Smooth algebras, convenient manifolds and differential linear logic

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ongoing discussions with
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Goals

- Develop a theory of (smooth) manifolds based on differential linear logic.
- Convenient vector spaces were recently shown to be a model.
- There is a well-developed theory of convenient manifolds, including infinite-dimensional manifolds.
- Convenient manifolds reveal additional structure not seen in finite dimensions.

One place to start might be the C^∞-algebras of Lawvere. These algebras are in particular monoids, so we should be able to define similar stuctures in models of differential linear logic.

C^∞-algebras

The algebra of continuous complex-valued functions $C(X)$ on a space X reveals structure about the space. For example, if X is a compact, hausdorff space, $C(X)$ is a unital commutative C^*-algebra.

Conversely:

Theorem (Gelfand,Naimark)

Given a commutative, unital C^*-algebra $\mathcal{C}$, there exists a compact hausdorff space X such that $C(X) \cong \mathcal{C}$. This process induces a contravariant equivalence of categories.

We want to axiomatize the algebra of smooth real-valued functions on a manifold M, and derive a similar result.

C^∞-algebras II

Define a category $\mathcal{POLY}$ whose objects are Euclidean spaces $\mathbb{R}^n$. An arrow $\mathbb{R}^n \to \mathbb{R}^m$ is an m-tuple of polynomials with variables in the set $x_1, x_2, \ldots, x_n$. Composition is substitution.

Lemma

A (real) associative commutative algebra is the same thing as a product-preserving functor from the category $\mathcal{POLY}$ to the category of sets and functions.

- If A is such a functor, the algebra structure is on $\mathrm{A} = A(\mathbb{R})$.
- If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is an arrow, we have $A(f)\colon \mathrm{A}^n \to \mathrm{A}^m$.
- Addition on A is given by the interpretation of $f(x, y) = x + y$.
- Multiplication on A is given by the interpretation of $g(x, y) = xy$.
- Scalars are the constant polynomials.

C^∞-algebras III

We would like to similarly interpret the larger collection of smooth maps, not just polynomials.

Definition

- Let $\mathcal{SM}$ be the category whose objects are Euclidean spaces $\mathbb{R}^n$, and an arrow $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is just a smooth map.
- A C^∞-*algebra* is a product-preserving functor from $\mathcal{SM}$ to the category of sets.

So, a C^∞-algebra A is a commutative, associative algebra (since polynomials are smooth) such that furthermore:

- Given a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, there is a map $A(f)\colon \mathrm{A}^n \to \mathrm{A}^m$.

Examples of C^∞-algebras

- $C^\infty(U)$, the set of smooth functions from U to $\mathbb{R}$, with U an open subset of Euclidean space.
- $C^\infty(M)$, with M a manifold.
- $\mathbb{R}[[x_1, x_2, \ldots, x_n]]$, the formal power series ring.
- For a fixed point p in $\mathbb{R}^n$, $C_p^\infty(\mathbb{R}^n)$, the ring of germs of smooth functions at p.
- $R[\varepsilon]$, where $\varepsilon^2 = 0$.

This last example is one of the key ideas in synthetic differential geometry. (See Moerdijk and Reyes-*Models for smooth infinitessimal analysis*.)

But this leaves open the question of how to recover the manifold from the algebra. One approach can be found in Nestruev, "Smooth algebras and observables", due to ??.

Spectrum of an algebra

Let F be an associative, commutative algebra. We wish to view F as an algebra of functions on a space. That space will be the *spectrum of* F.

$$\mathrm{Spec}(\mathsf{F}) = Hom_{Alg}(\mathsf{F}, \mathbb{R})$$

Why is this a sensible choice? We have a canonical map:

$$\delta\colon U \to \mathrm{Spec}(C^\infty(U)) \quad \text{where } U \text{ is an open subset of } \mathbb{R}^n.$$

where $\delta(x)(f) = f(x)$

Theorem

The above map δ is a bijection.

Geometric algebras

Note that we have a pairing map:

$$< -, - >: \mathsf{Spec}(\mathsf{F}) \times \mathsf{F} \to \mathbb{R}$$

defined by $< x, f >= x(f)$. We will denote this as $f(x)$ to match intuition.

Definition

The algebra F *is* geometric *if, for any* $f_1, f_2 \in \mathsf{F}$, *if* $f_1 \neq f_2$, *there is an* $x \in \mathsf{Spec}(\mathsf{F})$ *with* $f_1(x) \neq f_2(x)$.

The nongeometric case is still of great interest, especially in algebraic geometry.

(González & Sancho de Salas-C^∞*-differentiable spaces*)

Lemmas and examples

Lemma

F *is geometric if and only if* $\bigcap_{x \in \mathrm{Spec}(\mathrm{F})} ker(< x, - >) = \{0\}$

Lemma

If $\mathrm{Spec}(\mathrm{F})$ *is topologized with the weakest topology making all functionals of the form* $< -, f >$ *with* $f \in \mathrm{F}$ *continuous, then* $\mathrm{Spec}(\mathrm{F})$ *is a hausdorff space.*

- Let $\mathrm{F} = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be the polynomial algebra. Then $\mathrm{Spec}(\mathrm{F}) = \mathbb{R}^n$ with its usual topology.
- Let $\mathrm{F} = \mathcal{C}^\infty(U)$ be the algebra of real-valued smooth functions on U, an open subset of $\mathbb{R}^n$. Then the map

$$\delta \colon U \longrightarrow \mathrm{Spec}(\mathrm{F})$$

is a homeomorphism.

Restrictions and completeness

Definition

Suppose that F is a geometric algebra and $A \subseteq \mathrm{Spec}(\mathrm{F})$ is any subset of the space of points. The *restriction* $\mathrm{F}\,|_A$ is defined to be the set of all functions $f \colon A \to \mathbb{R}$ such that for all points $a \in A$, there is a neighborhood $U \subseteq A$ of a and an element $\overline{f} \in \mathrm{F}$ such that the restriction of $\overline{f}$ to U is equal to f restricted to U.

Lemma

Let $U, V \subseteq \mathbb{R}^n$ be open subsets with $V \subseteq U$. Let $\mathrm{F} = \mathcal{C}^\infty(U)$. Recalling that $\mathrm{Spec}(\mathrm{F}) \cong U$, then $\mathrm{F}\,|_V = \mathcal{C}^\infty(V)$.

Restrictions and completeness II

If $A \subseteq \mathrm{Spec}(\mathsf{F})$ and $f \in \mathsf{F}$, then we can restrict f to A, and denote this $f\,|_A$. This map is denoted

$$\rho = \rho_A \colon \mathsf{F} \to \mathsf{F}\,|_A$$

Note that ρ_A may not be surjective, even if $A = \mathrm{Spec}(\mathsf{F})$.

Definition

A geometric algebra is *complete* if the map

$$\rho \colon \mathsf{F} \to \mathsf{F}\,|_{\mathrm{Spec}(\mathsf{F})}$$

is surjective.

Lemma

The algebra $\mathsf{F} = \mathcal{C}^\infty(U)$ is complete, but the algebra of bounded smooth functions is not.

Definition

A complete, geometric algebra F is *smooth* if there is a countable covering of Spec(F), denoted $\{U_i\}_{i \in I}$ such that for every $i \in I$, there is an algebra isomorphism θ_i: F $|_{U_i} \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)$.

Lemma

Smooth algebras are C^∞-algebras, i.e. a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has an interpretation $A(f)$ on the algebra.

The correctness of these axioms is stated in a series of theorems. See Nestruev.

Smooth algebras II

Theorem

Suppose $\mathsf{F} = \mathcal{C}^\infty(M)$, *with* M *a smooth, n-dimensional manifold. Then* F *is a smooth, n-dimensional algebra, and the map*

$$\delta\colon M \to \mathrm{Spec}(\mathsf{F}) \qquad \delta(p)(f) = f(p)$$

is a homeomorphism.

Theorem

Suppose F *is a smooth, n-dimensional algebra. Then there exists a smooth atlas on the dual space* $M = \mathrm{Spec}(\mathsf{F})$ *such that the map*

$$\varphi\colon \mathsf{F} \to \mathcal{C}^\infty(M) \qquad \varphi(f)(p) = p(f)$$

is an algebra isomorphism.

Smooth algebras III

Finally, one can verify that the construction works properly on arrows. In other words, smooth maps between manifolds correspond precisely to algebra homomorphisms under this construction.

Corollary

There is a contravariant equivalence between the category of smooth manifolds and smooth maps, and the category of smooth algebras and algebra homomorphisms.

Convenient vector spaces (Frölicher,Kriegl)

Definition

A vector space is *locally convex* if it is equipped with a topology such that each point has a neighborhood basis of convex sets, and addition and scalar multiplication are continuous.

- Locally convex spaces are the most well-behaved topological vector spaces, and most studied in functional analysis.
- Note that in any topological vector space, one can take limits and hence talk about derivatives of curves. A curve is *smooth* if it has derivatives of all orders.
- The analogue of Cauchy sequences in locally convex spaces are called *Mackey-Cauchy sequences.*
- The convergence of Mackey-Cauchy sequences implies the convergence of all Mackey-Cauchy nets.

The following is taken from a long list of equivalences.

Convenient vector spaces II: Definition

Theorem

Let E be a locally convex vector space. The following statements are equivalent:

- *If $c\colon \mathbb{R} \to E$ is a curve such that $\ell \circ c\colon \mathbb{R} \to \mathbb{R}$ is smooth for every linear, continuous $\ell\colon E \to \mathbb{R}$, then c is smooth.*
- *Every Mackey-Cauchy sequence converges.*
- *Any smooth curve $c\colon \mathbb{R} \to E$ has a smooth antiderivative.*

Definition

Such a vector space is called a *convenient* vector space.

Convenient vector spaces III: Bornology

The theory of bornological spaces axiomatizes the notion of bounded sets.

Definition

A *convex bornology* on a vector space V is a set of subsets $\mathcal{B}$ (the bounded sets) such that

- $\mathcal{B}$ is closed under finite unions.
- $\mathcal{B}$ is downward closed with respect to inclusion.
- $\mathcal{B}$ contains all singletons.
- If $B \in \mathcal{B}$, then so are $2B$ and $-B$.
- $\mathcal{B}$ is closed under the convex hull operation.

A map between two such spaces is *bornological* if it takes bounded sets to bounded sets.

Convenient vector spaces IV: More bornology

- To any locally convex vector space V, we associate the *von Neumann bornology*. $B \subseteq V$ is bounded if for every neighborhood U of 0, there is a real number λ such that $B \subseteq \lambda U$.
- This is part of an adjunction between locally convex topological vector spaces and convex bornological vector spaces. The topology associated to a convex bornology is generated by *bornivorous disks*. See Frölicher and Kriegl.

Theorem

Convenient vector spaces can also be defined as the fixed points of these two operations, which satisfy Mackey-Cauchy completeness and a separation axiom.

- The category Con of convenient vector spaces and continuous linear maps forms a symmetric monoidal closed category. The tensor is a completion of the algebraic tensor. There is a convenient structure on the space of linear, continuous maps giving the **internal hom**.
- Since these are topological vector spaces, one can define smooth curves into them.

Definition

A function $f: E \to F$ with E, F being convenient vector spaces is *smooth* if it takes smooth curves in E to smooth curves in F.

Convenient vector spaces VI: More key points

- The category of convenient vector spaces and smooth maps is cartesian closed. This is an enormous advantage over Euclidean space, as it allows us to consider function spaces.
- There is a comonad on Con such that the smooth maps form the coKleisli category:

We have a map δ as before, but now the target is the larger space of linear, continuous maps:

$$\delta\colon E \rightarrow \mathrm{Con}(C^\infty(E), \mathbb{R}) \quad \delta(x)(f) = f(x)$$

Then we define $!E$ to be the closure of the span of the set $\delta(E)$.

Theorem (Frölicher,Kriegl)

- $!$ *is a comonad.*
- $!(E \oplus F) \cong\ !E \otimes\ !F.$
- *Each object* $!E$ *has canonical bialgebra structure.*

Convenient vector spaces VII: It's a model

Theorem (Frölicher,Kriegl)

The category of convenient vector spaces and smooth maps is the coKleisli category of the comonad $\mathbf{!}$.

One can then prove:

Theorem (RB, Ehrhard, Tasson)

Con *is a model of differential linear logic. In particular, it has a codereliction map given by:*

$$coder(v) = \lim_{t \to 0} \frac{\delta(tv) - \delta(0)}{t}$$

Codereliction

Using this codereliction map, we can build a more general differentiation operator by precomposition:

Consider $f\colon\ !E \to F$ then define $df\colon E \otimes\ !E \to F$ as the composite:

$$E \otimes\ !E \stackrel{coder \otimes id}{\longrightarrow}\ !E \otimes\ !E \stackrel{\nabla}{\longrightarrow}\ !E \stackrel{f}{\to} F$$

Theorem (Frölicher,Kriegl)

Let E and F be convenient vector spaces. The differentiation operator

$$d\colon \mathcal{C}^\infty(E, F) \to \mathcal{C}^\infty(E, \mathrm{Con}(E, F))$$

defined as

$$df(x)(v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

is linear and bounded. In particular, this limit exists and is linear in the variable v.

A convenient differential category

The above results show that Con really is an optimal differential category.

- The differential inference rule is really modelled by a directional derivative.
- The coKleisli category really is a category of smooth maps.
- Both the base category and the coKleisli category are closed, so we can consider function spaces.

This seems to be a great place to consider manifolds. There is a well-established theory.

Kriegl, Michor-*The convenient setting for global analysis*

Convenient manifolds

Definition

- A *chart* (U, u) on a set M is a bijection $u\colon U \to u(U) \subseteq E$ where E is a fixed convenient vector space, and $u(U)$ is an open subset.
- Given two charts (U_α, u_α) and (U_β, u_β), the mapping $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}$ is called a *chart-changing*.
- An *atlas* or *smooth atlas* is a family of charts whose union is all of M and all of whose chart-changings are smooth.
- A *(convenient) manifold* is a set M with an equivalence class of smooth atlases.
- Smooth maps are defined as usual.

Lemma

A function between convenient manifolds is smooth if and only if it takes smooth curves to smooth curves.

This is a complicated subject.

Definition

A manifold M is *smoothly hausdorff* if smooth real-valued functions separate points.

Note that this implies:

- M is hausdorff in its usual topology, **which implies:**
- The diagonal is closed in the manifold $M \times M$.

These three notions are equivalent in finite-dimensions. In the convenient setting, the reverse implications are open. Note that the product topology on $M \times M$ is different than the manifold topology! Also:

Lemma

There are smooth functions that are not continuous. (Seriously.)

Smooth real-compactness

As before, we have a map:

$$\delta \colon E \longrightarrow Hom_{Alg}(C^\infty(E), \mathbb{R})$$

It may or may not be a bijection. We say:

Definition

A convenient vector space is *smoothly real-compact*, if the above map is a bijection.

Theorem (Arias-de-Reyna,Kriegl,Michor)

The following classes of spaces are smoothly real-compact:

- *Separable Banach spaces.*
- *Arbitrary products of separable Fréchet spaces.*
- *Many more.*

Smoothly regular spaces

Definition

A convenient vector space V is *smoothly regular* if for every $x \in V$, for every neighborhood U of x, there is a smooth function $f\colon V \to \mathbb{R}$ such that $f(x) = 1$ and $f^{-1}(\mathbb{R} \setminus \{0\}) \subseteq U$.

Not even Banach spaces, let alone convenient vector spaces, necessarily satisfy this property.

- It is unknown whether the product of two such spaces is still smoothly regular.
- The same is true of smoothly real-compact spaces.

Definition

A complete, geometric algebra F is *conveniently smooth* if there is a covering of $|\mathsf{F}|$, denoted $\{U_i\}_{i\in I}$ such that for every $i \in I$, there is an algebra isomorphism $\theta_i\colon \mathsf{F}\mid_{U_i} \longrightarrow \mathcal{C}^\infty(E)$ for a fixed convenient vector space.

But to what extent do the above results recapturing the manifold from its algebra lift to this setting? Is this the right definition?

Convenient smooth algebras II

Open Questions

- If a convenient manifold is built using a smoothly real-compact vector space, does it satisfy the property of smooth real-compactness?
- Assuming the above, the program of smooth algebras should go through, but there are many details to check.
- In the case of a manifold built on a non smoothly real compact space, the algebra of functions is clearly not good enough. What is?

Tangent spaces

The many equivalent notions of tangent in finite-dimensions now become distinct. See Kriegl-Michor.

Definition

Let E be a convenient vector space, and let $a \in E$. A *kinematic tangent vector* at a is a pair (a, X) with $X \in E$. Let $T_aE = E$ be the space of all kinematic tangent vectors at a.

The above should be thought of as the set of all tangent vectors at a of all curves through the point a.

For the second definition, let $C^\infty_a(E)$ be the quotient of $C^\infty(E)$ by the ideal of those smooth functions vanishing on a neighborhood of a. Then:

Tangent spaces II

Definition

An *operational tangent vector* at a is a continuous derivation, i.e. a map

$$\partial\colon C_a^\infty(E) \to \mathbb{R}$$

such that

$$\partial(f \circ g) = \partial(f) \circ g(a) + f(a)\partial(g)$$

Note that every kinematic tangent vector induces an operational one via the formula

$$X_a(f) = df(a)(X)$$

where d is the directional derivative operator. Let $D_a E$ be the space of all such derivations.

Tangent spaces III

In finite dimensions, the above definitions are equivalent and the described operation provides the isomorphism. That is no longer the case here.

Let $Y \in E''$, the second dual space. Y canonically induces an element of $D_a E$ by the formula $Y_a(f) = Y(df(a))$. This gives us an injective map $E'' \to D_a E$. So we have:

$$T_a E \hookrightarrow E'' \hookrightarrow D_a E$$

Definition

E satisfies the *approximation property* if $E' \otimes E$ is dense in $Con(E, E)$ (This is basically the MIX map.).

Theorem (Kriegl,Michor)

If E satisfies the approximation property, then $E'' \cong D_a E$. If E is also reflexive, then $T_a E \cong D_a E$.

Some category theory

- A *differential category* (RB, Cockett, Seely) is a model of differential linear logic.

  In particular, it is symmetric monoidal closed with a comonad satisfying the usual properties and a codereliction operator $coder\colon E \to \,!\,E$.

- A *cartesian differential category* (RB, Cockett, Seely) is an axiomatization of the coKleisli category.

  In particular, it has finite products and an operator:

$$f\colon X \to Y \implies D(f)\colon X \times X \to Y$$

  satisfying usual equations. These were used by Bucciarelli, Ehrhard and Manzonetto in modelling the *resource λ-calculus*.

Some more category theory

- A *restriction category* (Cockett, Lack) is an axiomatization of a category of partial functions. In particular, there is an operator

$$f\colon X \to Y \implies \overline{f}\colon X \to X$$

$\overline{f}$ should be thought of as the inclusion of the domain of definition of f into the set X. There are 4 rules, including:
    - $f\overline{f} = f$
    - If $f\colon X \to Y$ and $g\colon X \to Z$, then $\overline{f}\overline{g} = \overline{g}\overline{f}$.

Theorem (Cockett, Lack)

Every restriction category embeds into a category of partial maps.

- A *differential restriction category* (Cockett, Crutwell, Gallagher) has the cartesian differential category operator and is a restriction category, and the two structures interact properly.

Still more category theory

Let C be a restriction category.

- An arrow $f\colon X \to Y$ is a *partial isomorphism* if there exists $g\colon Y \to X$ such that $gf = \overline{f}$ and $fg = \overline{g}$.
- Given two maps $f, g\colon X \to Y$, say $f \leq g$ if $\overline{f}g = f$. This says g is more defined than f, and they agree where both are defined.
- Write $f \smile g$ if $g\overline{f} = f\overline{g}$. This says f and g are compatible, i.e. they agree on the overlap.
- C is a *join restriction category* if every family of pairwise compatible arrows has a join.

Atlases categorically (Grandis, Cockett)

Definition

Let C be a join restriction category. An *atlas of objects* is a set of objects $\{X_i\}_{i\in I}$ with a series of maps $\varphi_{ij}\colon X_i \to X_j$ such that:

- $\varphi_{ij}\varphi_{ii} = \varphi_{ij}$
- $\varphi_{jk}\varphi_{ij} \leq \varphi_{ik}$
- φ_{ij} has partial inverse φ_{ji}

Definition

If (U_i, φ_{ij}) and (V_k, ψ_{kh}) are atlases, then an atlas morphism is a family of maps $A_{ik}\colon U_i \to V_k$ satisfying 3 equations. Composition uses the join structure.

Lemma

The resulting category, denoted $\mathrm{Atl}(C)$, *is a join restriction category.*

Tangents (Cockett, Crutwell)

Given a join restriction category C and an atlas $\mathrm{M} = (U_i, \varphi_{ij})$, define a new atlas $T\mathrm{M}$ as follows:

- The same index set as M.
- The charts are of the form $U_i \times U_i$
- The transition maps are $U_i \times U_i \xrightarrow{\langle D\varphi_{ij}, \varphi_{ij}\pi_1\rangle} U_j \times U_j$.

Then:

- One can also extend T to a functor on the atlas category.
- There is a projection $\pi\colon T\mathrm{M} \to \mathrm{M}$ giving the tangent bundle.
- The axioms of a cartesian differential category combine to give additive structure on tangent spaces.

Still to do:

- The above construction seems to capture the notion of kinematic tangent vector well. But what about operational tangent vectors?
- Will a theorem similar to Kriegl and Michor's relating kinematic and operational tangent vectors hold much more abstractly?
- Further develop the theory of conveniently smooth algebras. In the general case, replace algebras with ??
- What can one say syntactically about manifolds? In this talk we only consider semantics.
- Manifold invariants, like de Rham cohomology, should be considered at the level of differential categories.