MATH 1300D-MIDTERM2-2003
Solutions Version A

Multiple Choice Section

Question 1- Let $f(x) = \frac{x^2-1}{x+2}$, then $f'(x) =$

Solution: Use quotient rule.

$$f'(x) = \frac{2x \cdot (x+2) - 1 \cdot (x^2-1)}{(x+2)^2} = \frac{x^2+4x+1}{(x+2)^2}$$

Answer: A.

Question 2- Find $\frac{dy}{dx}$ for the equation $xy^3 - 3y = 2x$.

Solution: Implicit differentiation.

$$y^3 + x \cdot \left(3y^2 \frac{dy}{dx}\right) - 3\frac{dy}{dx} = 2$$
$$\left(3xy^2 - 3\right) \frac{dy}{dx} = 2 - y^3$$
$$\frac{dy}{dx} = \frac{2-y^3}{3xy^2-3}$$

Answer: B.

Question 3- What are the critical points of

$$f(x) = 3 - \frac{1}{x} + \frac{1}{x^2}?$$

Solution: We have:

$$f'(x) = \frac{1}{x^2} - \frac{2}{x^3} = \frac{1}{x^3}(x-2).$$

Thus, $f'(0)$ is undefined and $f'(2) = 0$. However, $x = 0$ is not in the domain of the *original* function $f(x)$. Thus, by definition, it is not a critical point. Answer: B.

Question 4- Let $f(x) = x^4 - 2x^2 + 3$ and let I be the interval $[-2, 3]$. Which of the following is true for the function f on I?

Solution: f is continuous and the domain I is a closed interval. Thus, we only need to plug in the critical points of f and endpoints of I into f and pick the largest and smallest values. Since $f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$, the critical points are $x = 0, \pm 1$. The endpoints of I are 3 and -2. We have:

x	-2	-1	0	1	3
$f(x)$	11	2	3	2	66

Answer: C.

Question 5- Let $f(x) = \ln\left(2x^3 - 3e^x\right)$. What is $f'(x)$?

Solution: We see a function $2x^3 - 3e^x$ inside the function ln. Thus, we must use the Chain Rule.

$$\frac{d}{dx}\ln\left(2x^3 - 3e^x\right) = \frac{1}{2x^3 - 3e^x} \cdot (6x^2 - 3e^x).$$

Answer: A.

Long Answer Questions

Question 1

Let a company's demand function p from selling x units of a product be given by

$$p = 20x + \sqrt{x-1}.$$

Assume that there is no cost.

a) Find the profit function.

Solution: Since there is no cost, we have $P = R - C = R$. By the definition of the demand function, revenue equals $xp(x)$. Answer: $x(20x + \sqrt{x-1})$.

b) Find the rate of change in the profit when 10 units have been sold.

Solution: The rate of change of the profit in this case is $P'(x) = R'(x) = \frac{d}{dx}xp(x)$.

$$P'(x) = \frac{d}{dx}\left(x(20x + \sqrt{x-1})\right) = \frac{d}{dx}\left(20x^2 + x\sqrt{x-1}\right)$$
$$= 40x + \left(\sqrt{x-1} + x \cdot \frac{1}{2\sqrt{x-1}}\right).$$

Thus, $P'(10) = 40 \cdot 10 + \sqrt{9} + 10/(2\sqrt{9}) = 400 + 3 + 5/3 = 404\frac{2}{3}$. Answer: $404\frac{2}{3}$.

Question 2

Let $f(x) = -2x^3 - 6x^2 + 1$.

a) Calculate $f'(x)$ and find all critical points.

Solution: $f'(x)$ is $-6x^2 - 12x$. Also, $-6x^2 - 12x = -6x(x+2) = 0$ when $x = 0, -2$.

Answer: $f'(x) = -6x^2 - 12x$ and $x = 0, -2$.

b) Find the intervals on which f is increasing and those on which f is decreasing, and give the relative minima and mixima.

Solution: From part a), we deduce that the intervals are: ($-\infty$,-2), (-2, 0), and (0, ∞).

Intervals	(-∞, -2)	(-2, 0)	(0, ∞)
Test values	-3	-1	1
$f'(x) =$	-18	6	-18
Sign of $f'(x)$	-	+	-
Conclusion	dec.	inc.	dec.

Answer: Increases on $(-2, 0)$ and decreases on $(-\infty, -2)$ and $(0, \infty)$. Relative minimum at $x = 0$ and relative maximum at $x = -2$.

c) Find the intervals where f is concave upward, downward. Give the possible points of inflection.

Solution: $f''(x)$ is $-12x - 12$. Since $-12x - 12 = -12(x+1) = 0$ when $x = -1$, this is the only possible inflection point. Then the intervals are $(-\infty, -1)$ and $(-1, \infty)$.

Intervals	$(-\infty, -1)$	$(-1, \infty)$
Test values	-2	0
$f''(x) =$	12	-12
Sign of $f''(x)$	+	-
Conclusion	up	down

Answer: Concave up on $(-\infty, -1)$ and concave down on $(-1, \infty)$. Inflection point: $x = -1$.

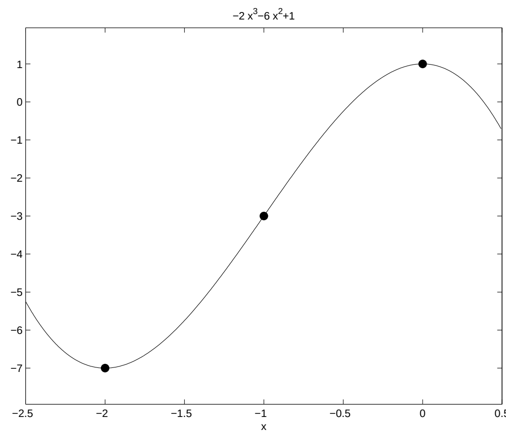
d) Using the information above, sketch the graph of f (Note: you don't have to calculate the x-intercepts).

Solution: Find the coordinates of relative min/max and inflection points.

$x =$	-2	-1	0
$y = f(x) =$	-7	-3	1

Your graph must reflect where the graph increases/decreases and where it's concave up/down.

Answer:



Question 3

After humans eliminated their predators, rabbit population in a certain region started growing exponentially. The population, which used to ge 2,000 when the last predator was hunted down, grew to 4,000 in 3 years.

Let $P(t)$ denote the size of the rabbit population, where t is the number of years since the elimination of the predators.

a) Give the formula for $P(t)$.

Solution: Since the growth is exponential, $P(t)$ has the form Ce^{kt}. By the information given, we have $P(0) = 2,000$ and $P(3) = 4,000$. So we have $P(0) = Ce^0 = C = 2,000$, and $P(3) = Ce^{3k} = 2,000e^{3k} = 4,000$. We can solve for k as follows:

$$2,000e^{3k} = 4,000 \quad \Rightarrow \quad e^{3k} = 2 \quad \Rightarrow \quad 3k = \ln 2 \quad \Rightarrow \quad k = \frac{\ln 2}{3}.$$

Answer: $P(t) = 2,000\, e^{\frac{\ln 2}{3}t}$.

Alternatively, we can use $P(t) = Cb^t$. Then $P(0) = C = 2,000$ and $P(3) = Cb^3 = 2,000b^3 = 4,000$. Then $b^3 = 2$ and $b = 2^{1/3}$.

Answer: $P(t) = 2,000\, 2^{t/3}$.

b) 10 years after the elimination of the predators, how many rabbits will there be?

Solution: Let $t = 10$ in the solution to part a).

Answer: $2,000\, e^{\frac{10 \ln 2}{3}}$ or $2,000\, 2^{10/3}$.

c) How long will it take for the population to double?

Solution: We already know that it took three years from when the predators were eliminated for the population to double.

Answer: 3 years.

But if you didn't see this, you could still compute it. At $t = 0$ there were 2,000. So we want to know how long it takes until there are 4,000.

$$4,000 = 2,000\, e^{\frac{\ln 2}{3} t} \quad \Rightarrow \quad 2 = e^{\frac{\ln 2}{3} t} \quad \Rightarrow \quad \ln 2 = \frac{\ln 2}{3} t \quad \Rightarrow \quad t = 3$$

(Note: Since the growth is exponential, it takes 3 years for the population to double regardless of when you start counting.)