MATH 1300-MIDTERM-2004

NAME and I.D.# Solutions - Version 1

Instructions– This exam consists of 6 multiple choice questions and 2 long answer questions. The multiple choice questions are worth 6 points each, and the long answer questions are as indicated. The total value of the exam is 60 points.

Place your answers to the multiple choice questions in the boxes below. All your work on the long answer questions must be clearly marked. You may use the backs of pages. **If you need additional scrap paper, it will be provided by the proctors.**

NO CALCULATORS. NO BOOKS. NO NOTES.

On the long answer questions, you must show your work.

Answers:

#1 #2 #3 #4

#5 #6

Multiple Choice Section-Question 1-
Consider the following function:

$$f(x) = \begin{cases} x^2 - x + 1 & \text{if } x < 4 \\ ax^2 + 2x - 3 & \text{otherwise} \end{cases}$$

What value must the constant a be for the function to be continuous at $x = 4$?
A) $\frac{1}{3}$ **B)** $\frac{1}{2}$ **C)** $\frac{-1}{4}$ **D)** $\frac{-1}{2}$ **E)** $\frac{2}{3}$

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} x^2 - x + 1 = 13$

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} ax^2 + 2x - 3 = 16a + 5$

So we must have $13 = 16a + 5$. So $\boxed{a = 1/2}$

Question 2- Find the equation of the tangent line of the function $f(x) = (x^3 + 7)^{\frac{2}{3}}$ at $x = 1$.

A) $y = \frac{x}{3} + \frac{11}{3}$ **B)** $y = \frac{x}{4} + \frac{15}{4}$ **C)** $y = \frac{x}{2} + \frac{7}{2}$
D) $y = 2x + 2$ **E)** $y = x + 3$

$f'(x) = \frac{2}{3}(x^3+7)^{-1/3} \cdot 3x^2$ $\quad f'(1) = \frac{2}{3} \cdot (8)^{-1/3} \cdot 3 = \frac{2}{3} \cdot \frac{1}{2} \cdot 3 = 1$

The point $(1,4)$ is on the line, So $y = 1x + b$

$4 = 1 \cdot 1 + b$ or $b = 3$.

So $\boxed{y = x + 3}$

Question 3-
Calculate:

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$$

A) The limit does not exist. B) 2 C) 3 D) 4 E) 5

$$\frac{x-1}{\sqrt{x}-1} = \frac{x-1}{\sqrt{x}-1}\,\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} = \frac{x-1\,(\sqrt{x}+1)}{x-1} = \sqrt{x}+1$$

So $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \sqrt{x}+1 = 2$

Question 4-Solve

$$16^{x+2} = 64^{2x-1}$$

A) $x = \frac{2}{3}$ B) $x = \frac{2}{5}$ C) $x = \frac{7}{4}$ D) $x = \frac{3}{5}$ E) $x = \frac{3}{4}$

$16^{x+2} = (2^4)^{x+2} = 2^{4x+8}$

$64^{2x-1} = (2^6)^{2x-1} = 2^{12x-6}$.

So we have $4x+8 = 12x-6$

$14 = 8x$

$x = \frac{7}{4}$

Question 5

Let $f(x) = x + \frac{3}{2-4x}$. Find $f'(x)$ at $x = 0$.

A) 1 B) 2 C) 3 D) 4 E) 5

$f(x) = x + 3(2-4x)^{-1}$

$f'(x) = 1 + \frac{3(-1)(-4)}{(2-4x)^2}$

$f'(0) = 1 + \frac{3(4)}{4} = 4$

Question 6 Solve the following inequality:

$$|2x-1| < 3$$

A) $(-\infty,-1)$ and $(2,\infty)$ B) $(-1,4)$ C) $(-2,1)$ D) $(-1,2)$ E) $(-\infty,-2)$ and $(1,\infty)$

$|2x-1| < 3$

$-3 < 2x-1 < 3$

$-2 < 2x < 4$

$-1 < x < 2$

So $x \in (-1,2)$

Long Answer Questions-Question 1 (12 points)

The population of a certain country t years from now is given by $P(t) = 50e^{.2t}$ million.

- (2 points) What is the population now (the initial population)?
- (5 points) How long will it take for the population to double?
- (5 points) How long will it take for the population to triple?

1) Initial Population $= P_0 = P(0) = 50e^0$ million

$= 50$ million

2) We must solve $P(t) = 100$ million for t.

$$100 \text{ million} = 50\, e^{.2t} \text{ million}$$

$$2 = e^{.2t}$$

$$\ln(2) = .2t \Rightarrow t = 5\ln(2)$$

3) Similarly, to triple $t = 5\ln(3)$

Question 2 (12 points)

- (4 points) Let $f(x)$ be a function. State the definition of the *derivative* of $f(x)$ in terms of limits.
- (8 points) Using only the definition of derivative in terms of limits, calculate the derivative of $f(x) = x^2 + 3x$.

- $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- If $f(x) = x^2 + 3x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 3(x+\Delta x) - x^2 - 3x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 3x + 3\Delta x - x^2 - 3x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 + 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x + 3 = 2x+3$$

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NAME and I.D.# Solutions - Version 2

Instructions– This exam consists of 6 multiple choice questions and 2 long answer questions. The multiple choice questions are worth 6 points each, and the long answer questions are as indicated. The total value of the exam is 60 points.

Place your answers to the multiple choice questions in the boxes below. All your work on the long answer questions must be clearly marked. You may use the backs of pages. **If you need additional scrap paper, it will be provided by the proctors.**

NO CALCULATORS. NO BOOKS. NO NOTES.

On the long answer questions, you must show your work.

Answers:

#1 #2 #3 #4

#5 #6

Multiple Choice Section-Question 1-
Consider the following function:

$$f(x) = \begin{cases} x^2 - x & \text{if } x < 3 \\ ax^2 + 2x - 3 & \text{otherwise} \end{cases}$$

What value must the constant a be for the function to be continuous at $x = 3$?

A) $\frac{7}{3}$ **B)** $\frac{1}{2}$ **C)** $\frac{-1}{4}$ **D)** $\frac{-1}{2}$ **E)** $\frac{2}{3}$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - x = 6$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} ax^2 + 2x - 3 = 9a + 3$$

So $6 = 9a + 3$ or $a = 1/3$

Question 2- Find the equation of the tangent line of the function $f(x) = (2x+7)^{\frac{1}{2}}$ at $x = 1$.

A) $y = \frac{x}{3} + \frac{8}{3}$ **B)** $y = \frac{x}{4} + \frac{11}{4}$ **C)** $y = \frac{x}{2} + \frac{5}{2}$

D) $y = -x + 4$ **E)** $y = \frac{2x}{3} + \frac{7}{3}$

$$f'(x) = \frac{1}{2}(2x+7)^{-1/2}(2) = \frac{1}{\sqrt{2x+7}} \qquad f'(1) = \frac{1}{3}$$

So $y = mx + b = \frac{1}{3}x + b$. The point $(1,3)$ is on the line, so $3 = \frac{1}{3}(1) + b$, or $b = 8/3$.

So $\boxed{y = \frac{1}{3}x + 8/3}$

Question 3-
Calculate:

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$$

A) The limit does not exist. B) 2 C) 3 D) 4 E) 5

$$\frac{x-4}{\sqrt{x}-2} = \frac{x-4}{\sqrt{x}-2}\,\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)} = \frac{x-4\,(\sqrt{x}+2)}{x-4} = \sqrt{x}+2$$

So $\lim_{x \to 4} \frac{x-4}{\sqrt{x}+2} = \lim_{x \to 4} \sqrt{x}+2 = 4$

Question 4-Solve

$$16^{x+1} = 8^x$$

A) $x = \frac{2}{3}$ B) $x = \frac{4}{7}$ C) $x = \frac{7}{4}$ D) $x = \frac{3}{5}$ E) $x = \frac{3}{4}$

$$16^{-x+1} = (2^4)^{-x+1} = 2^{-4x+4}$$

$$8^x = (2^3)^x = 2^{3x}$$

So $-4x+4 = 3x \Rightarrow x = 4/7$

Question 5
Let $f(x) = x^2 + \frac{3}{2-4x}$. Find $f'(x)$ at $x = 0$.

A) 1 **B)** 2 **C)** 3 **D)** 4 **E)** 5

$$f'(x) = 2x + \frac{3(-1)(-4)}{(2-4x)^2}$$

So $f'(0) = 3$

Question 6 Solve the following inequality:

$$|2x+1| < 3$$

A) $(-\infty, -1)$ and $(2, \infty)$ **B)** $(-1, 4)$ **C)** $(-2, 1)$ **D)** $(-1, 2)$ **E)** $(-\infty, -2)$ and $(1, \infty)$

$|2x+1| < 3 \Rightarrow -3 < 2x+1 < 3 \Rightarrow -4 < 2x < 2$

$\Rightarrow -2 < x < 1$, So

$x \in (-2, 1)$

Long Answer Questions-Question 1 (12 points)

The population of a certain country t years from now is given by $P(t) = 40e^{.5t}$ million.

- (2 points) What is the population now (the initial population)?
- (5 points) How long will it take for the population to double?
- (5 points) How long will it take for the population to triple?

- Initial Population = $P(0) = 40e^{0} = 40$ million
- We must solve $P(t) = 80$ million for t

$$80 = 40e^{.5t}$$

$$2 = e^{.5t}$$

$$\ln(2) = .5t \Rightarrow t = 2\ln(2)$$

- Similarly to triple population $t = 2\ln(3)$

Question 2 (12 points)

- (4 points) Let $f(x)$ be a function. State the definition of the *derivative* of $f(x)$ in terms of limits.
- (8 points) Using only the definition of derivative in terms of limits, calculate the derivative of $f(x) = x^2 \quad 2x$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $f(x) = x^2 - 2x$,

$$f'(y) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 2(x+\Delta x) - (x^2 - 2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 2x - 2\Delta x - x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x - 2 = 2x - 2$$