Introduction To Cartesian Bicategories.
Initial Paper by Carboni & Walters

Consider the category $\mathbf{Rel}$, of sets & relations.

Objects: Sets
Arrows: Relations $R : X \to Y$ is a subset $R \subseteq X \times Y$

Compositions:
$$R \circ S : X \to Y \to Z$$
$(x, z) \in R \circ S$ if $\exists y \in Y$ such that $(x, y) \in R$ & $(y, z) \in S$.

We would like an axiomatization of $\mathbf{Rel}$
similar to the way the elementary topos
axiomatizes $\mathbf{Set}$, the category of sets
and functions.

What structure do we have?
1) Every hom-set is a poset ordered
   by inclusion. The ordering is preserved
   by composition in each variable.
Defn: A bicategory $\mathcal{B}$ consists of

1) A class of objects or 0-cells $X, Y, Z$
2) For cells $X, Y$ a category $\mathcal{B}(X, Y)$.
3) Composition functors
   \[ c_{x,y,z} : \mathcal{B}(X, Y) \times \mathcal{B}(Y, Z) \rightarrow \mathcal{B}(X, Z) \]
4) For each 0-cell $X$, a 1-cell
   \[ I_X \in \mathcal{B}(X, X) \]

These must satisfy a number of axioms.

The composition must be associative, up to isomorphism. The 1-cell $I_X$
must act as an identity up to isomorphism.

Note: It makes sense to talk about the operations having an appropriate
property up to isomorphism because of the existence of 2-cells.
In Rel, each $B(x,y)$ is just a poset. Such a bicategory is locally posetal. Next, Rel has a tensor product, i.e. is monoidal, and symmetric.

3) The general definition of symmetric monoidal bicategory is nightmarish.

**PAUSE**

$\otimes : B \times B \to B$

But in the locally posetal case, it's straightforward. See [CWJ] for details, or Evangelia Aleiferi's thesis.

48) In bicategories, one can talk about one cells having adjoints. In the locally posetal case, this amounts to

**Defn**: A 1-cell $f : X \to Y$ has a right adjoint (is a left adjoint)

if $I_x \leq f \cdot f^\ast$ and $f^\ast \cdot f \leq I_y$
Lemma: In a Rel, the left adjoints are precisely the functions viewed as relations.

In general, the left adjoints are called maps.

Map(B) is the subbicategory consisting of maps out the 2-cell between them.

Definition: Let B be a locally posetal bicategory. A cartesian structure for B consists of:

1) A symmetric monoidal structure

2) For every 0-cell X, a cocommutative comonoidal structure, i.e.,

\[ d_x : X \rightarrow X \otimes X \]

\[ e_x : X \rightarrow I \]

such that

\[ X \xrightarrow{d} X \otimes X \]

\[ X \xrightarrow{\Delta} (X \otimes X) \otimes X \]

\[ d \otimes I_X \]

\[ I_x \otimes d_x \]

\[ X \otimes (X \otimes X) \]
3) Every 1-cell is a lax comonad homomorphism, i.e.,

\[ F : X \to Y \]

\[ F \downarrow \cong \downarrow F \otimes F \]

\[ Y \otimes Y \]
4) For all O-cells X, $d X$ and $t X$ have right adjoints, i.e. they are maps.

**EX: Rel:** Verify this if you’ve never checked the details.

**EX:** A regular category is a category with structure that ensures monics satisfy nice factorization properties, which ensure one can form a category of relations.

**Lemmas:** If $E$ is a regular category, $\text{Rel}(E)$ is a cartesian bicategory.

**EX:** Let $\mathcal{B}$ be a bicategory. Define the Karoubi bicategory of $\mathcal{B}$, denoted $\text{Ker}(\mathcal{B})$, as follows:

- O-cells are pairs $(A, g)$ where $A$ is an O-cell of $\mathcal{B}$ and $g : A \to A$ is a 1-cell s.t. $g^2 = g$, i.e. $g$ is idempotent.
If \((X,x)\) and \((A,a)\) are 0-cells, a 1-cell is a 1-cell of \(B\) \(R : X \to A\) s.t. \(aRx = R\) or equivalently \(aR = R = Rx\) 2-cells are as in \(B\).

Thm (Aleiferi): If \(B\) is a locally posetal cartesian bicategory, so is \(\text{Kar}(B)\).

**Ex:** Let \(B = \text{Rel}\), a relation \(\leq_x : X \to X\) is idempotent if it is
- 1) transitive \(\forall x, y, z : x \leq_x y \land y \leq_x z \Rightarrow x \leq_x z\)
- 2) interpolative \(\exists z : x \leq_x y \Rightarrow \exists z : x \leq z \leq x\)

A morphism \(R : (X, \leq_x) \to (A, \leq_a)\) is a relation \(R : X \to A\) s.t.

\[
P^R_x \cdot \leq_x \cdot = \leq_R = \leq_a^R\]
In a locally ordered bicategory $B$, a monad is a pair $(A, c)$ where $A$ is a 0-cell and $c : A \to A$ is a 1-cell such that

$$1_A \leq c \& c c \leq c$$

A module $m : (A,g) \to (B,f)$ is an arrow $m : A \to B$ s.t. $c_j m = m = m_j b$

Thm (Aleiferi) If $B$ is a locally posetal cartesian bicategory, then so is $\text{Mod}(B)$

If you apply this to Rel, you get the following:

Objects are preorders
Arrows are order ideals, i.e.

$$R : (P, \leq_P) \to (Q, \leq_Q)$$

is a subset of $P \times Q$ s.t.
\[ x' \leq p \times R q \Rightarrow x' R q \]
\[ x R q \leq q' \Rightarrow x R q' \]

So this is a cartesian bicategory.

Next issue: Generalizing to arbitrary bicategories. Quite difficult with the present definition.

Remark: Cartesian Bicategories I came out in 1987.

Cartesian Bicategories II came out in 2008

\text{Rel} has the following structures:

1) \( \text{Map}(\text{Rel}) \cong \text{Set} \) and so has finite products.

2) If \( X, Y \) are sets, then \( \text{Hom}(X, Y) \cong \mathcal{P}(X \times Y) \) which has finite products.

We say that \( \text{Rel} \) has \underline{local finite products}. 
Lemma [Carboni & Walters]

If $B$ is a locally ordered bicategory with a tensor product, then that tensor is the product (in the sense of bicategories) if and only if every object has a cocommutative comonoid structure and every arrow $F : X \to Y$ is a strict comonoid homomorphism.

Lemma [CW]

If $B$ is a locally ordered cartesian bicategory, then $\text{Map}(B)$ has finite products.

All of this leads to an alternative characterization of cartesian bicategories,
Thm\textsubscript{[CWJ]}

Suppose $B$ is a locally ordered cartesian bicategory. Then:

1) $\text{Map}(B)$ has finite products. In particular, the tensor is the product in $\text{Map}(B)$. We'll denote this product by $\times$, and it has projections $p$ and $q$.

2) Each hom category $B(X,Y)$ has finite products. We'll denote this product by $\wedge$.

3) For all one-cells $F : X \to Y$, $G : W \to Z$,

\[ F \otimes G = (p_\ast F \ast q_\ast) \wedge (r_\ast G \ast r_\ast) \]

This is happening in $B(X \otimes W, Y \otimes Z)$.
Conversely, suppose:

1) Map (B) has finite products
2) Each B(X,Y) has finite products
3) The formula (*) defines a tensor product on B, then B is cartesian.

This allows us to generalize from locally posetal bicategories. Key example is the bicategory of spans:

A span from X to Y is a diagram:

\[ z \rightarrow y \leftarrow x \]

Spans can be composed if we have pullbacks.
This is a bicategory since a morphism of spans is:

There can be more than one morphism between 2 spans.