

# Von Neumann Categories

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- Algebraic quantum field theory (AQFT) is a mathematically rigorous framework for modelling the interaction of quantum mechanics and relativity.
- It is also explicitly category-theoretic; essentially an AQFT is a well-behaved functor.
- We consider Minkowski space as an ordered set with the causal ordering and then take the set of relatively compact opens. These opens form a directed poset under inclusion.

- An AQFT is then a functorial assignment of a  $C^*$ -algebra to each open. So we have a map:

$$U \mapsto \mathcal{A}(U)$$

The algebras  $\mathcal{A}(U)$  are called *local algebras*. They are the algebras of observables local to that region. Then, as in the  $C^*$ -algebraic interpretation of QM, a local *state* is then a positive linear functional.

- Since this set of opens in Minkowski space is directed, one can form the directed colimit of the local algebras. It will just be the closure of the union. The result is denoted  $\hat{\mathcal{A}}$ , and called the *quasilocal algebra*.

- Then the crucial condition says there can be no influence between spacelike separated regions:
- (Einstein Causality) If  $U$  and  $V$  are spacelike separated regions, i.e. there can be no causal influence in either direction, then the local algebras  $\mathcal{A}(U)$  and  $\mathcal{A}(V)$  pairwise commute in the quasilocal algebra.
- One typically adds further conditions, such as invariance with respect to the action of the Poincaré group, but we'll ignore this.

- Quantum mechanics is reformulated away from the notion of  $C^*$ -algebras and expressed in abstract, categorical terms.
- The categorical structure in question is that of a *compact closed dagger category*. So we have a symmetric monoidal category, with *dual objects*, i.e. an  $A^*$  with arrows

$$I \rightarrow A \otimes A^* \text{ and } A^* \otimes A \rightarrow I$$

inducing closed structure (Kelly).

- We furthermore assume an involutive contravariant endofunctor  $\dagger: \mathcal{C}^{op} \rightarrow \mathcal{C}$  which is the identity on objects, and interacts correctly with the monoidal structure.
- The primary examples are Rel, the category of sets and relations, and  $\text{Hilb}_{fd}$ , the category of finite-dimensional Hilbert spaces.

- The authors show for example that compact closed dagger categories provide sufficient structure to model protocols such as quantum teleportation or entanglement swapping. The correctness of the interpretation basically just amounts to the coherence equations of the theory.
- **But**, this encoding does not take into account that protocols take place in space-time, and relativistic effects may be significant.
- A straightforward modification of the definition of AQFT would be to assign to each open in spacetime a compact closed dagger category. Much of the above structure is easily lifted to this level.
- The problem is with expressing Einstein Causality.

- We propose modifying the usual notion of compact closed dagger category by replacing the monoidal structure with premonoidal structure (Power-Robinson).
- The lack of bifactoriality in the definition of premonoidal category will allow us to express a categorical version of the above causality condition.
- Most  $C^*$ -algebras that arise in AQFT are *von Neumann algebras*. The notion of commutant in premonoidal categories allows us to define a categorification of von Neumann algebra.

## Definition

A **binoidal** category consists of a category  $\mathcal{C}$  and functors  $H_B: \mathcal{C} \rightarrow \mathcal{C}$  and  $K_B: \mathcal{C} \rightarrow \mathcal{C}$  for all objects  $B$  in  $\mathcal{C}$  and satisfying  $H_B(C) = K_C(B)$  for all pairs of objects.

- In a binoidal category the object  $H_B(C) = K_C(B)$  is denoted  $B \otimes C$  and for any arrow  $f: X \rightarrow Y$  we write  $B \otimes f$  for  $H_B(f)$  and  $f \otimes B$  for  $K_B(f)$ .
- Thus in this new notation  $H_B = B \otimes -$  and  $K_B = - \otimes B$ . Notice that  $- \otimes -$  is only a functor when one of the arguments is fixed, i.e. it is not a bifunctor.

## Definition

A *premonoidal category* consists of a binoidal category  $\mathcal{C}$  together with a distinguished object  $I \in |\mathcal{C}|$  and natural isomorphisms  $\alpha$ ,  $\lambda$  and  $\rho$   
 $\alpha: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ ,  $\lambda: I \otimes A \longrightarrow A$ , and  $\rho: A \otimes I \longrightarrow A$ .  
These structural isomorphisms must satisfy coherence conditions.

## Definition

- If  $\mathcal{C}$  is a binoidal category and  $f: A \longrightarrow C$  is an arrow, then  $f$  is *central* if for all arrows  $g: B \longrightarrow D$ ,

$$(f \otimes id_B); (id_C \otimes g) = (id_A \otimes g); (f \otimes id_D)$$

and symmetrically for the two composites  $B \otimes A \rightarrow D \otimes C$ . The results will be denoted  $f \otimes g$  and  $g \otimes f$ .

- More generally, for any  $f, g$  as above, we write  $f \perp g$  if the above equations hold.
- The *centre* of  $\mathcal{C}$  is the category  $\mathcal{Z}(\mathcal{C})$  with objects the same as those of  $\mathcal{C}$  and its arrows are the central maps in  $\mathcal{C}$ .

## Proposition

*The centre  $\mathcal{Z}(\mathcal{C})$  of a premonoidal category  $\mathcal{C}$  is a monoidal category.*

## Example

*If  $M$  is a monoid, then  $M$  is a one object premonoidal category.*

## Example

*Let  $\mathcal{C}$  be a symmetric monoidal category and let  $S$  be a fixed object. Define  $\mathcal{C}_S$  as follows: the objects are those of  $\mathcal{C}$  and*

$$\mathcal{C}_S(X, Y) = \mathcal{C}(X \otimes S, Y \otimes S)$$

There is a canonical functor  $I_S: \mathcal{C} \rightarrow \mathcal{C}_S$ , which is the identity on objects and sends  $f$  to  $f \otimes S$ . All maps in the image of this functor are central. Sometimes the converse holds as well.

## Theorem

*If  $H$  is a Hilbert space with  $\dim H \geq 1$  then  $\mathcal{Z}(\mathbf{Hilb}_H) \simeq \mathbf{Hilb}$ .*

## Lemma

*The directed colimit of a class of symmetric premonoidal dagger categories is a symmetric premonoidal dagger category.*

## Definition

A *premonoidal quantum field theory*  $\mathcal{A}$  is an assignment of a symmetric premonoidal dagger category to each open in Minkowski space. We require the relativistic assumption that if  $U$  and  $V$  are spacelike separated systems then

$$\text{Ar}(\mathcal{A}(U)) \perp \text{Ar}(\mathcal{A}(V))$$

In applications, we will typically assume that the premonoidal categories are all premonoidal dagger closed subcategories of some  $\mathbf{Hilb}_H$  especially *von Neumann categories*, to be defined next.

## Quote (Rudolf Haag, *Local Quantum Physics*)

From the previous chapters of this book it is evidently not obvious how to achieve a division of the world into parts to which one can assign individuality. . . Instead we used a division according to regions in space-time. This leads in general to open systems. . .

In the monoidal approach to modeling quantum processes in spacetime, a process would be modelled by a map  $f : A \rightarrow B$ .

An advantage of our premonoidal approach is that we should be able to explicitly model the interaction of the environment on processes. In our setting, a process will be modelled by an arrow  $f : A \otimes H \rightarrow B \otimes H$ .

The Hilbert space  $H$  can be chosen so that it represents states of the entire system, and then the map  $f$  will contain information about the interaction of the process with the environment.

## Definition

Let  $\mathcal{B}(H)$  be the  $C^*$ -algebra of bounded linear operators on a Hilbert space  $H$ . A subset  $A \subseteq \mathcal{B}(H)$  closed under the  $*$ -operation is a *von Neumann algebra* if  $A = A''$ , where  $B'$  is the *commutant* of  $B$ , i.e.

$$B' = \{f \in \mathcal{B}(H) \mid fb = bf \text{ for all } f \in B\}$$

There are several equivalent definitions, some of which do not depend on a choice of  $H$ .

## Example

$\mathcal{B}(H)$  itself is a von Neumann algebra.

## Example

A finite-dimensional  $*$ -subalgebra of  $\mathcal{B}(H)$  closed under 1 is a von Neumann algebra.

# Von Neumann Categories

Our intuition is that to categorify the notion of von Neumann algebra, we will replace the  $C^*$ -algebra  $\mathcal{B}(H)$  with the premonoidal category  $\mathbf{Hilb}_H$ .

Note that  $\mathbf{Hilb}_H$  has a compatible dagger operation, in addition to being premonoidal.

## Definition

Let  $A$  be a collection of arrows in the category  $\mathbf{Hilb}_H$ . Then we define the *commutant* of  $A$ , denoted  $A'$ , to be the collection of all arrows

$$\{g \in \text{Ar}(\mathcal{C}) \mid f \perp g, \text{ for all } f \in A\}$$

## Lemma

*For any dagger-closed class of arrows  $A$ , we have  $A'$  is a premonoidal dagger subcategory of  $\mathbf{Hilb}_H$ .*

## Definition

A symmetric monoidal dagger-subcategory  $\mathcal{C}$  of  $\mathbf{Hilb}_H$  is a *von Neumann category* if  $\mathcal{C} = \mathcal{C}''$ .

- The primary example of a von Neumann category we have at the moment is  $\mathbf{Hilb}_H$ . It remains to find a rich class of examples analogous to the operator-theoretic setting.
- It is ongoing work to examine the equivalent definitions of von Neumann algebra, and determine if they lift to this categorical setting.

## Definition

A *factor* is a von Neumann algebra  $\mathcal{A}$  with trivial center, i.e.

$$\mathcal{Z}(\mathcal{A}) = \{c1_{\mathcal{A}} \mid c \in \mathbb{C}\}.$$

Extending our analogy between von Neumann algebras and von Neumann categories, the analogue of the base field  $\mathbb{C}$  is the category **Hilb**.

## Definition

A *factor* is a von Neumann category  $\mathcal{C}$  with trivial center, i.e.

$$\mathcal{Z}(\mathcal{C}) \cong \mathbf{Hilb}.$$

Ultimately, we have the goal of extending the notion of *type of a factor* to the categorified setting.

One way to construct von Neumann algebras of various types is the *crossed product*.

Let  $M \subseteq \mathcal{B}(H)$  be a von Neumann algebra and suppose  $\alpha: G \rightarrow \text{Aut}(M)$  be an action of a discrete group  $G$ . Define the *crossed product*  $G \times_{\alpha} M$  as follows. First let

$$\tilde{H} = \{\zeta: G \rightarrow H \mid \sum_{g \in G} \|\zeta(g)\| < \infty\}$$

Then define maps

$$\pi: M \rightarrow \mathcal{B}(\tilde{H}) \quad \pi(a)(\zeta)(g) = (g^{-1} \cdot a)(\zeta(g))$$

$$\lambda: G \rightarrow \mathcal{B}(\tilde{H}) \quad \lambda(g)(\zeta)(u) = \zeta(g^{-1}u)$$

Then  $G \times_{\alpha} M = [\pi(M) \cup \lambda(G)]'' \subseteq \mathcal{B}(\tilde{H})$ . This satisfies some canonical commutation relations.

# Crossed Products of Von Neumann Categories

Let  $G$  be a discrete group, viewed as a premonoidal category. Let  $\mathcal{C} \subseteq \text{Hilb}_H$  be a von Neumann category. We define a  $G$ -action on  $\mathcal{C}$  to be a premonoidal functor

$$\alpha: G \times \mathcal{C} \rightarrow \mathcal{C}$$

So, given an arrow  $f: K \rightarrow K'$  in  $\mathcal{C}$  and  $g \in G$ , we have an action  $g \cdot f: K \rightarrow K'$ .

We have a premonoidal functor  $\lambda: G \rightarrow \text{Hilb}_{\tilde{H}}$  given by the same formula. ( $G$  is a 1-object category, and  $\text{Hom}_{\text{Hilb}_{\tilde{H}}}(I, I) = \mathcal{B}(\tilde{H})$ .)

The functor  $\pi: \mathcal{C} \rightarrow \text{Hilb}_{\tilde{H}}$  should be defined similarly. Then, as above,

$$G \times_{\alpha} \mathcal{C} = [\pi(\mathcal{C}) \cup \lambda(G)]''$$

- Work out details of the cross product construction, and use it to generate new examples.
- Find a premonoidal version of the *Doplicher-Roberts Theorem*:

## Theorem

*Every compact closed  $\mathcal{C}^*$ -category (essentially a dagger category with compatible normed structure) is equivalent to the category of finite-dimensional unitary representations of a unique compact group.*

- What is the appropriate analogue of compact group in the DR-theorem to obtain premonoidal categories?