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3 Nuclear ideals

3.1 Compact closed categories and dagger categories

We assume that the reader is familiar with the notion of monoidal, symmetric monoidal and symmetric monoidal closed category. An appropriate reference is [25].

Definition 3.1 *A* compact closed category *is a symmetric monoidal category such that for each object A there exists a dual object A^*, and canonical morphisms:*

$$\nu\colon I \to A \otimes A^*$$

$$\psi : A^* \otimes A \to I$$

such that the evident equations hold. In the case of a strict monoidal category, these equations reduce to the usual adjunction triangles. One of the two equations says that the following composite must be the identity:

$$A \xrightarrow{\cong} I \otimes A \xrightarrow{\nu \otimes id} (A \otimes A^*) \otimes A \xrightarrow{\cong} A \otimes (A^* \otimes A) \xrightarrow{id \otimes \psi} A \otimes I \xrightarrow{\cong} A.$$

The other is a similar equation for A^*. It is easy to see that a compact closed category is indeed closed and that the internal hom satisfies $A \multimap B \cong A^* \otimes B$. Note that the operation $*$ is only defined on objects, but extends to a contravariant functor.

We now describe the prototypical example, the *category of relations*.

Definition 3.2 The *category of relations*, **Rel**, has sets as objects, and a morphism from X to Y will be a relation on $X \times Y$, with the usual relational composition.

In what follows, X, Y, Z will denote sets, and x, y, z will denote elements. A binary relation on $X \times Y$ will be denoted $x\mathcal{R}y$. Given a relation $\mathcal{R}: X \to Y$, we let $\mathcal{R}^\dagger: Y \to X$ denote the converse relation. The tensor product $\otimes$ in **Rel** is given by taking the cartesian products of sets, and on morphisms, we have:

$$\mathcal{R}: X \to Y \qquad \mathcal{S}: X' \to Y'$$

$$(x, x')\mathcal{R} \otimes \mathcal{S}(y, y') \text{ if and only if } x\mathcal{R}y \text{ and } x'\mathcal{S}y'.$$

The unit for the tensor is given by any one point set. There is a functor $(\ )^*: \mathbf{Rel} \to \mathbf{Rel}$ by:

$$X^* = X \qquad \mathcal{R}^* = \mathcal{R}^\dagger$$

There is a natural bijection of the form:

$$Hom(X, Y) \cong Hom(I, X^* \otimes Y).$$

Or, more generally:

$$Hom(X \otimes Z, Y) \cong Hom(Z, X^* \otimes Y).$$

Thus **Rel** is a compact closed category. **Rel** and the category of finite-dimensional Hilbert spaces have an additional structure not shared by all compact closed categories. The dual functor splits into two functors, a contravariant functor which is the identity on objects and a covariant functor. This is the basis for the definition of *tensored $*$-category* as introduced in [10] and used as the original basis for the definition of nuclear ideal in [2]. More recently it has been the basis for the axiomatization of *symmetric monoidal dagger category*, as introduced in [5,33]. This is a strictly stronger notion, and we will adopt the theory of nuclear ideals to this axiomatization below.

We begin by establishing notation and conventions for the category of Hilbert spaces and bounded linear maps, which will serve as our motivating example. Our notation will be as follows. If H is a Hilbert space, then H^* will denote the conjugate space. We will only consider the usual tensor product of Hilbert spaces, as described, for example, in [23]. The *adjoint* of $f: \mathcal{H} \to \mathcal{K}$, denoted $f^\dagger$ as usual, is defined to be the unique bounded linear map $f^\dagger: \mathcal{K} \to \mathcal{H}$ such that, for all $a \in \mathcal{H}, \quad b \in \mathcal{K}$, we have:

$$\langle a, f^\dagger(b) \rangle = \langle f(a), b \rangle$$

Hilb will denote the category of Hilbert spaces and bounded linear maps. The adjoint construction endows **Hilb** with a contravariant involutive functor which is the identity on objects. With this functor, **Hilb** becomes a *symmetric monoidal dagger category.* A suitable reference for basic Hilbert space theory is [23]. In the following, we use the terminology of Selinger in [33].

Definition 3.3 *A category $\mathcal{C}$ is a* dagger category *if it is equipped with a functor $(-)^\dagger\colon \mathcal{C}^{op} \to \mathcal{C}$, which is strictly involutive and the identity on objects. In such a category, a morphism f is* unitary *if it is an isomorphism and $f^{-1} = f^\dagger$. An endomorphism is* hermitian *if $f = f^\dagger$. A* symmetric monoidal dagger category *is one in which all of the structural morphisms in the definition of symmetric monoidal category [25] are unitary and dagger commutes with the tensor product.*

We will require further structure not necessary in [5,33]; those authors were considering *compact closed* dagger categories. In a compact closed category, the duality operation $A \mapsto A^*$ can be assumed to exist only for objects. Then, one proves that it extends to a contravariant functor. We also note that, for our examples, the duality decomposes into a covariant functor and a contravariant identity-on-objects functor (the already defined dagger). With this in mind, we define:

Definition 3.4 *A symmetric monoidal dagger category $\mathcal{C}$ is said to* have conjugation *if equipped with a covariant functor $(\ )^*\colon \mathcal{C} \to \mathcal{C}$ (called* conjugation*) which is strictly involutive and commutes with both the symmetric monoidal structure and the dagger operation. Since we have a covariant functor, we denote its action on arrows as follows:*

$$f\colon A \to B \longmapsto f_*\colon A^* \to B^*$$

This is in line with the notation of [33].

So in particular, our $*$-functor satisfies

$$(f_*)^\dagger = (f^\dagger)_*\colon B^* \to A^*$$

3.2 Hilbert-Schmidt maps

While the tensor in **Hilb** does not have an adjoint, the category **Hilb** does contain a large class of morphisms which have something like an adjointness structure with respect to the tensor. These are the *Hilbert-Schmidt maps*. The material in this section can be found in [23].

Definition 3.5 *If $f\colon \mathcal{H} \to \mathcal{K}$ is a bounded linear map, we call f a* Hilbert-Schmidt map *if the sum*

$$\sum_{i\in I} ||f(e_i)||^2$$

is finite for an orthonormal basis $\{e_i\}_{i\in I}$. The sum is independent of the basis chosen.

One can easily verify that the Hilbert-Schmidt operators on a space form a 2-sided ideal in the set of all bounded linear operators. Furthermore, if $\mathsf{HSO}(\mathcal{H},\mathcal{K})$ denotes the set of Hilbert-Schmidt maps from $\mathcal{H}$ to $\mathcal{K}$, then $\mathsf{HSO}(\mathcal{H},\mathcal{K})$ is a Hilbert space, when endowed with an appropriate norm.

We now discuss the adjointness properties that Hilbert-Schmidt maps satisfy:

Theorem 3.6 *Define a linear mapping $U\colon \mathcal{H}^* \otimes \mathcal{K} \to \mathsf{HSO}(\mathcal{H},\mathcal{K})$ by $U(x \otimes y)(u) = \langle x, u\rangle y$, where $x \otimes y \in \mathcal{H}^* \otimes \mathcal{K}$. Then U is a unitary transformation of $\mathcal{H}^* \otimes \mathcal{K}$ onto $\mathsf{HSO}(\mathcal{H},\mathcal{K})$. In particular, we note that the morphism U is a linear bijection.*

Expressed more categorically, there is a bijective correspondence:

$$\mathsf{HSO}(\mathcal{H},\mathcal{K}) \cong Hom(I, \mathcal{H}^* \otimes \mathcal{K})$$

This bijection is the basis for the definition of nuclear ideal.

In a general category (with sufficient structure), given the above type of isomorphism, maps $f: X \to Y$ which correspond to maps $\hat{f}: I \to X^* \otimes Y$ are said to *have a transpose*. So in **Hilb**, the Hilbert-Schmidt maps have a transpose, while in **Rel**, all maps have a transpose.

Another important property of **Hilb** is that it has a class of maps which have a trace, much like the trace in finite-dimensional vector spces. But, like transposition, trace is only a partial operation. One abstract way to define this class is as follows (See [23].):

Definition 3.7 *A morphism $f: \mathcal{H} \to \mathcal{H}$ between Hilbert spaces is* trace class *if it can be written $f = gh$, where $h: \mathcal{H} \to \mathcal{K}$ and $g: \mathcal{K} \to \mathcal{H}$, with g, h are Hilbert-Schmidt maps.*

This definition is suitable for both defining a notion of trace class given any nuclear ideal, and can be used for defining a trace operation, which mimics what happens in **Hilb** for the trace class, see [2].

3.3 Nuclear ideals

The notion of symmetric monoidal dagger category can be viewed as simultaneously axiomatizing the crucial structure of the category of Hilbert spaces, and the category of sets and relations. Both categories are symmetric monoidal, and have an involutive functor which is the identity on objects. One of the key aspects of the category of sets and relations is that one has "transfer of variables" i.e. one can use the closed structure and the involution to move variables from "input" to "output". Intuitively speaking, this reflects the idea that the source and target of a binary relation are a matter of convention and a binary relation is an inherently symmetric object.

The category of Hilbert spaces does not allow such transfer of variables arbitrarily. Instead as we have seen, one has a large class of morphisms which can be transposed in this fashion, the Hilbert-Schmidt maps. To axiomatize this "partial transpose", the paper [2] introduced the new notions of *nuclear ideal* and *nuclear morphism*. This idea and the terminology were suggested by the definition of a nuclear morphism between Banach spaces, due to Grothendieck [17], and subsequent work of Higgs and Rowe [19]. The concept of nuclearity in analysis can be viewed as describing when one can think of linear maps as matrices. Of course, in the finite-dimensional case one can always do this and it will be the case that all maps between finite-dimensional vector spaces are nuclear. The Higgs-Rowe theory applies only to autonomous (symmetric monoidal closed) categories, while our definition applies to the somewhat different setting of symmetric monoidal dagger categories with conjugation; in our earlier paper [2] we worked with tensored $*$-categories.

Definition 3.8 (i) *Let $\mathcal{C}$ be a symmetric monoidal dagger category with conjugation. A* nuclear ideal *for $\mathcal{C}$ consists of the following structure:*

- *For all objects A, $B \in \mathcal{C}$, a subset $\mathcal{N}(A, B) \subseteq Hom(A, B)$. We will refer to the union of these subsets as $\mathcal{N}(\mathcal{C})$ or $\mathcal{N}$. We will refer to the elements of $\mathcal{N}$ as* nuclear maps*. The class $\mathcal{N}$ must be closed under composition with arbitrary $\mathcal{C}$-morphisms, closed under $\otimes$, closed under $(\,)^*$ and $(\,)^\dagger$*
- *A bijection $\theta: \mathcal{N}(A, B) \to Hom(I, A^* \otimes B)$. The bijection θ must be natural and preserve the structure of $(\,)^*$ in an evident sense, see [2].*

(ii) *Suppose $\mathcal{C}$ and $\mathcal{C}'$ are symmetric monoidal dagger categories with conjugation equipped with nuclear ideals. Then a functor $F: \mathcal{C} \to \mathcal{C}'$ is* nuclear *if it commutes with all relevant structure.*

i.e it is strong symmetric monoidal, commutes with $()^*$ and $()^\dagger$, and takes nuclear maps to nuclear maps.

3.4 Examples

- The category **Rel** of sets and relations is a tensored $*$-category for which the entire category forms a nuclear ideal.
- The category of Hilbert spaces and bounded linear maps maps is a well-known tensored $*$-category, which, in fact, led to the axiomatization [10]. Then the Hilbert-Schmidt maps form a nuclear ideal [2]
- The category **DRel** of tame distributions on Euclidean space [2] is a tensored $*$-category. The ideal of test functions (viewed as distributions) is a nuclear ideal.
- We will define a subcategory of **Rel** called the category of *locally finite relations.* Let $R: A \to B$ be a binary relation and $a \in A$. Then $R_a = \{b \in B | aRb\}$. Define R_b similarly for $b \in B$. Then we say that a relation is *locally finite* if, for all $a \in A, b \in B$, R_a, R_b are finite sets.Then it is straightforward to verify that we have a tensored $*$-category which is no longer compact closed. It is also easy to verify that the finite relations form a nuclear ideal.

4 Topological and conformal field theory, functorially

In this section, we give a slightly informal, pictorial description of the geometric category that is the basis of the functorial approach to field theory. We begin with the two-dimensional topological case. Objects of the geometric category 2Cob are finite families of circles. A morphism is an equivalence class of 2-manifolds with boundary. Such an equivalence class is called a *cobordism*, and a precise definition can be found in [28] for example. Composition is obtained by gluing along boundaries. Identities are families of cylinders. It is a property of the cobordism equivalence that gluing a cylinder along a boundary preserves equivalence classes. More details can be found for example in [7]. In this paper, we prefer to just exhibit the structure pictorially, as in Figure 1, which exhibits a morphism from two circles to one. Figure 2 is a composite of two morphisms, the

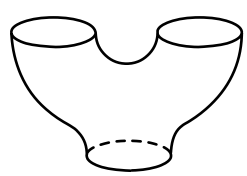

Fig. 1. A sample cobordism

second morphism being a morphism from one circle to two.

The category 2Cob is symmetric monoidal, indeed it is compact closed. Then a *2-dimensional topological quantum field theory* is a symmetric monoidal functor from 2Cob to the category of finite-dimensional Hilbert spaces. (We note that a functor between compact closed categories that preserves the symmetric monoidal structure automatically preserves the compact closed structure.)

Segal's notion of conformal field theory begins by replacing smooth manifolds with Riemann surfaces with boundary. Equivalence is then an appropriate notion of conformal equivalence. He

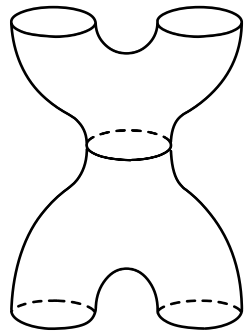

Fig. 2. A composite of two cobordisms

then allows the field theory functor to take as values infinite-dimensional Hilbert spaces. He further requires the image of a morphism under the field theory functor to be trace class.

This is an extremely interesting idea with many important consequences. In particular, it provides the basis for the elliptic cohomology of Hu and Kriz [20]. However, the resulting geometric structure lacks identities as conformal equivalence does not allow for the attaching of cylinders.

We argue in this paper that if topological quantum field theory is at the categorical level fundamentally about compact closed categories, then conformal field theory is about nuclear ideals. Furthermore, given a symmetric monoidal dagger category with conjugation containing a nuclear ideal, one can reasonably define morphisms not in the ideal to be *singular*. This definition captures for example the singularity of the Dirac delta as well as the failure of the identity on an infinite-dimensional Hilbert space to have a trace. In the present context, it is saying that the circle, which is what would act as identity in Segal's structure, is a singular morphism. This is very much in line with the idea that the circle is a singular point in the corresponding moduli space.

5 Segal's definition of conformal field theory

Segal's definition of Conformal Field Theory [31,32] actually predates TQFT, and in part inspired Atiyah's axioms. We begin by roughly describing the idea and then present an alternative formulation due to Neretin [29].

Define a *precategory* to be a category, except for the requirement of the existence of identities. Then define a precategory whose objects are of the form C_n, where n is a natural number and C_n is a family of n parametrized circles. A morphism $\mathsf{C}_n \to \mathsf{C}_m$ is a Riemann surface X, with boundary ∂X and an identification:

$$\partial X \cong \mathsf{C}_n^* \coprod \mathsf{C}_m$$

Here C_n^* refers to C_n, but with the parametrizations of the circles reversed. Let C_{nm} denote the moduli space of all morphisms $\mathsf{C}_n \to \mathsf{C}_m$. Composition is defined as in the cobordism category, and the result is a monoidal precategory, with monoidal structure also as in the cobordism category. Segal calls this precategory simply $\mathcal{C}$. A conformal field theory is then defined to be a tensor preserving functor to the category of Hilbert spaces such that each Hilbert morphism in the image of the functor is trace class. (Segal actually defines a CFT as a continuous, projective functor from $\mathcal{C}$ into the category of topological vector spaces. The values of a projective functor can be thought of as depending on the Riemann surface together with a chosen metric which is compatible with the conformal structure. For the purpose of this paper, we will not take projectivity or continuity into consideration.)

To make this idea more precise and to view CFT as part of a genuine categorical structure, we will describe a category extending a precategory introduced by Neretin [29]. Neretin's construction can be seen as an implementation of Segal's precategory $\mathcal{C}$, which makes precise how one can view surfaces with boundaries as Riemann surfaces (with a conformal structure). It is also described in such a way that it can naturally be extended to become a category. We describe this category, which we call **Pants**, and show that Neretin's structure is a nuclear ideal within this tensored $*$-category. We then show that in this context Segal's definition can be equivalently formulated as a nuclear functor from the pants category to the category of Hilbert spaces, equipped with its usual nuclear ideal.

To give an idea of the additional complexity entailed by this category, we refer to the discussion in Segal's papers. For example, one connected component of C_{01} consists of all surfaces which are topologically disks. In the topological category, there is only one disk, but in conformal field theory, the set of all disks has the structure of a complex manifold. By a standard argument, this manifold is isomorphic to Diff $(S^1)/PSL_2(\mathsf{R})$.

5.1 The category of pants

We write S^1 for the unit circle in the complex plane $\mathbb{C}$, and $\overline{\mathbb{C}}$ for the extended complex plane. Let D_+ and D_- be the domains defined by

$$D_+ = \{z \in \mathbb{C} |\, |z| \leq 1\} \text{ and } D_- = \{z \in \overline{\mathbb{C}} |\, |z| \geq 1\};$$

write D_+^o and D_-^o for their respective interiors. The domains D_+ and D_- are homeomorphic closed subsets of the Riemann sphere, with the property that the function $z \longrightarrow \frac{1}{z}$ gives a homeomorphism which reverses the orientation of the boundary circle.

Let Diff be the group of analytic orientation-preserving automorphisms of S^1. For a closed domain B, the words 'the function f is holomorphic (single-valued) in B up to the boundary' will mean that f extends holomorphically (single-valuedly) to some neighborhood of B.

Definition 5.1 *The category **Pants** is defined as follows:*

Objects *are finite disjoint unions of oriented circles. We represent these by pairs of natural numbers (n_1, n_2) where n_1 is the number of positively (counter-clockwise) oriented circles and n_2 is the number of negatively (clockwise) oriented circles.*

A morphism *from (m_1, m_2) to (n_1, n_2) consists of a collection*

$$(R, r_i^+, r_j^-), \quad 1 \leq i \leq m_1 + m_2, \quad 1 \leq j \leq n_1 + n_2,$$

*where R is a compact closed (possibly disconnected) oriented Riemann surface; moreover, the $r_i^+ \colon D_+ \to R$ and $r_j^- \colon D_- \to R$ are holomorphic functions which are single-valued up to the boundary, and for each pair i, j, the intersection $r_i^+(D_+) \cap r_j^-(D_-)$ is empty or S^1 (the boundary of the images of the two disks). Moreover, the $r_i^+(D_+)$ do not intersect pairwise, and the $r_j^-(D_-)$ do not intersect pairwise. Note that the r_i^+ and r_j^- induce maps $S^1 \hookrightarrow D_\pm \to R$. We require that for the $r_i^+(S^1)$, the orientation inherited from the domain circles (*i.e., *positive for the first m_1 and negative for the last m_2) is opposite to the orientation as subsets of R; for the codomain circles we require that the two orientations agree.*

Two such morphisms (R, r_i^+, r_j^-) and (P, p_i^+, p_j^-) are considered to be equivalent if there exists a biholomorphic orientation-preserving isomorphism $\rho\colon R \to P$ such that $\rho \circ r_i^+ = p_i^+$ and $\rho \circ r_i^- = p_i^-$. This is obviously the conformal analogue of cobordism.

Consider arrows:

$$(P, p_i^+, p_j^-) \in \textbf{\textit{Pants}}((m_1, m_2), (n_1, n_2)) \text{ and } (Q, q_j^+, q_l^-) \in \textbf{\textit{Pants}}((n_1, n_2), (k_1, k_2)).$$

Their composition

$$(R, r_i^+, r_l^-) = (Q, q_j^+, q_l^-) \circ (P, p_i^+, p_j^-) \in \textbf{\textit{Pants}}((m_1, m_2), (k_1, k_2))$$

is defined as follows. The surface R is obtained from the disjoint union of $P - \bigcup p_j^-(D_-^o)$ and $Q - \bigcup q_j^+(D_+^o)$ by identifying the points $p_j^-(e^{i\varphi})$ and $q_j^+(e^{i\varphi})$, where $j = 1, \cdots, n_1 + n_2$ and $\varphi \in [0, 2\pi]$. Further $r_i^+ = p_i^+$ and $r_l^- = q_l^-$. More precisely, note that $p_i^+(D_+) \subseteq P - \bigcup p_j^-(D_-^o) \subseteq R$ and $q_l^-(D_-) \subseteq Q - \bigcup q_j^+(D_+^o) \subseteq R$ so there are induced maps $r_i^+\colon D_+ \to R$ and $r_l^-\colon D_- \to R$. Note that the surface R inherits a well-defined orientation from the oriented surfaces P and Q.

Identities correspond to spheres in which the intersection $r_i^+(D_+) \cap r_j^-(D_-)$ is homeomorphic to S^1 and $r_i^+(e^{i\varphi}) = r_j^-(e^{i\varphi})$.

We note that this category is a *symmetric monoidal dagger category with conjugation*, where the tensor is defined on objects by

$$(m_1, m_2) \otimes (n_1, n_2) = (m_1 + n_1, m_2 + n_2)$$

and $I = (0, 0)$. To define $\otimes$ on arrows, let $(P, p_i^+, p_j^-)\colon (m_1, m_2) \to (m_1', m_2')$ and $(Q, q_k^+, q_l^-)\colon (n_1, n_2) \to (n_1', n_2')$; then

$$(P, p_i^+, p_j^-) \otimes (Q, q_k^+, q_l^-) = (R, r_i^+, r_j^-)\colon (m_1 + n_1, m_2 + n_2) \to (m_1' + n_1', m_2' + n_2'),$$

where $R = P \cup Q$, the disjoint union of P and Q, and

$$\begin{aligned}&(r_1^+, \ldots, r_{m_1+m_2+n_1+n_2}^+) = \\ &= (p_1^+, \ldots, p_{m_1}^+, q_1^+, \ldots, q_{n_1}^+, p_{m_1+1}^+, \ldots, p_{m_1+m_2}^+, q_{n_1+1}^+, \ldots, q_{n_1+n_2}^+),\end{aligned}$$

and

$$\begin{aligned}&(r_1^-, \ldots, r_{m_1'+m_2'+n_1'+n_2'}^-) = \\ &= (p_1^-, \ldots, p_{m_1'}^-, q_1^-, \ldots, q_{n_1'}^-, p_{m_1'+1}^-, \ldots, p_{m_1'+m_2'}^-, q_{n_1'+1}^-, \ldots, q_{n_1'+n_2'}^-).\end{aligned}$$

The †-*operator* $(-)^\dagger\colon \mathbf{Pants} \to \mathbf{Pants}^{\mathrm{op}}$ is the identity on objects and is defined on arrows by

$$(P, p_i^+, p_j^-)^* = (P, (p')_j^+, (p')_i^-).$$

The maps are defined by

(1) $$(p')_j^+(z) = p_j^-(z^{-1}) \text{ and } (p')_i^-(z) = p_i^+(z^{-1}).$$

Note that if $p_j^-\colon D_- \to P$, then $(p')_j^+\colon D_+ \to P$ and it induces the opposite orientation on the image of S^1.

Conjugation $(-)^*\colon \mathbf{Pants} \to \mathbf{Pants}$ is defined by reverse of orientation on both objects and arrows, *i.e.*, $(n_1, n_2)^* = (n_2, n_1)$ and if $\mathcal{P} = (P, p_i^+, p_j^-)$, then

$$\mathcal{P}_* = (P', p_{m_1+1}^+, \ldots, p_{m_1+m_2}^+, p_1^+, \ldots, p_{m_1}^+, p_{n_1+1}^-, \ldots, p_{n_1+n_2}^-, p_1^-, \ldots, p_{n_1}^-),$$

where P' has the same underlying surface as P, but opposite orientation.

Definition 5.2 *We will say that an arrow (R, r_i^+, r_j^-) has* positive volume *if for all i and j, the intersection $r_i^+(D_+) \cap r_j^-(D_-) = \emptyset$;* i.e., *every connected component of $R - (\bigcup r_i^+(D_+^o) \cup \bigcup r_i^-(D_-^o))$ has positive volume. Let $\mathbb{P}((n_1, n_2), (m_1, m_2))$ indicate the set of arrows with positive volume.*

Theorem 5.3 *The morphisms (R, r_i^+, r_j^-) with positive volume form a* nuclear ideal *in this category.*

Proof. The bijection

$$\theta\colon \mathbb{P}((m_1, m_2), (n_1, n_2)) \to \mathbf{Pants}((0, 0), (m_2 + n_1, m_1 + n_2))$$

is constructed as follows. Let (R, r_i^+, r_j^-) be an arrow in $\mathbb{P}((m_1, m_2), (n_1, n_2))$. Then $r_i^+(D_+) \cap r_j^-(D_-) = \emptyset$ for all i and j, so we can define

$$\theta(R, r_i^+, r_j^-) = (R, (r'_{m_1+1})^-, \ldots, (r'_{m_1+m_2})^-, r_1^-, \ldots, r_{n_1}^-, (r'_1)^-, \ldots, (r'_{m_1})^-, r_{n_1+1}^-, \ldots, r_{n_1+n_2}^-)$$

where r'_j is defined as in (1). Note that the orientation on R remains the same.

Finally, note that the class $\mathbb{P}$ is closed under tensor, dual, conjugation, and composition with other morphisms, and hence we have a two-sided ideal. □

5.2 An extension of the category of pants

Again following Neretin, the category of pants can be extended with the following structure to obtain the category $\widetilde{\mathbf{Pants}}$ with the same objects. Arrows in this new category are of the form

$$\mathcal{R} = (R, r_i^+, r_j^-, \pi), \qquad 1 \le i \le m_1 + m_2, \qquad 1 \le j \le n_1 + n_2,$$

where π is a maximal isotropic lattice in the first integral homology group of $R - \bigcup r_\alpha^\pm(D_\pm)$ (*i.e.*, a maximal collection of homotopy classes of loops in $R - \bigcup r_\alpha^\pm(D_\pm)$ such that the intersection index of any two of them is zero).

This is a symmetric monoidal dagger category with conjugation equipped with a nuclear ideal of morphisms (R, r_i^+, r_j^-, π) such that (R, r_i^+, r_j^-) has positive volume.

5.3 Segal's definition revisited

We are now in a position to propose an alternative to Segal's definition.

Definition 5.4 *A* conformal field theory (CFT) *is a nuclear functor from either **Pants** or $\widetilde{\mathbf{Pants}}$ to **Hilb**, each equipped with its usual nuclear structure.*

Note that this is justified by the fact that any CFT according to this definition restricts to one according to Segal's notion. This follows from the fact that in **Pants**, every nuclear map can be written as the composite of two nuclear maps, and hence a nuclear functor must take a nuclear map in **Pants** to a trace-class map in **Hilb** .

In the reverse direction, Segal has an additional requirement of continuity (which we have not dealt with, but can be adapted to our setting). This will ensure that his CFTs can be extended to functors from either **Pants** or $\widetilde{\textbf{Pants}}$ to **Hilb**.

As for examples, we will describe a generalized CFT in the next section. For the moment, we note that there is an evident forgetful functor from **Pants** or $\widetilde{\textbf{Pants}}$ to the category of 1-dimensional cobordisms, and so every TQFT yields a CFT, by composition.

5.4 Generalized conformal field theory

This then leads to the following obvious definition.

Definition 5.5 *A* generalized conformal field theory *is a nuclear functor from either* ***Pants*** *or* $\widetilde{\textbf{Pants}}$ *to any symmetric monoidal dagger category with conjugation and equipped with a nuclear ideal.*

In the next section, we will describe a nontrivial example of such a field theory, due to Neretin [29].

6 Neretin's example of a generalized conformal field theory

6.1 Correct linear relations

In this section we generalize the definition of the Ol'shanskii symplectic semigroup (cf.[29]) to a category of Hilbert spaces and correct linear relations. Let $V_{\mathbb{R}}$ be a complex Hilbert space with inner product $(\cdot,\cdot)$. Consider this as a real Hilbert space with a complex structure operator I such that $I^2 = -1$. Now complexify this space: $V = V_{\mathbb{R}} \oplus JV_{\mathbb{R}}$. The real and imaginary parts of the inner product on $V_{\mathbb{R}}$ can be extended in various ways to linear forms on V.

Extending the form $\mathrm{Re}(\cdot,\cdot)$ by sesquilinearity to the form $\langle\cdot,\cdot\rangle$ gives V a Hilbert space structure. Extending the form $i\,\mathrm{Im}(\cdot,\cdot)$ by bilinearity to $\{\cdot,\cdot\}$ makes V into a symplectic space. Moreover, extending $-i\,\mathrm{Im}(\cdot,\cdot)$ by sesquilinearity equips V with an indefinite Hermitian form $\Lambda(\cdot,\cdot)$. So V is a symplectic Hilbert space with an indefinite Hermitian form. V is also polarized in the following way. Define

$$V_{\pm} = \mathrm{Ker}(J \pm I).$$

Then V_+ and V_- are maximal subspaces in V which are isotropic (or Lagrange) with respect to the form $\{\cdot,\cdot\}$ (*i.e.*, $\{v,w\} = 0$ when $v,w \in V_+$ or $v,w \in V_-$). Note that V_+ and V_- can also be viewed as the positive and negative eigenspaces for the operator $IJ\colon V \to V$ as in [30].

Note that any $v \in V_+$ can be written as $v = Iu - Ju$ for some $u \in V_{\mathbb{R}}$ and analogously, any $v \in V_-$ can be written as $v = Iu + Ju$. One easily verifies that $\langle v,w\rangle = 0$ when $v \in V_+$ and $w \in V_-$. Moreover, any element $u \in V$ can be written in a unique way as $u = v+w$ with $v \in V_+$ and $w \in V_-$. So $V = V_+ \oplus V_-$.

For an element $v \in V$ we will write $v = (v_1, v_2)$ with $v_1 \in V_+$ and $v_2 \in V_-$. Note that we use the same notation for these pairs as for the inner product on $V_{\mathbb{R}}$. It will always be clear from the context which parentheses are meant. Conjugation on V is defined as conjugation on the complexification of $V_{\mathbb{R}}$. It is obvious that $V_+ = V_-^*$.

We will use an overline to view a vector as an element of the conjugate space; thus $\overline{v}$ is the same vector as v, but in the conjugate space. The transpose of an operator on V is defined as

$$A^t v = \overline{A^\dagger \overline{v}}.$$

For $v \in V$, we define v^t to be the element with the property that $(v, w) = \Lambda(v^t, w)$ for any $w \in V$.

Lemma 6.1 *If $v \in V_+$ and $w \in V_-$ then $\Lambda(v, v) = \langle v, v\rangle$ and $\Lambda(w, w) = -\langle w, w\rangle$.*

Proof. For arbitrary $u \in V$, write $u = u_1 + Ju_2$ with $u_1, u_2 \in V_{\mathbb{R}}$. Then

$$\begin{aligned}\langle u, u\rangle + \Lambda(u, u) &= \mathrm{Re}(u_1, u_1) + i\,\mathrm{Re}(u_2, u_1) - i\,\mathrm{Re}(u_1, u_2) + \\ &\quad + \mathrm{Re}(u_2, u_2) + \mathrm{Im}(u_2, u_1) - \mathrm{Im}(u_1, u_2) \\ &= (u_1, u_1) + (u_2, u_2) - 2\,\mathrm{Im}(u_1, u_2),\end{aligned}$$

so

$$\Lambda(u, u) = -\langle u, u\rangle + (u_1, u_1) + (u_2, u_2) - 2\,\mathrm{Im}(u_1, u_2).$$

For $v \in V_+$, write $v = Iv' - Jv'$ (with $v' \in V_{\mathbb{R}}$). Then the equation above implies that

$$\begin{aligned}
&(2)\quad \Lambda(v, v) = -\langle v, v\rangle + (Iv', Iv') + (-v', -v') - 2\,\mathrm{Im}(Iv', -v') \\
&(3)\quad = -\langle v, v\rangle + 2(v', v') + 2\,\mathrm{Im}(Iv', v') \\
&(4)\quad = -\langle v, v\rangle + 2(v', v') + 2\,\mathrm{Im}[i(v', v')] \\
&(5)\quad = -\langle v, v\rangle + 4(v', v').
\end{aligned}$$

We have also

$$\langle v, v\rangle = 2\,\mathrm{Re}(u, u) = 2(u, u). \tag{6}$$

From (5) and (6) we get that $\Lambda(v, v) = \langle v, v\rangle$, as required.

For $w = Iw' + Jw' \in V_-$, we get

$$\begin{aligned}\Lambda(w, w) &= -\langle w, w\rangle + (Iw', Iw') + (w', w') - 2\mathrm{Im}(Iw', w') \\ &= -\langle w, w\rangle + 2(w', w') - 2\,\mathrm{Im}[i(w', w')] \\ &= -\langle w, w\rangle,\end{aligned}$$

as required. □

Sums

Note that when V and W are both symplectic Hilbert spaces with indefinite Hermitian forms, this structure can be extended to the sum $V \oplus W$. The inner product is defined as the sum of the inner products:

$$\langle (v_1, w_1), (v_2, w_2)\rangle = \langle v_1, v_2\rangle + \langle w_1, w_2\rangle.$$

The symplectic and indefinite Hermitian forms are defined as differences of the original forms:

$$\begin{aligned}
&(7)\quad \{(v_1, w_1), (v_2, w_2)\}' = \{v_1, v_2\} - \{w_1, w_2\} \\
&(8)\quad \Lambda'((v_1, w_1), (v_2, w_2)) = \Lambda(v_1, v_2) - \Lambda(w_1, w_2)
\end{aligned}$$

Definition 6.2 (i) *A* linear relation $V \to W$ *is a subspace $P \subseteq V \oplus W$, which is Lagrange (*i.e.*, maximally isotropic) with respect to the symplectic form $\{\cdot, \cdot\}'$.*

(ii) *A linear relation P is called* correct *if it is the graph of an operator*

$$\Omega_P: V_+ \oplus W_- \to V_- \oplus W_+$$

where the matrix

$$\Omega_P = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

has the following properties:
(a) $K = -K^t$ *and* $M = -M^t$*;*
(b) $||\Omega_P|| \leq 1$*;*
(c) $||K|| < 1$ *and* $||M|| < 1$*;*
(d) K *and* M *are Hilbert-Schmidt operators.*

We say that the matrix Ω_P is associated with the relation P.

Lemma 6.3 *Condition (a) in Definition 6.2 is equivalent to requiring that the subspace P is Lagrange with respect to $\{\cdot,\cdot\}'$.*

Proof. We show first that P is isotropic with respect to $\{\cdot,\cdot\}'$ if and only if $K = -K^t$ and $M = -M^t$. Note that $\{\overline{v}, v'\} = \overline{-i\{v, \overline{v'}\}}$. So $\{L^t v_1, w_2\} = \{v_1, Lw_2\}$. We use this to calculate:

$$\begin{aligned}
&\{(v_1 + Kv_1 + Lw_2, L^t v_1 + Mw_2 + w_2), (v_1' + Kv_1' + Lw_2', L^t v_1' + Mw_2' + w_2')\}' = \\
&= \{v_1 + Kv_1 + Lw_2, v_1' + Kv_1' + Lw_2'\} - \{L^t v_1 + Mw_2 + w_2, L^t v_1' + Mw_2' + w_2'\} \\
&= \{v_1, Kv_1'\} + \{v_1, Lw_2'\} + \{Kv_1, v_1'\} + \{Lw_2, v_1'\} - \{L^t v_1, w_2'\} - \\
&\quad -\{w_2, L^t v_1'\} - \{Mw_2, w_2'\} - \{w_2, Mw_2'\} \\
&= \{v_1, Kv_1'\} + \{Kv_1, v_1'\} - \{Mw_2, w_2'\} - \{w_2, Mw_2'\}
\end{aligned}$$

This is equal to zero for all v_1, v_1', w_2, w_2' only when $\{v_1, Kv_1\} = -\{Kv_1, v_1'\}$ for all v_1, v_1' and $\{Mw_2, w_2'\} = -\{w_2, Mw_2'\}$ for all w_2, w_2', *i.e.*, when $K = -K^t$ and $M = -M^t$.

It remains to show that P is *maximally* isotropic under these conditions. Let $(v, w) = (v_1 + v_2, w_1 + w_2) \in V_+ \oplus V_- \oplus W_+ \oplus W_-$. Suppose that $\{(v, w), (v', w')\}' = 0$ for all $(v', w') = (v_1' + Kv_1' + Lw_2', L^t v_1' + Mw_2' + w_2') \in P$. Note that

$$\begin{aligned}
\{(v, w), (v', w')\}' &= \{v_1, Kv_1'\} + \{v_1, Lw_2'\} + \{v_2, v_1'\} - \{w_1, w_2'\} - \\
&\quad - \{w_2, L^t v_1'\} - \{w_2, Mw_2'\} \\
&= \{K^t v_1 + v_2 - Lw_2, v_1'\} + \{L^t v_1 - w_1 - M^t w_2, w_2'\}
\end{aligned}$$

So $\{(v, w), (v', w')\}' = 0$ for all v_1' and w_2' if and only if $K^t v_1 + v_2 - Lw_2 = 0$ and $L^t v_1 - w_1 - M^t w_2 = 0$, *i.e.*, $(v, w) \in P$. □

Lemma 6.4 *Condition (b) in Definition 6.2 implies that Λ' is nonnegative on P.*

Proof. Let $(v, w) = (v_1 + \Omega_P(v_1, w_2)_1, \Omega_P(v_1, w_2)_2 + w_2) \in P$. One easily verifies that $\Lambda(u, u') = 0$ when $u \in V_+$ and $u' \in V_-$. We use this together with Lemma 6.1 to calculate

$$\begin{aligned}
\Lambda'((v, w), (v, w)) &= \Lambda(v, v) - \Lambda(w, w) \\
&= \Lambda(v_1, v_1) + \Lambda(\Omega_P(v_1, w_2)_1, \Omega_P(v_1, w_2)_1) - \\
&\quad -\Lambda(\Omega_P(v_1, w_2)_2, \Omega_P(v_1, w_2)_2) - \Lambda(w_2, w_2)
\end{aligned}$$

$$
\begin{aligned}
&= \langle v_1, v_1\rangle + \langle w_2, w_2\rangle - \langle \Omega_P(v_1,w_2)_1, \Omega_P(v_1,w_2)_1\rangle \\
&\quad -\langle \Omega_P(v_1,w_2)_2, \Omega_P(v_1,w_2)_2\rangle \\
&= ||(v_1,w_2)||^2 - ||(\Omega_P(v_1,w_2)||^2
\end{aligned}
$$

So Λ' is nonnegative on P precisely when $||\Omega_P|| \leq 1$. □

Lemma 6.5 *Condition (c) in Definition 6.2 means that Λ is positive definite on $P\cap V$ and $P\cap W$.*

Proof. Recall that

$$P \cap V = \{(v_1, Kv_1, 0, 0)|L^t v_1 = 0\} \text{ and } P \cap W = \{(0, 0, Mw_2, w_2)|Lw_2 = 0\}.$$

For $P \cap V$, we calculate

$$
\begin{aligned}
\Lambda'(v_1, Kv_1, 0, 0) &= \Lambda(v_1, v_1) + \Lambda(Kv_1, Kv_1) \\
&= ||v_1||^2 - ||Kv_1||^2.
\end{aligned}
$$

For $P \cap W$ we calculate

$$
\begin{aligned}
\Lambda'(0, 0, Mw_2, w_2) &= -\Lambda(Mw_2, Mw_2) - \Lambda(w_2, w_2) \\
&= -||Mw_2||^2 + ||w_2||^2,
\end{aligned}
$$

by Lemma 6.1. It is obvious that Λ' is positive definite on these subspaces precisely when $||K|| < 1$ and $||M|| < 1$. □

6.2 *The category CLR*

We now define a composition for correct linear relations. We will see that composition in this category is closely related to Girard's execution formula in his *geometry of interaction*.

Let $P: U \to V$ and $Q: V \to W$ be correct linear relations with associated matrices $\Omega_P: U_+ \oplus V_- \to U_- \oplus V_+$ and $\Omega_Q: V_+ \oplus W_- \to V_- \oplus W_+$, say

$$\Omega_P = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \quad \text{and} \quad \Omega_Q = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}. \tag{9}$$

Then the composition $Q \circ P$ corresponds to the product of linear relations and the associated matrix $\Omega_P * \Omega_Q: U_+ \oplus W_- \to U_- \oplus W_+$ is

$$\begin{pmatrix} A & B \\ B^t & C \end{pmatrix} * \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} = \tag{10}$$

$$\begin{pmatrix} A + BK(1-CK)^{-1}B^t & B(1-KC)^{-1}L \\ L^t(1-CK)^{-1}B^t & M + L^t(1-CK)^{-1}CL \end{pmatrix}$$

(Note that since $||C|| < 1$, and $||K|| < 1$, the operators $1 - CK$ and $1 - KC$ are invertible.)

Lemma 6.6 *Let P and Q be correct linear relations as above. If $(u_1, u_2, v_1, v_2) \in P$ and $(v_1, v_2, w_1, w_2) \in Q$, then $\Omega_P * \Omega_Q(u_1, w_2) = (u_2, w_1)$.*

Proof. Let $(u_1, u_2, v_1, v_2) \in P$ and $(v_1, v_2, w_1, w_2) \in Q$. This implies that

$$\begin{cases} u_2 = Au_1 + Bv_2 \\ v_1 = B^t u_1 + Cv_2 \\ v_2 = Kv_1 + Lw_2 \\ w_1 = L^t v_1 + Mw_2 \end{cases}$$

The second and third equation can be rewritten as

$$\begin{cases} v_1 - Cv_2 = B^t u_1 \\ -Kv_1 + v_2 = Lw_2 \end{cases}$$

Multiplying the second equation with C gives

$$\begin{cases} v_1 - Cv_2 = B^t u_1 \\ -CKv_1 + Cv_2 = CLw_2 \end{cases}$$

Adding these equations gives

$$\begin{cases} v_1 - Cv_2 = B^t u_1 \\ (1 - CK)v_1 = B^t u_1 + CLw_2 \end{cases}$$

Since $1 - CK$ is invertible his implies that

$$v_1 = (1 - CK)^{-1}(B^t u_1 + CLw_2) = (1 - CK)^{-1}B^t u_1 + (1 - CK)^{-1}CLw_2.$$

Substituting this back into the original equations gives

$$\begin{aligned} v_2 &= Kv_1 + Lw_2 \\ &= K(1 - CK)^{-1}B^t u_1 + K(1 - CK)^{-1}CLw_2 + Lw_2 \\ &= K(1 - CK)^{-1}B^t u_1 + (K(1 - CK)^{-1}C + 1)Lw_2 \\ &= K(1 - CK)^{-1}B^t u_1 + (1 - KC)^{-1}Lw_2, \end{aligned}$$

$$\begin{aligned} u_2 &= Au_1 + Bv_2 \\ &= Au_1 + BK(1 - CK)^{-1}B^t u_1 + B(1 - KC)^{-1}Lw_2 \\ &= (A + BK(1 - CK)^{-1}B^t)u_1 + B(1 - KC)^{-1}Lw_2, \end{aligned}$$

and

$$\begin{aligned} w_1 &= L^t v_1 + Mw_2 \\ &= L^t(1 - CK)^{-1}B^t u_1 + L^t(1 - CK)^{-1}CLw_2 + Mw_2 \\ &= L^t(1 - CK)^{-1}B^t u_1 + (L^t(1 - CK)^{-1}CL + M)w_2. \end{aligned}$$

We conclude that $\Omega_P * \Omega_Q(u_1, w_2) = (u_2, w_1)$. □

Lemma 6.7 *If P and Q, represented by Ω_P and Ω_Q respectively, are composable correct linear relations as above, then $Q \circ P$ represented by $\Omega_P * \Omega_Q$ is a correct linear relation.*

Proof. We check all four conditions in Definition 6.2. Condition (a) follows from the fact that $[K(1-CK)^{-1}]^t = K^t(1-C^tK^t)^{-1}) = -K(1-CK)^{-1}$.

To prove condition (b), note that this condition applied to P and Q translates into

$$\Lambda((u_1, Au_1 + Bv_2), (u_1, Au_1 + Bv_2)) - \Lambda((B^t u_1 + Cv_2, v_2), (B^t u_1 + Cv_2, v_2)) \geq 0 \tag{11}$$

and

$$\Lambda((v_1, Kv_1 + Lw_2), (v_1, Kv_1 + Lw_2)) - \Lambda((L^t v_1 + Mw_2, w_2), (L^t v_1 + Mw_2, w_2)) \geq 0. \tag{12}$$

To show that $||\Omega_{Q\circ P}|| \leq 1$, we need to show that

$$\begin{aligned}
&\Lambda((u_1, (A + BK(1-CK)^{-1}B^t)u_1 + B(1-KC)^{-1}Lw_2),\\
&\quad (u_1, (A + BK(1-CK)^{-1}B^t)u_1 + B(1-KC)^{-1}Lw_2))\\
&- \Lambda((L^t(1-CK)^{-1}B^t u_1 + (M + L^t(1-CK)^{-1}CLw_2, w_2),\\
&\quad (L^t(1-CK)^{-1}B^t u_1 + (M + L^t(1-CK)^{-1}CLw_2, w_2)))
\end{aligned}$$

is nonnegative. This expression can be rewritten as the left hand side of the following inequality

$$\begin{aligned}
&\Lambda((u_1, Au_1 + B[K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2]),\\
&\quad (u_1, Au_1 + B[K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2]))\\
&- \Lambda((L^t[(1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2] + Mw_2, w_2),\\
&\quad (L^t[(1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2] + Mw_2, w_2))\\
&\geq \Lambda((B^t u_1 + C[K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2],\\
&\qquad [K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2]),\\
&\quad (B^t u_1 + C[K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2],\\
&\qquad [K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2]))\\
&- \Lambda(((1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2,\\
&\qquad K[(1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2] + Lw_2),\\
&\quad ((1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2,\\
&\qquad K[(1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2] + Lw_2)).
\end{aligned}$$

This inequality follows from (11) and (12) by taking

$$v_1 = (1-CK)^{-1}B^t u_1 + (1-CK)^{-1}CLw_2$$

and

$$v_2 = K(1-CK)^{-1}B^t u_1 + (1-KC)^{-1}Lw_2.$$

By linearity, the right hand side of the last inequality can be rewritten as

$$\begin{aligned}
&\Lambda((B^t u_1 + CK(1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1),\\
&\quad (B^t u_1 + CK(1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1))\\
&+ \Lambda((B^t u_1 + CK(1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1),\\
&\quad (C(1-KC)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1))\\
&+ \Lambda((C(1-KC)^{-1}Lw_2, (1-KC)^{-1}Lw_2),
\end{aligned}$$

$$
\begin{aligned}
&\quad (B^t u_1 + CK(1-CK)^{-1}B^t u_1, K((1-CK)^{-1}B^t u_1)) \\
&+ \Lambda((C(1-KC)^{-1}Lw_2, (1-KC)^{-1}Lw_2), \\
&\quad (C(1-KC)^{-1}Lw_2, (1-KC)^{-1}Lw_2)) \\
&- \Lambda(((1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1), \\
&\quad ((1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1)) \\
&- \Lambda(((1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1), \\
&\quad ((1-CK)^{-1}CLw_2, K(1-CK)^{-1}CLw_2 + Lw_2)) \\
&- \Lambda(((1-CK)^{-1}CLw_2, K(1-CK)^{-1}CLw_2 + Lw_2), \\
&\quad (1-CK)^{-1}CLw_2, K(1-CK)^{-1}CLw_2 + Lw_2)) \\
&- \Lambda(((1-CK)^{-1}CLw_2, K(1-CK)^{-1}CLw_2 + Lw_2), \\
&\quad ((1-CK)^{-1}B^t u_1, K(1-CK)^{-1}B^t u_1)) \\
&= 0.
\end{aligned}
$$

This last equality follows from the fact that

$$B^t u_1 + CK(1-CK)^{-1}B^t u_1 = (1-CK)^{-1}B^t u_1,$$

$$C(1-KC)^{-1}Lw_2 = (1-KC)^{-1}CLw_2,$$

and

$$K(1-CK)^{-1}CLw_2 + Lw_2 = (1-KC)^{-1}Lw_2.$$

We conclude that

$$
\begin{aligned}
&\Lambda((u_1, (A + BK(1-CK)^{-1}B^t)u_1 + B(1-KC)^{-1}Lw_2), \\
&\quad (u_1, (A + BK(1-CK)^{-1}B^t)u_1 + B(1-KC)^{-1}Lw_2)) \\
&- \Lambda((L^t(1-CK)^{-1}B^t u_1 + (M + L^t(1-CK)^{-1}CLw_2, w_2), \\
&\quad (L^t(1-CK)^{-1}B^t u_1 + (M + L^t(1-CK)^{-1}CLw_2, w_2))) \geq 0,
\end{aligned}
$$

as required.

We check condition (c) for the operator $A + BK(1-CK)^{-1}B^t$. The other operator can be treated in a similar fashion. Since $||\Omega_P|| \leq 1$, we have that for any $u \in U_+$,

$$
\begin{aligned}
&||Au + B[K(1-CK)^{-1}B^t u]||^2 + ||B^t u + C[K(1-CK)^{-1}B^t u]||^2 \\
&\leq ||u||^2 + ||K(1-CK)^{-1}B^t u||^2.
\end{aligned}
$$

So

$$
\begin{aligned}
&||[A + BK(1-CK)^{-1}B^t](u)||^2 \leq \\
&\leq ||u||^2 + ||K(1-CK)^{-1}B^t u||^2 - ||B^t u + C[K(1-CK)^{-1}B^t u]||^2 \\
&= ||u||^2 + ||K(1-CK)^{-1}B^t u||^2 - ||[(1-CK) + CK](1-CK)^{-1}B^t u||^2 \\
&= ||u||^2 + ||K(1-CK)^{-1}B^t u||^2 - ||(1-CK)^{-1}B^t u||^2 \\
&< ||u||^2 + ||(1-CK)^{-1}B^t u||^2 - ||(1-CK)^{-1}B^t u||^2 \\
&= ||u||^2.
\end{aligned}
$$

(The inequality follows from the fact that $||K|| < 1$.) So $||A + BK(1-CK)^{-1}B^t|| < 1$.

Condition (d) follows from the fact that A, B, K, and M are Hilbert Schmidt operators, and that these operators form an ideal. □

The identity arrow $V \to V$ has associated matrix $\Omega_V\colon V_+ \oplus V_- \to V_- \oplus V_+$ with

$$\Omega_V = \begin{pmatrix} 0 & I_{V_-} \\ I_{V_+} & 0 \end{pmatrix}.$$

Definition 6.8 *Let* CLR *be the category of Hilbert spaces and correct linear relations.*

The category CLR is symmetric monoidal with $\oplus$ as tensor-product and 0 (the zero-space) as unit for the tensor. Note that $(U \oplus U')_+ = U_+ \oplus U'_+$ and $(U \oplus U')_- = U_- \oplus U'_-$. The tensor of two correct linear relations $P \subseteq U \oplus V$ and $Q \subseteq U' \oplus V'$ with associated matrices

$$\Omega_P = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \quad \text{and} \quad \Omega_Q = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}.$$

is represented by the subspace

$$\{(u, u', v, v') | (u, v) \in P, (u', v') \in Q\} \in U \oplus U' \oplus V \oplus V'.$$

This is a correct linear relation with associated matrix

$$\Omega_{P \oplus Q} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & K & 0 & L \\ B^t & 0 & C & 0 \\ 0 & L^t & 0 & M \end{pmatrix}$$

Lemma 6.9 *The category* CLR *forms a symmetric monoidal dagger category with conjugation.*

Proof. In order to prove this lemma, we need to define a functor

$$(-)^\dagger\colon \mathsf{CLR}^{\mathrm{op}} \to \mathsf{CLR},$$

which is the identity on objects, and a conjugate functor

$$(-)^*\colon \mathsf{CLR} \to \mathsf{CLR}.$$

Let $P \subseteq V \oplus W$ be a correct linear relation with associated matrix

$$\Omega_P = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}.$$

The linear operator $\Omega_{P^\dagger}\colon W_+ \oplus V_- \to W_- \oplus V_+$ for the *adjoint relation* $P^\dagger \subseteq W \oplus V$ has matrix

$$\Omega_{P^\dagger} = \begin{pmatrix} \overline{M} & \overline{L}^t \\ \overline{L} & \overline{K} \end{pmatrix},$$

where $\overline{M}(v) = \overline{M\overline{v}}$. Note that

(13) $$(v_1, v_2, w_1, w_2) \in P \text{ if and only if } (\overline{w_2}, \overline{w_1}, \overline{v_2}, \overline{v_1}) \in P^\dagger.$$

The conjugation functor is defined on objects by conjugation in the complexification. As remarked above, this means that $V_+^* = V_-$ and $V_-^* = V_+$. The *conjugate relation* $P_* \subseteq V^* \oplus W^*$ has associated matrix

$$\Omega_{P_*} = \begin{pmatrix} \overline{K} & \overline{L} \\ \overline{L}^t & \overline{M} \end{pmatrix} : V_+^* \oplus W_-^* = V_- \oplus W_+ \to V_-^* \oplus W_+^* = V_+ \oplus W_-.$$

Note that

(14) $$(v_1, v_2, w_1, w_2) \in P \text{ if and only if } (\overline{v_1}, \overline{v_2}, \overline{w_1}, \overline{w_2}) \in P^*.$$

It is obvious that these functors satisfy the required properties. □

6.3 Positive relations

We may now want to ask ourselves which correct linear relations allow for a transfer of variables; *i.e.*, for which correct linear relations $P: U \to V$, represented by $P \subseteq U \oplus V$, there is a corresponding relation $\theta(P) \subseteq 0 \oplus (U^* \oplus V)$ representing $\theta(P): 0 \to U^* \oplus V$. At first glance, one might be tempted to think that since $0 \oplus U^* \oplus V \cong U^* \oplus V \cong U \oplus V$, this is the case for all correct linear relations. However, a more careful examination of the conditions on the corresponding linear operator $\Omega_{\theta(P)}$ shows that this operator only represents a correct linear relation when $||\Omega_P|| < 1$. (A complete proof will be given below.)

Definition 6.10 *We call a correct linear relation* positive *when* $||\Omega_P|| < 1$*, or equivalently, when the form* Λ' *as defined in (8) is positive on* P*.*

We shall show that the positive correct linear relations form a nuclear ideal in the category CLR.

Lemma 6.11 *The class of positive correct linear relations is closed under composition with arbitrary correct linear relations.*

Proof. Let P and Q be correct linear relations as defined above, and suppose that P is positive. Then we have that

(15) $$\Lambda((u_1, Au_1 + Bv_2), (u_1, Au_1 + Bv_2)) - \Lambda((B^t u_1 + Cv_2, v_2), (B^t u_1 + Cv_2, v_2)) > 0$$

and

(16) $$\Lambda((v_1, Kv_1 + Lw_2), (v_1, Kv_1 + Lw_2)) - \Lambda((L^t v_1 + Mw_2, w_2), (L^t v_1 + Mw_2, w_2)) \geq 0.$$

The proof that $Q \circ P$ is positive follows from (15) and (16) in the same way as the proof of Condition (b) in Lemma 6.7. Since the inequality in (15) is strict, the inequalities in the rest of that proof become strict too. □

It is straightforward to verify that positive correct linear relations are closed under $\oplus$, conjugation and duality. Finally, we need to construct a bijection $\theta: \mathcal{N}(U, V) \to \mathrm{Hom}\,(I, U^* \oplus V) =$

$\mathrm{Hom}\,(0, U^* \oplus V)$. Let $P \subseteq U \oplus V$ be a positive correct linear relation, with associated matrix

$$\Omega_P = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

Then $\theta(P) \subseteq 0 \oplus U^* \oplus V$ is represented by the linear operator:

$$\Omega_{\theta(P)}: 0 \oplus (U^* \oplus V)_- = 0 \oplus (U_+ \oplus V_-) \to 0 \oplus (U^* \oplus V)_+ = 0 \oplus (U_- \oplus V_+),$$

with matrix

$$\Omega_{\theta(P)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & L \\ 0 & L^t & M \end{pmatrix}$$

Note that this is a correct linear relation because P is positive. The function θ is a bijection, since for any matrix

$$\Omega_Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_1 & M_2 \\ 0 & M_3 & M_4 \end{pmatrix}$$

representing a linear operator $0 \oplus (U_+ \oplus V_-) \to 0 \oplus (U_- \oplus V_+)$, we have that if Ω_Q is associated with a correct linear relation $Q \subseteq 0 \oplus U^* \oplus V$, the matrix

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

is associated to a positive correct linear relation in $U \oplus V$. We conclude:

Theorem 6.12 *The class of positive correct linear relations forms a nuclear ideal in* CLR.

6.4 The functor T

In [29] Neretin constructs a presentation of his precategory **Shtan** (which is our nuclear ideal of pants with positive volume, but without orientations) in the category of correct linear relations. We will now show that his presentation can be extended to a nuclear functor $\widetilde{\mathbf{Pants}} \to \mathsf{CLR}$.

To define the functor T on objects, we start with the construction of the space $V = T(1, 0)$. Let $H = C^\infty(S^1, \mathbb{R})$ with nonnegative quadratic form

$$(f, g) = \frac{1}{\pi^2} \int_0^{2\pi} \text{p.v.} \int_0^{2\pi} \cot\left(\frac{\varphi - \psi}{2}\right) f(\varphi) g'(\psi) d\varphi d\psi.$$

Let $V_{\mathbb{R}}$ be the completion (w.r.t. the inner product) of H quotiented by the space of constant functions. This is a complex Hilbert space with complex structure operator I, given by the generalized

Hilbert transform

$$If(\varphi) = \frac{1}{\pi}\int_0^{2\pi} \cot\left(\frac{\varphi-\psi}{2}\right) f(\psi)d\psi. \tag{17}$$

The inner product on $V_{\mathbb{R}}$ is defined by

$$\langle f, g\rangle = (f, g) + i(f, Ig).$$

Apply the procedure described in Section 6.1 to obtain $V = V_{\mathbb{R}} \oplus JV_{\mathbb{R}}$ with the decomposition $V = V_+ \oplus V_-$. Then V is the space of complex-valued functions on S^1 determined up to addition of a constant. The space V_+ consists of functions which extend holomorphically into the interior of the disk D_+. The space V_- consists of functions which extend holomorphically into the exterior of the disk D_+.

For $f, g \in V_\pm$ we have

$$\langle f, g\rangle = \mp\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(\varphi)\overline{g'(\psi)}d\varphi d\psi.$$

The symplectic form is

$$\{f(z), g(z)\} = \frac{1}{2\pi i}\int_{|z|=1} f(z)dg(z),$$

and the indefinite Hermitian form is

$$\Lambda(f, g) = \int_{|z|=1} f(z)d\overline{g(z)}.$$

Note that since T has to preserve conjugation, the space $T(0,1) = V^*$ is constructed in the same way as V except that the complex structure operator in (17) is now defined with a minus-sign. Finally, since T should be monoidal, we define:

$$T(n_1, n_2) = V^{\oplus n_1} \oplus (V^*)^{\oplus n_2}.$$

Let $\mathcal{P} = (P, p_i^+, p_j^-, \pi)\colon (m_1, m_2) \to (n_1, n_2)$ be a morphism in $\widetilde{\mathbf{Pants}}$. Define $T(\mathcal{P}) \subseteq V^{\oplus m_1} \oplus \overline{V}^{\oplus m_2} \oplus V^{\oplus n_1} \oplus (V^*)^{\oplus n_2}$ by:

$$(f_1, \ldots, f_{m_1+m_2}, g_1, \ldots, g_{n_1+n_2}) \in T(\mathcal{P}) \tag{18}$$

if there exists a holomorphic 1-form F on $P - \bigcup p_\alpha^\pm(D_\pm)$ such that $(p_\alpha^+)^*F = df_\alpha$ and $(p_\alpha^-)^*F = dg_\alpha$ and such that the integrals of F over the cycles in π are all 0.

In our calculations we will write $f_\alpha = ((f_\alpha)_+, (f_\alpha)_-)$ and $g_\alpha = ((g_\alpha)_+, (g_\alpha)_-)$, where $(f_\alpha)_+ \in V_+ = V_-^*$ (*i.e.*, it extends holomorphically into D_+^o), and $(f_\alpha)_- \in V_- = V_+^*$ (*i.e.*, it extends holomorphically into D_-^o). In the notation of Section 6.2, (18) would read

$$\begin{aligned}((f_1)_+, \ldots, (f_{m_1})_+, (f_{m_1+1})_-, \ldots, (f_{m_1+m_2})_-, (f_1)_-, \ldots, (f_{m_1})_-,\\ (f_{m_1+1})_+, \ldots, (f_{m_1+m_2})_+, (g_1)_+, \ldots, (g_{n_1})_+, (g_{n_1+1})_-, \ldots, (g_{n_1+n_2})_-,\\ (g_1)_-, \ldots, (g_{n_1})_-, (g_{n_1+1})_+, \ldots, (g_{n_1+n_2})_+) \in T(\mathcal{P}).\end{aligned}$$

The linear relation $T(\mathcal{P})$ satisfies the conditions to be correct. The following theorem (cf. [29] or [27]), implies that the form Λ' is nonnegative on $T(\mathcal{P})$.

Theorem 6.13 ***[The Area Theorem]*** *Let μ be a holomorphic 1-form on*

$$P - \bigcup p_i(D^o_+).$$

*Suppose that every component of $P - \bigcup p_i(D^o_+)$ has positive volume (*i.e.*, there are no components which are homeomorphic to S^1), and the integrals of μ over all the cycles in the lattice π are* 0*. Then*

$$\sum_i \Lambda(p_i^*\mu, p_i^*\mu) > 0.$$

Thus defined, T is a functor of symmetric monoidal dagger categories with conjugation. Let

$$\mathcal{P} = (P, p_1^+, \ldots, p^+_{m_1+m_2}, p_1^-, \ldots, p^-_{n_1+n_2}, \pi)\colon (m_1, m_2) \to (n_1, n_2)$$

be an arrow in $\widetilde{\mathbf{Pants}}$. We first show that T preserves adjunction, *i.e.*, $T(\mathcal{P}^\dagger) = T(\mathcal{P})^\dagger$. If

$$\begin{aligned}((f_1)_+, \ldots, (f_{n_1})_+, (f_{n_1+1})_-, \ldots, (f_{n_1+n_2})_-, (f_1)_-, \ldots, (f_{n_1})_-,\\ (f_{n_1+1})_+, \ldots, (f_{n_1+n_2})_+, (g_1)_+, \ldots, (g_{m_1})_+, (g_{m_1+1})_-, \ldots, (g_{m_1+m_2})_-,\\ (g_1)_-, \ldots, (g_{m_1})_-, (g_{m_1+1})_+, \ldots, (g_{m_1+m_2})_+) \in T(\mathcal{P}^*),\end{aligned}$$

then there is a holomorphic 1-form F on P satisfying

(i) $[(p'_j)^+]^*F = df_j$, where $f_j = ((f_j)_+, (f_j)_-)$, for $j = 1, \ldots, n_1$;

(ii) $[(p'_j)^+]^*F = df_j$, where $f_j = ((f_j)_-, (f_j)_+)$, for $j = n_1, \ldots, n_1 + n_2$;

(iii) $[(p'_i)^-]^*F = dg_i$, where $g_i = ((g_i)_+, (g_i)_-)$, for $i = 1, \ldots, m_1$;

(iv) $[(p'_i)^-]^*F = dg_i$, where $g_i = ((g_i)_-, (g_i)_+)$, for $i = m_1, \ldots, m_1 + m_2$.

Note that if $[(p'_\alpha)^\pm]^*F = df$, then $[p^\mp_\alpha]^*F = d\overline{f}$, where $\overline{f}(z) = f(z^{-1})$ and if $f = (f_+, f_-)$, then $\overline{f} = (\overline{f_+}, \overline{f_-})$. So we have that

(i) $[(p_j)^-]^*F = d\overline{f_j}$, where $\overline{f_j^+} = (\overline{(f_j)_-}, \overline{(f_j)_+})$, for $j = 1, \ldots, n_1$;

(ii) $[(p_j)^-]^*F = d\overline{f_j}$, where $\overline{f_j} = (\overline{(f_j)_+}, \overline{(f_j)_-})$, for $j = n_1 + 1, \ldots, n_1 + n_2$;

(iii) $[(p_i)^+]^*F = d\overline{g_i}$, where $\overline{g_i} = (\overline{(g_i)_-}, \overline{(g_i)_+})$, for $i = 1, \ldots, m_1$;

(iv) $[(p_i)^+]^*F = d\overline{g_i}$, where $\overline{g_i} = (\overline{(g_i)_+}, \overline{(g_i)_-})$, for $i = m_1 + 1, \ldots, m_1 + m_2$.

This implies that

$$\begin{aligned}(\overline{(g_1)_-}, \ldots, \overline{(g_{m_1})_-}, \overline{(g_{m_1+1})_+}, \ldots, \overline{(g_{m_1+m_2})_+},\\ \overline{(g_1)_+}, \ldots, \overline{(g_{m_1})_+}, \overline{(g_{m_1+1})_-}, \ldots, \overline{(g_{m_1+m_2})_-},\\ \overline{(f_1)_-}, \ldots, \overline{(f_{n_1})_-}, \overline{(f_{n_1+1})_+}, \ldots, \overline{(f_{n_1+n_2})_+},\\ \overline{(f_1)_+}, \ldots, \overline{(f_{n_1})_+}, \overline{(f_{n_1+1})_-}, \ldots, \overline{(f_{n_1+n_2})_-}) \in T(\mathcal{P})\end{aligned}$$

By (13), this implies that

$$\begin{aligned}((f_1)_+, \ldots, (f_{n_1})_+, (f_{n_1+1})_-, \ldots, (f_{n_1+n_2})_-,\\ (f_1)_-, \ldots, (f_{n_1})_-, (f_{n_1+1})_+, \ldots, (f_{n_1+n_2})_+,\end{aligned}$$

$$(g_1)_+, \ldots, (g_{m_1})_+, (g_{m_1+1})_-, \ldots, (g_{m_1+m_2})_-,$$
$$(g_1)_-, \ldots, (g_{m_1})_-, (g_{m_1+1})_+, \ldots, (g_{m_1+m_2})_+) \in T(\mathcal{P})^\dagger,$$

as required. The proof that T preserves conjugation $(-)^*$ goes similarly.

Finally, note that T sends the nuclear ideal of pants with positive volume to the nuclear ideal of positive linear relations, so $T : \widetilde{\mathbf{Pants}} \to \mathsf{CLR}$ is a nuclear functor. Thus we have:

Theorem 6.14 *The functor* $T : \widetilde{\mathbf{Pants}} \to \mathsf{CLR}$ *is a generalized conformal field theory.*

7 Geometry of Interaction

We now give a brief overview of Girard's geometry of interaction program, and show how Neretin's formula for composition of correct linear relations is an instance of Girard's execution formula in the geometry of interaction. For a more extensive discussion of GoI, one can consider Girard's original paper [15]. Girard's work was reformulated by Abramsky and Jagadeesan [1], and Abramsky's paper [4] was the first to establish an explicit connection to traced monoidal categories. Haghverdi in his thesis [18], looks at the extension to the exponential fragment of linear logic and Abramsky elaborates further on these issues in [3].

The basic principle behind the traditional approach to categorical semantics is that one builds a category whose objects are formulas in the logic being modeled, and whose morphisms are equivalence classes of proofs. The equivalence relation is established in such a way that each equivalence class has within it a cut-free proof, *i.e.*, a proof which does not make use of the *cut-rule*:

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}\ CUT$$

So Gentzen's *cut-elimination theorem*, stated categorically, becomes the statement that every proof is equivalent to a cut-free proof.

The proof of the cut-elimination theorem is typically algorithmic in nature. By an iterative process, one generates a cut-free proof. At each step in the process, one replaces a cut with a "smaller" cut or cuts. There is a measure one assigns to a sequent proof, and one must verify that all of the cut-elimination rewrites reduce this measure. Categorically, we view all of the rewrites in the procedure as equations, thereby ensuring the desired result that every proof is equivalent to a cut-free one.

While this approach has been quite fruitful, Girard argues that it obscures the fact that cut-elimination is a dynamic process, and furthermore understanding this dynamics is fundamental in the theory of computation. This is due to the *Curry-Howard* isomorphism [16] which asserts an equivalence between cut-elimination (normalization) and computation.

Girard's geometry of interaction program moved beyond traditional categorical semantics by precisely capturing the dynamics of the cut-elimination process. GoI models can be thought of as dynamical systems or processes in which information is represented as a token which traces a path through a network. See [26] for an example of this application to computing. Of particular interest is the paper [11] which looks at proofs geometrically and the token-passing intuition becomes precise by considering paths through proofs (or more precisely proof nets) in linear logic.

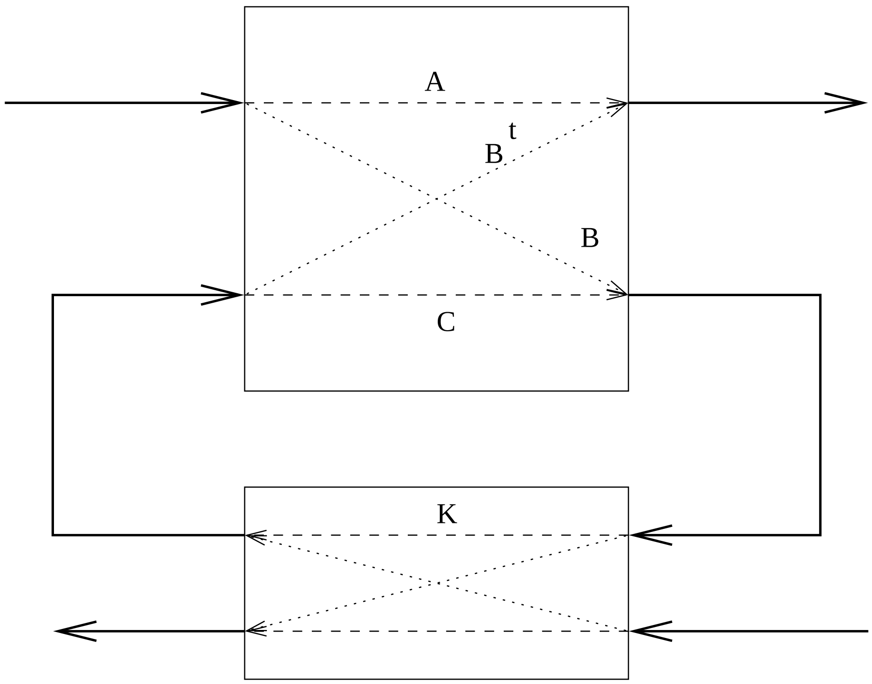

Fig. 3. Connecting Processes

In Girard's original formulation, a proof was represented as an operator on a Hilbert space, and cut-elimination was an operation on this space of operators. The operation was defined iteratively as an infinite sum, and convergence of the sum corresponded precisely to normalization of the proof. This sum is known as the *execution formula*. Subsequent work abstracted away from the original Hilbert space framework. See for example [3,4] for one line of development.

We will demonstrate a version of the execution formula using matrices as suggested by the INT-construction of [22], and show that it is precisely the composition of Neretin's category of correct linear relations.

Consider the matrices Ω_P and Ω_Q of section 6.2.

(19)
$$\Omega_P = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \quad \text{and} \quad \Omega_Q = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}.$$

For the purposes of this exposition, it is a sound intuition to think of the matrices as two-input, two-output processes, P and Q, and the entries as probabilities. If we were interpreting proofs, the input/outputs would be labeled by logical formulas.

For example, the entry A would represent the probability that a token entered the process P via the first (upper) left port and left via the first right port, and so on. Now suppose we allow these processes to interact as in Figure 3.

A straightforward calculation reveals that the probability of a token entering this composite process via the left upper port and leaving via the right upper port is given by $A+BK(1-CK)^{-1}B^t$. Similarly one can verify that with this intuition, one precisely recovers Neretin's definition of composition of correct linear relations. So we make the following observation:

Theorem 7.1 *Composition in the category* CLR *corresponds precisely to the execution formula of geometry of interaction.*

This is really little more than an observation, but we are hopeful that it is also the beginning of a fruitful line of research. The categorical reformulation of geometry of interaction is, in essence, the construction of an adjunction. We have an inclusion of the category of compact closed categories into the category of traced monoidal categories and geometry of interaction provides an adjoint to the inclusion. An evident question is to formulate the corresponding adjunction in the nuclear ideal setting. However, straightforward attempts to do this do not work and there is something new that needs to be understood. It is possible that this will reveal new structure in some of the categories considered here.

8 Conclusions

We have given a presentation of conformal field theory as a functor preserving nuclear ideal structure. This definition is essentially due to Segal, but in his paper he remarks, in a footnote, that the "category" of Riemann surfaces does not have identities and invites readers to supply their own remedy. We feel that the treatment of the present paper does that, but, in addition, brings out the beautiful fact that certain structures are "compact closed categories" even when they are not categories!

By making the link with nuclearity we have allowed one to consider other nuclear ideal systems as conformal field theories. For example, there are such structures arising in the categorical theory of stochastic processes. It would be fascinating if some of the connections to statistical mechanics could fit in this framework.

Also of interest would be to consider the recently defined *shape theory* for nuclear ideals of [8]. In shape theory, one has a categorical notion of approximation, and a canonical way in which more complex objects or morphisms are approximated by simpler ones. As a simple example, Hilbert-Schmidt maps are approximated by finite rank maps. This may allow for the possibility of more complex field theories to be approximated by simpler ones.

It will also be important to compare our notion of conformal field theory to other approaches to defining such theories. Perhaps most important is the work of Hu and Kriz [20,21], as examined extensively by Fiore [12,13]. Their approach makes use of higher-order categorical structure. They begin with the groupoid whose objects are Riemann surfaces with boundary and morphisms are isomorphisms respecting the boundaries. The operation of disjoint union makes this groupoid a symmetric monoidal category in which one can define the notion of a lax algebraic structure. The appropriate algebraic structure is the theory of commutative monoids with cancellation. Their geometric category then is a stack of lax commutative monoids with cancellation. Fiore [13] also deals directly with the lack of indentities in Segal's geometric category by adding "infinitely thin annuli". He then shows that the resulting category is a *Frobenius symmetric monoidal category.* The relationship of the work of Hu-Kriz and Fiore to the present work requires further research.

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