Improper Integrals, Type I

\[
\int_{1}^{\infty} \frac{1}{x^6} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^6} \, dx
\]

First calculate

\[
\int_{1}^{b} \frac{1}{x^6} \, dx = \int_{1}^{b} x^{-6} \, dx = \left. \frac{x^{-5}}{-5} \right|_{1}^{b} = -\frac{1}{5} \frac{1}{b^5} + \frac{1}{5}
\]

So

\[
\int_{1}^{\infty} \frac{1}{x^6} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{5} \frac{1}{b^5} + \frac{1}{5} \right] = \frac{1}{5}
\]

Harder example

\[
\int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x \ln(x)} \, dx
\]

Note

\[
\int \frac{1}{x \ln(x)} \, dx = \int \frac{1}{u} \, du = \ln(u) = \ln(\ln(x))
\]

\[
\text{with } u = \ln(x), \quad du = \frac{1}{x} \, dx
\]

So

\[
\int_{2}^{b} \frac{1}{x \ln(x)} \, dx = \ln(\ln(b)) - \ln(\ln(2))
\]

\[
\lim_{b \to \infty} \ln(\ln(b)) - \ln(\ln(2)) = \infty
\]

So the integral is divergent.
Improper Integrals - Type II

\[
\int_{0}^{3} \frac{1}{x^5} \, dx
\]

It's improper because one of the endpoints is a V.A.

\[
= \lim_{b \to 0^+} \int_{b}^{3} x^{-5} \, dx = \lim_{b \to 0^+} \frac{1}{4b^4} - \frac{1}{32} = \infty
\]

Integral is divergent.

Note

\[
\int_{0}^{b} \frac{1}{x^5} \, dx = \lim_{b \to 0^-} \int_{-3}^{b} x^{-5} \, dx
\]

In both cases, it is a one-sided limit.

But notice how we determined which side.

End Ch 6  Now Ch 7, which is easier!
Functions of several variables

In any reasonable, realistic scenario, profit, revenue and cost will be functions of many variables. We wish to therefore do calculus on functions that look like

\[ P(x_1, x_2, \ldots, x_n) \]
\[ R(x_1, x_2, \ldots, x_n) \]
\[ C(x_1, x_2, \ldots, x_n) \]

We will focus on functions of 2 variables.

**Ex.**

\[ f(x,y) = x^2 + y^3 \]
\[ g(y) = x^2 - \sqrt{xy^2} \]
\[ \text{etc} \]

Stuff we want to know:

- graphs
- domains

Ultimately, we will want to optimize which requires derivatives.
Graphing \( z = f(x, y) \) requires 3 axes.

Think of how the \( x \)-axis is coming straight out of the page.

Planes: There are 3 crucial planes:
- \( xy \) plane
- \( xz \) plane
- \( yz \) plane
but these are planes of any angle.

Play around with software or online websites (not graphing fun (hmm)).

Q1: Recall from Ch.1 that if \( y = f(x) \), then the domain of \( f \) is the set of all \( x \) for which the function is defined.

EX: If \( f(x) = \sqrt{1 - x^2} \), domain is \( -1 \leq x \leq 1 \).
The graph of \( y = \sqrt{11-x^2} \) is

The domain corresponds to those points \( x \)-axis for which there is a point on the graph.

In the case of a function of 2 variables, the domain will be a subset of the xy plane. For functions such as

\[ f(x, y) = x^2 + y^2 + xy \]

\[ g(x, y) = x^2 \]

will have the whole plane as domain.

A more interesting examples:

\[ f(x, y) = \sqrt{\text{?} - x^2 - y^2} \]

Since we have a square root, we must have \( 16 - x^2 - y^2 \geq 0 \) or \( 16 \geq x^2 + y^2 \). This is the following disk of radius 4 centered at origin.

Graph looks like
Let \( f(x,y) = \frac{4}{(x-2)(y-3)} \)

This function is defined unless \( x = 2 \) or \( y = 3 \)

So the domain will be the entire \( xy \) plane except for the following 2 lines:

There will possibly be a domain question on the final.

Q: A store \( \ast \) carries 2 brands of shoes. The demand for each increases not just when its own price decreases, but when the price of the other increases.

Suppose that when the prices of brand A and brand B are \( p_A \) and \( p_B \) dollars, and the demand for brand A
is \((800 - 30P_A + P_B^2)\). In other words, if the prices are fixed at \(P_A\) and \(P_B\), then this many units will sell. Suppose the demand for bread \(B\) is \((600 - 40P_B + 25P_A)\).

Find the total revenue in terms of \(P_A\) and \(P_B\).

\[
R(P_A, P_B) = \frac{(800 - 30P_A + P_B^2)P_A}{\# \text{ of units of } A} + \frac{(600 - 40P_B + 25P_A)P_B}{\# \text{ of units of } B}
\]

So total revenue = revenue from \(A\) + revenue from \(B\).

Ultimately, we would like to be able to find maxs and mins for such functions.

**Level Curves**

Let \(z = f(x, y)\) be a function.

Given a \(c\), the set of all points satisfying the equation \(f(x, y) = c\) is called the level curve of \(f\) at height \(c\). If you understand the level curves of a function, you can usually piece this info together to understand the whole graph.

**Example:** \(f(x, y) = x^2 + y^2\). Find level
Level curves for $c = 0, 1, 4, 9, 16$

When $c < 0$, there is nothing. These piece together to give a hyperboloid.