Lecture 1

Goal: Read Cartesian bicategories carefully. But we'll do so slowly and look at various other papers along the way.

Today: Bicategories

References:
1) Benabou: Introduction to Bicategories
2) Barbosa: A brief introduction to bicategories.

A bicategory consists of (Coll₁ §)

1) A set |S| of objects.
2) For each pair of objects, A, B ∈ |S|, a category $\mathcal{S}(A, B)$.
   An object of $\mathcal{S}(A, B)$ is called a \textit{1-cell}, and denoted $A \rightarrow B$.
   An arrow of $\mathcal{S}(A, B)$ is called a \textit{2-cell}.
We'll denote 
\[ A \xrightarrow{f} B \xrightarrow{g} C. \]

3) A composition functor
\[ C : S(A,B) \times S(B,C) \rightarrow S(A,C) \]
\[ f \xrightarrow{g} C \mapsto f \circ g : A \rightarrow C \text{ on objects} \]
\[ f \xrightarrow{g} C \mapsto A \xrightarrow{f \circ g} C \text{ on morphisms}. \]

This is an example of pasting.

4) \( \forall A \in \mathcal{S}, \) an object \( I_A : A \rightarrow A \)

5) \( \forall A, B, C, D \in \mathcal{S}, \) an associativity isomorphism.
Make sure you understand what this is. Both legs of this square are functors, from the category
\[ S(A, B) \times S(B, C) \times S(C, D) \]
to the category
\[ S(A, D) \]
So a must be a natural transformation. Since it is an isomorphism, there must be a \( NT a^{-1} \) in the other direction.

6) \( \forall A, B \in \text{Set}, I \text{ need two isos} \)

\[ 1 \times S(A, B) \xrightarrow{I_a \times 1d} S(A, A) \times S(A, B) \]

\[ S(A, B) \quad \underline{\lambda} \quad S(A, A) \times S(A, B) \]

\[ S(A, B) \times 1 \rightarrow S(A, A) \times S(A, B) \]

\[ S(A, B) \quad \underline{\gamma} \quad S(A, A) \times S(A, B) \]
These must satisfy coherence conditions

**COHERENCE**

\[ S : A \rightarrow B \]
\[ T : B \rightarrow C \]
\[ U : C \rightarrow D \]
\[ V : D \rightarrow E \]

\[
\begin{align*}
\left( (S \circ T) \circ U \right) \circ V & \xrightarrow{a \circ Id} \left( S \circ (T \circ U) \right) \circ V \\
\downarrow & \\
\left( S \circ T \right) \circ (U \circ V) & \xrightarrow{a} \left( S \circ (T \circ U \circ V) \right) \\
\end{align*}
\]

\[
\begin{align*}
(S \circ T) \circ U & \xrightarrow{a} S \circ (T \circ U \circ V) \\
S \circ T \circ U & \xrightarrow{a} S \circ (T \circ U \circ V) \\
\end{align*}
\]

**Alternative Presentation**

A bigraph \( \Sigma \) is a diagram of sets and maps.
\[ \Sigma_0 \rightarrow \Sigma_1 \rightarrow \Sigma_2 \]

s.t. 2 equations hold

Think of \( \Sigma_0 \) as O-cells, etc.

To compose, I need

\[ \Sigma_2 \times \Sigma_1 \rightarrow \Sigma_2 \]

\[ PB \rightarrow \Sigma_2 \]

\[ \Sigma_2 \rightarrow \Sigma_1 \]

For identities, I need

\[ \Sigma_0 \rightarrow \Sigma_1 \rightarrow \Sigma_2 \]

These pick out identities.

\[ PB \rightarrow \Sigma_2 \]

\[ \Sigma_2 \rightarrow \Sigma_1 \]

A lot of equations have to hold.

Thus: This is equivalent to original definition.

Shows relationship with simplicial sets.
EX 0 CAT: The category of categories, functions and natural transformations.

Defn: This is a 2-category, since all \( a, b, c \) are equalities.

2) Monoidal categories can be seen things as 1-object bicategories.

3) Rel

Object are sets, arrows are binary relations \( \text{Hom}(X, Y) \) is a poset under inclusion. Composition is functorial in both variables. That's all we need. Coherence is straightforward. Why?

4) Span

Let \( C \) be any category with pullbacks.

A span is a diagram of the form

\[
\begin{array}{ccc}
  & f & \rightarrow & s \\
 V & \downarrow & & \downarrow & \downarrow \\
 Y & \rightarrow & X & \rightarrow & Z
\end{array}
\]
One can compose spans using pullbacks:

\[ X \times_Z W \]

So we view a span as an arrow (1-cell)

\[ Y \to Z \]

The 2-cells are:

\[
\begin{array}{ccc}
X & \to & Z \\
\downarrow & & \downarrow \\
X & \to & Z \\
\end{array}
\]

Checking all axioms is tedious but straightforward.

**Thm:** Relations are special cases of spans on sets.

Q: How can you characterize them abstractly?

There is a condition called **jointly monotone**.
5) Let \( \mathcal{M} \) be a monoidal category and let \( \mathcal{C} \) be any category. A left action of \( \mathcal{M} \) on \( \mathcal{C} \) is

1) A functor

\[ \otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{C} \]

2) Two natural transformations

\[ \alpha : (A_1 \otimes A_2) \otimes X \to A_1 \otimes (A_2 \otimes X) \]

for \( A_1, A_2 \in \mathcal{M}, X \in \mathcal{C} \)

\[ \eta : I \otimes X \to X \]

\( I \) is the unit for \( \mathcal{M} \)

\[ X \in \mathcal{C} \]

satisfying obvious axioms.

From this, one can build a bicategory as follows:

- **O-cells**: \( 0,1 \)

\[ S(0,0) = \mathcal{M} \]

\[ S(0,1) = \mathcal{E} \]

\[ S(1,1) = 1 \]

\[ S(1,0) = \emptyset \]

Fill in the rest of the details.
Bimodules

0-cells : Rings
1-cells are bimodules

So if Rs on rings. A 1-cell is a bimodule M, which is an R-S bimodule.
So I have maps

\[ R \times M \rightarrow M \]
\[ M \times S \rightarrow M \]

denote the \( R \) s satisfying obvious actions.

2-cells are bimodule maps.

Then given

\[ R \times S = \text{and} \quad S \times T \]

one forms

\[ R \times M \otimes S N \]

for composition.

This sets us to p23 of Benabou.