Lecture 3

Let's back up.

**Defn:** A binary relation from \(X\) to \(Y\) is a subset \(R \subseteq X \times Y\). We write \(R : X \rightarrow Y\).

If \((x,y) \in R\), we'll write \(xRy\).

One can compose relations

\[ X \xrightarrow{R} Y \xrightarrow{S} Z \]

Say, if \(\exists y\) s.t. \(xRy\) and \(ySz\),

\[ x((R \circ S))z \]

Every function \(f : X \rightarrow Y\) can be viewed as a relation by considering its graph \(\text{Gr}(f) \subseteq X \times Y\):

\[ \text{Gr}(f) = \{ (x,f(x)) \mid x \in X \} \]

In this case, relational composition corresponds to functional composition

\[ X \xrightarrow{f} Y \xrightarrow{S} Z \]

\[ \text{Gr}((f \circ g)) = \text{Gr}(f) \circ \text{Gr}(g) \]
Relation are a special case of spans.
A span from \( X \) to \( Y \) consists of 2 arrows

\[
\begin{array}{c}
S \\
\downarrow \\
X & \rightarrow & Y
\end{array}
\]

In the case of a relation, we have

\[
\begin{array}{c}
R \\
\downarrow \pi_1 & \rightarrow & \downarrow \pi_2 \\
X & \rightarrow & Y
\end{array}
\]

where \( \pi_1 \) and \( \pi_2 \) are the projections, i.e.

if \( XR_\gamma Y \), then \( \pi_1(X\gamma) = X \), etc.

One can "compose" 2 spans using pullbacks

\[
\begin{array}{c}
S_1 \\
\downarrow S_2 \\
\downarrow \\
X & \rightarrow & Y & \rightarrow & Z
\end{array}
\]

The problem here is that pullbacks are only unique up to isomorphism.
In Set, the category of sets and functions, there is a canonical choice for representing pullback:

![Diagram](PB \rightarrow S_1 \downarrow \rightarrow S_2)

$$PB \subseteq S_1 \times S_2 = \{(s_1, s_2) \mid f(s_1) = g(s_2)\}$$

If you use this choice of pullback, then composition of spans = composition of relations. Check this!

In a more general category, there is no canonical choice. There are various solutions.

If relations are a special case of spans, how do I distinguish the relations?
A span $S$ is jointly monic if, given 2 arrows $a, b : U \to S$, if $g \circ a = g \circ b$, then $a = b$.

Lemmas: The jointly monic spans correspond precisely to the relations.

It seems like we should be able to do the following:
Suppose $E$ is a category with pullbacks. Can I form a category of relations in $E$?
Well, that works! Set, then is

Mike's Bar

in regular connections, a concept.

All of this can be solved if one works

be jointly manik

of jointly nonzero spans may not

of jointly nonzero spans may not

is also a problem for the span compared

can only operate at some parallel, then

Aside from this problem, that problem

There: This doesn't work in general.

Then use composition of spans, to compose

arrows a jointly nonzero span

objects is some as E

This would consist of E.
Back to our ordinary category, $\text{Rel}$.

It has some additional structure. $\text{Hom}(X,Y) = \emptyset (X,Y)$ and is hence a partially ordered set under inclusion. We also have $\text{Rel} \times \text{Rel} \to \text{Rel}$ with $R \sqsubseteq R'$, then $R' \sqsubseteq R$.

Similarly with composition on the other side. This makes $\text{Rel}$ a particularly simple ordered category.

Is there a way to pick out those relations that are Hasse graphs of functions?

Yes.
Let $c$ be an ordered category. Let $f: A \to B$ be an arrow. Then $f$ has a right adjoint if there exists an arrow $f^* : B \to A$ such that

$$f^* \circ f \leq \text{Id}_B \quad \text{and} \quad \text{Id}_A \leq f \circ f^*$$

Thm: In $\text{Rel}$, the arrows with right adjoints are precisely the relations which are graphs of functions. Prove this!

$\text{Rel}$ has some additional structure.

1) It has binary products.

On objects, it is the disjoint union of the sets.

The projections $\Pi_1, \Pi_2 : A \times B \to A, B$ are

$$\Pi_1(a, b) = a \quad \text{and} \quad \Pi_2(a, b) = b$$

Similarly for $\Pi_2$.
2) The empty set is the terminal object.

So Rel has all finite products.

Similarly Rel has all finite coproducts.

(Exercise) In fact as far as objects, products and coproducts coincide.

One issue: Rel doesn't have most equalizers or pullbacks. (Exercise: Explore this problem.)

\[
\text{Rel is monoidal. On objects,}\]

\[A\otimes B = A \times B\]

\[\text{the cartesian product of the sets.}\]

Rel has a functor \((\_)^\ast : \text{Rel}^{op} \to \text{Rel}\)
which is the identity on objects, but takes the reciprocal relation

\[R : A \to B\]

\[R^\ast : B \to A\]
It satisfies $(\cdot)^{**} = \text{Id}$

This gives us a **tensored** $\times$-category.

We'll define a category of **preminimata** spaces.

An object is a pair $(X, U)$ where $X$ is a set and $U$ is a set of subsets $U \subseteq P(X)$ ($U$ is arbitrary, we'll assume nothing about it.)

A morphism $(X, U) \to (Y, V)$ is a relation $R : X \to Y$ which takes elements of $U$ to elements of $V$.

So, if $u \in U$, define

$$R(u) = \{ y \in Y | \exists x \in u \text{ with } xRy \}$$

One way to think of this is as the restriction of $R$ to $u$.

So we require that for all $u \in U$, we have $R(u) \subseteq V$.

Then: This is a category.