Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

FINAL EXAM-MATH 1300-2002

A/B
Blute/Stanley

December, 2002

Name(Print LEGIBLY) Solutions

I.D. Number

Instructions-This final examination consists of 15 multiple choice questions worth 4 points each. Your answers to the multiple choice questions must be clearly marked in the squares below. There are also 3 long answer questions worth a total of 40 points. For the long answer questions, you must show your work **on the exam itself** and clearly display your answers. **Do not unstaple these pages.**

NO CALCULATORS. NO NOTES OR BOOKS.

Each student will receive one booklet for scratch work only. **Scratch work will not be graded.** Do not unstaple the booklet or remove pages. Your name, student number and seat number must appear on the booklet, and you must turn the booklet in at the end of the exam.

Multiple Choice Answers:

#1	#2	#3	#4
#5	#6	#7	#8
#9	#10	#11	#12
#13	#14	#15	

Question 1- Calculate $f'(0)$, when $f(x)$ is given by the following function:

$$f(x) = \frac{4x+11}{x^2-3}$$

A) $\frac{1}{9}$ **B)** $-\frac{3}{8}$ **C)** $-\frac{1}{32}$ **D)** $-\frac{4}{3}$ **E)** 2

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$f'(x) = \frac{(x^2-3)(4x+11)' - (4x+11)(x^2-3)'}{(x^2-3)^2} = \frac{(x^2-3)(4) - (4x+11)(2x)}{(x^2-3)^2}$$

$$\text{So } f'(0) = \frac{(-3)(4) - (11)(0)}{(-3)^2} = -4/3$$

Question 2- Calculate:

$$\lim_{x \to 2} \frac{x^2+x-6}{x-2}$$

A) 3 **B)** -2 **C)** 9 **D)** -15 **E)** 5

$$\frac{x^2+x-6}{x-2} = \frac{(x+3)(x-2)}{(x-2)}. \text{ So } \lim_{x\to 2} \frac{x^2+x-6}{x-2}$$

$$= \lim_{x \to 2} (x+3) = 5$$

Question 3- Find the interval on which the function $f(x) = x^4 - 8x^3 + 18x^2 - 11x + 21$ is concave down.

A) $(1,3)$ **B)** $(-\infty, 1)$ **C)** $(-\infty, 3)$
D) $(-2,3)$ **E)** $(0,1)$.

$f'(x) = 4x^3 - 24x^2 + 36x - 11$

$f''(x) = 12x^2 - 48x + 36$

$= 12(x^2 - 4x + 3)$

$= 12(x-3)(x-1)$

This is negative when $x \in (1,3)$

Question 4- Find the equation of the tangent line to the graph of $y = x^4 - 5x^3 + 2$ when $x = 2$.

A) $y = -28x + 26$ **B)** $y = -32x + 42$ **C)** $y = -32x + 28$ **D)** $y = -32x + 54$

E) $y = -28x + 34$

Note that when $x = 2$, $y = -22$

$y' = 4x^3 - 15x^2$. $y'(2) = -28$

Line equation is $y = -28x + b$. Now plug in $(2, -22)$

$-22 = -28(2) + b$. So $b = 34$

$\boxed{y = -28x + 34}$

Question 5- Suppose a demand function for a product is given by $p(x) = 216 - 2x^2$. What is the elasticity of demand when $x = 10$? Is demand elastic at this point?

A) $-\frac{3}{50}$, inelastic **B)** $-\frac{3}{50}$, elastic **C)** $-\frac{1}{25}$, inelastic **D)** $-\frac{1}{25}$, elastic **E)** $-\frac{5}{12}$, elastic

$\eta = \dfrac{p/x}{dp/dx}$. When $x=10$, $p=16$

$\dfrac{dp}{dx} = -4x$. When $x=10$, $\dfrac{dp}{dx} = -40$.

So $\eta = \dfrac{16/10}{-40} = \dfrac{16}{10} \cdot \dfrac{-1}{40} = \dfrac{-1}{25}$. Demand is inelastic

Question 6- Which of the following statements is correct for the function

$$f(x) = (x-3)(x+1)^3$$

A) $x = 0$ is a local max. **B)** $x = 0$ is a local min. **C)** $x = 2$ is a local max.
D) $x = 2$ is a local min. **E)** $x = 3$ is a local min.

$f'(x) = (x+1)^3 + (x-3)3(x+1)^2 = (x+1)^2[(x+1)+3(x-3)]$

$= (x+1)^2[4x-8]$. So there is a CP at $x=-1$ and at $x=2$

Apply first derivative test

		$x=-1$		$x=2$	
$f'(x)$	$-$	0	$-$	0	$+$
$f(x)$	↘		↘		↗

So $x=2$ is a local max.

Question 7- Suppose y is defined implicitly as a function of x by the equation:

$$x + \sqrt{xy} = y^2$$

Find $\frac{dy}{dx}$ at the point $(2, 2)$.

A) $-\frac{1}{7}$ B) $\frac{1}{3}$ C) $\frac{3}{2}$ D) $\frac{5}{2}$ E) $\frac{3}{7}$

$1 + \frac{1}{2}(xy)^{-1/2}\left[y + xy'\right] = 2yy'$. Now plug in $(2,2)$

$1 + \frac{1}{2}\left(\frac{1}{2}\right)\left[2 + 2y'\right] = 4y' \Rightarrow 1 + \frac{1}{2} + \frac{1}{2}y' = 4y'$

$\Rightarrow \frac{3}{2} = \frac{7}{2}y' \Rightarrow y' = \frac{3}{7}$

Question 8- If $f(x)$ is a function such that $f'(x) = x^{\frac{2}{3}}$ and $f(1) = 1$, find $f(8)$.

A) $\frac{2}{5}$ B) $\frac{98}{5}$ C) $\frac{52}{5}$ D) $\frac{2}{3}$ E) $\frac{64}{3}$

$\int x^{2/3}\,dx = \frac{3}{5}x^{5/3} + C = f(x)$

$f(1) = \frac{3}{5} + C = 1$, So $C = 2/5$. So $f(x) = \frac{3}{5}x^{5/3} + \frac{2}{5}$

So $f(8) = \frac{3}{5}(32) + \frac{2}{5} = \frac{98}{5}$

Question 9- Calculate:

$$\int_{\sqrt{5}}^{\sqrt{6}} x\sqrt{x^2-5}dx$$

A) $\frac{-2}{3}$ B) $-\frac{1}{2}$ C) $-\frac{2}{5}$ D) $\frac{1}{15}$ E) $\frac{1}{3}$

$$\int x\sqrt{x^2-5}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}(x^2-5)^{3/2}$$

$u = x^2-5$

$du = 2x\,dx$

$$\frac{1}{3}(x^2-5)^{3/2}\Big|_{\sqrt{5}}^{\sqrt{6}} = \frac{1}{3} - 0 = \frac{1}{3}$$

Question 10-

$$\int_1^e x\ln(x)dx =$$

A) $\frac{4e^2+1}{5}$ B) $\frac{3e}{4}$ C) $\frac{e^2+1}{4}$ D) $\frac{e^2-1}{3}$ E) $\frac{e}{3}$

$$\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \int\frac{x}{2}\,dx$$

$u = \ln(x)$ $\quad v = \frac{x^2}{2}$

$du = \frac{1}{x}dx$ $\quad dv = x\,dx$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4}\Big|_1^e$$

$$= \left(\frac{e^2}{2}(1) - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{e^2+1}{4}$$

Question 11-Calculate:

$$\int_0^{\infty} 2e^{-4x}\,dx$$

A) $\frac{1}{2}$ B) $\frac{3}{4}$ C) $\frac{7}{2}$ D) The integral diverges. E) $\frac{5}{4}$

$\int 2e^{-4x}\,dx = -\frac{1}{2}e^{-4x}$

So $\int_0^{\infty} 2e^{-4x}\,dx = \lim_{b\to\infty} -\frac{1}{2}e^{-4x}\Big|_1^b = \cancel{-\frac{1}{2}e^{-4b}} \to 0 - (-\frac{1}{2})$

$= \frac{1}{2}$

Question 12- Suppose that for a certain product, the demand function is given by $D(x) = 21 - 3x$ and the supply function is given by $S(x) = x^2 + x$. Calculate the producer surplus.

A) 3 B) $\frac{11}{3}$ C) $\frac{45}{2}$ D) 1 E) $\frac{27}{2}$

Equilibrium Point

$21 - 3x = x^2 + x$

$0 = x^2 + 4x - 21$

$0 = (x+7)(x-3)$

$x = 3$ is the E.P.

$D(3) = S(3) = 12$

So producer Surplus is:

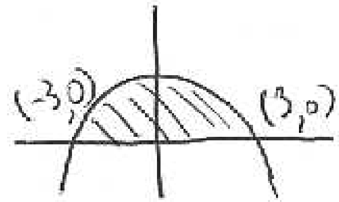
$\int_0^3 12 - (x^2 + x)\,dx = \int_0^3 12 - x^2 - x\,dx$

$= 12x - \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^3 = 36 - 9 - 9/2$

$= \frac{45}{2}$

Question 13- Calculate the area below the curve $f(x) = 9 - x^2$ and above the x-axis.
A) 20 B) 24 C) 28 D) 36 E) 40

$$\int_{-3}^{3} (9 - x^2)\, dx$$



$$= 9x - \frac{x^3}{3}\Big|_{-3}^{3} = (27-9) - (-27+9) = 36$$

Question 14- Find $f_{xy}(1, 1)$ for $f(x, y) = -4x^3 - 3x^2y^3 + 2y^2 + 11$.

A)-9 B) 16 C) -18 D) 10 E) -7

$$f_x = -12x^2 - 6xy^3$$

$$f_{xy} = -18xy^2$$

$$f_{xy}(1,1) = -18$$

Question 15- Consider the function $f(x,y) = e^{xy}$. Which of the following statements is correct?

A) $f(x,y)$ has a local minimum at (0,0).
B) $f(x,y)$ has a local maximum at (0,0).
C) $f(x,y)$ has a saddle point at (0,0).
D) $f(x,y)$ has a saddle point at (-1,-1).
E) $f(x,y)$ has a saddle point at (1,-1).

$$\left.\begin{aligned} f_x &= ye^{xy} \\ f_y &= xe^{xy} \end{aligned}\right\}$$ So there is one CP at $(0,0)$

$f_{xx} = y^2 e^{xy}$ $\qquad f_{xx}(0,0) = 0$

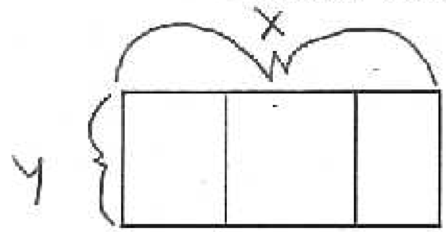
$f_{yy} = x^2 e^{xy}$ $\qquad f_{yy}(0,0) = 0$

$f_{xy} = e^{xy} + xye^{xy}$ $\qquad f_{xy}(0,0) = 1$

So $D = -1$, and $(0,0)$ is a saddle point

Long Answer Question 1 (15 points)

A pig farmer must construct suitable fenced-in regions to contain his pigs. He must construct a rectangular area with two additional fences across its width, as shown in the following picture. Find the dimensions which maximize the area he can enclose with 3600 feet of fencing. Be sure to explain why your answer is an absolute max. You may assume all three rectangles have the same dimensions, but this will not affect the answer.



Let x & y be as labelled. Since there is 3600 ft of fencing, we have $2x+4y=3600$ or $x=1800-2y$

We are trying to maximize $\text{Area}=xy=(1800-2y)y$

So $A(y)=1800y-2y^2$

$A'(y)=1800-4y$. So there is one CP when $y=450$

If $y=450$, then $x=900$. This CP is an <u>absolute</u> max, since the function $A(y)$ is a concave down parabola.

Question 2 (15 points)

Consider the two functions:

$f(x) = x$ and $g(x) = x^2 - 2x$

(a)(3 points) Find the intersection points of the graphs of the two functions.

(b)(6 points) On the next page, graph these functions, and shade the region bounded by $f(x), g(x), x = 0$ and $x = 4$.

(c)(6 points) Find the area of the shaded region.

a) We have $x = x^2 - 2x$ or $x^2 - 3x = 0$ or $x(x-3) = 0$

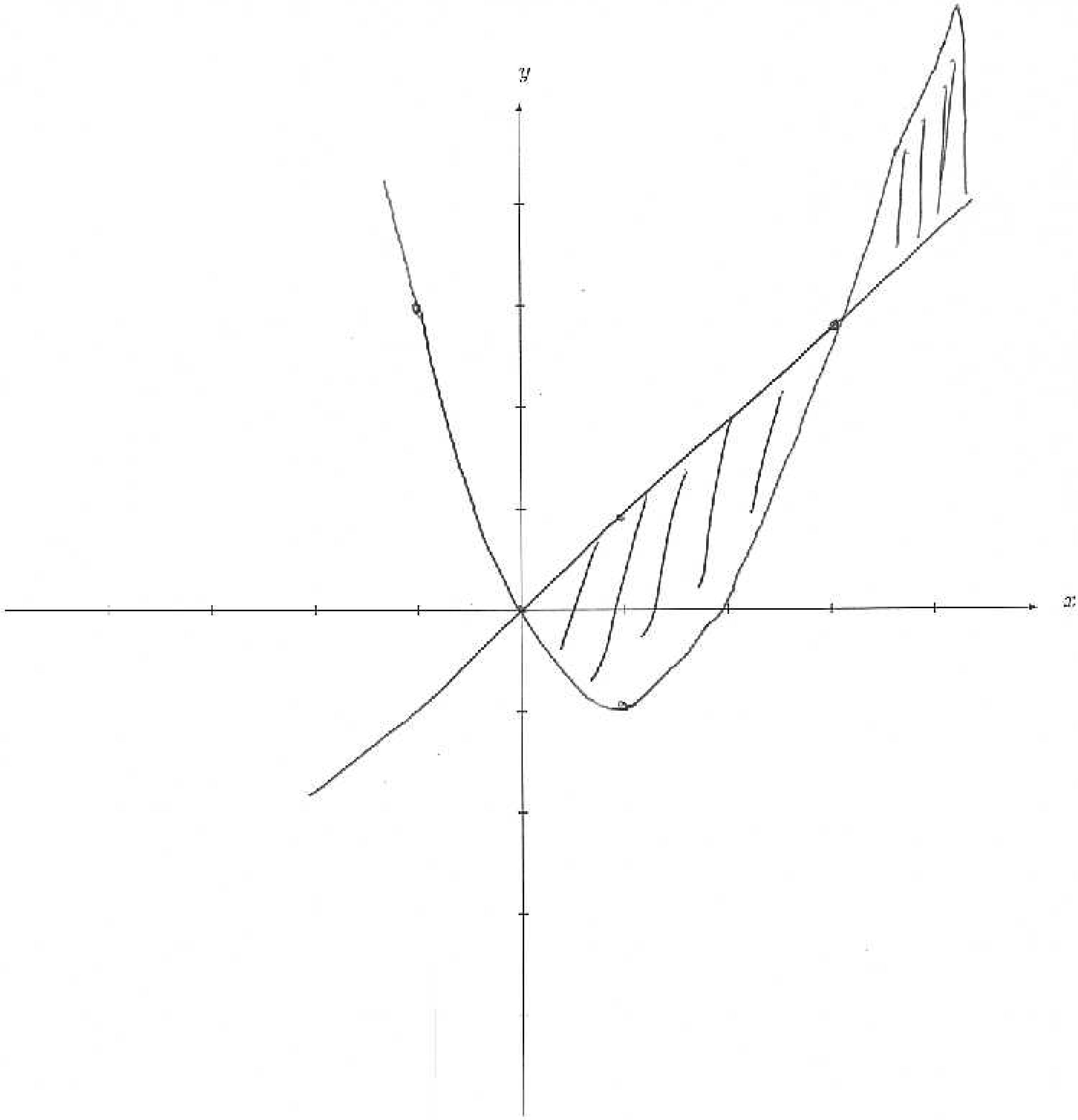
So $x=0$ and $x=3$ are intersection points.

c) Area $= \int_0^3 [x - (x^2 - 2x)]\,dx + \int_3^4 [(x^2 - 2x) - x]\,dx$

$= \int_0^3 (-x^2 + 3x)\,dx + \int_3^4 (x^2 - 3x)\,dx = -\frac{x^3}{3} + \frac{3}{2}x^2 \Big|_0^3$

$+ \frac{x^3}{3} - \frac{3}{2}x^2 \Big|_3^4 = \frac{9}{2} + \left(\frac{64}{3} - -\frac{9}{2}\right) = \frac{18}{2} + \frac{64}{3} = \frac{54}{6} + \frac{128}{6}$

$= \frac{182}{6} = \frac{91}{3}$


y
x

Question 3 (10 points)

Consider the function of two variables $f(x, y) = 2x^3 + 3y^2 - 12xy + 11$. Find all critical points and identify their type.

① $f_x = 6x^2 - 12y = 0$

② $f_y = 6y - 12x = 0 \Rightarrow y = 2x$. Sub into ①

$6x^2 - 24x = 0$

$6x(x-4) = 0$

So $x = 0$ or $x = 4$. So there are 2 CP's at $(0,0)$ and $(4,8)$

$$\left.\begin{aligned} f_{xx} &= 12x \\ f_{yy} &= 6 \\ f_{xy} &= -12 \end{aligned}\right\} \Rightarrow D = 72x - 144$$

For $(0,0)$, $D < 0$. So this is a saddle point

For $(4,8)$, $D > 0$ and $f_{xx} > 0$, so $(4,8)$ is a local min